Subdivisions of a large clique in C_6-free graphs

József Balogh * Hong Liu † Maryam Sharifzadeh ‡

December 17, 2013

Abstract

Mader conjectured that every C_4-free graph has a subdivision of a clique of order linear in its average degree. We show that every C_6-free graph has such a subdivision of a large clique.

We also prove the dense case of Mader's conjecture in a stronger sense, i.e. for every c, there is a c' such that every C_4-free graph with average degree $cn^{1/2}$ has a subdivision of a clique K_ℓ with $\ell = \lfloor c'n^{1/2} \rfloor$ where every edge is subdivided exactly 3 times.

1 Introduction

A subdivision of a clique K_ℓ, denoted by TK_ℓ, is a graph obtained from K_ℓ by subdividing each of its edges into internally vertex-disjoint paths. The celebrated result of Bollobás and Thomasson [2], and independently of Komlós and Szemerédi [9], states that every graph of average degree d contains a subdivision of a large clique of order $\Omega(\sqrt{d})$. Mader [10] conjectured that if a graph is C_4-free, then one can find a subdivision of a much larger clique, of order linear in its average degree. Kühn and Osthus proved two weaker statements: in [5], they showed that the conjecture is true if the graph has girth at least 15; in [6], they showed that one can find a subdivision of a clique of order almost linear, $\Omega(d/\log^{12} d)$, in any C_4-free graph with average degree d.

Extending ideas in [8] and [9], we prove that every C_6-free graph has such a subdivision of a large clique.

*Department of Mathematical Sciences, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA jobal@math.uiuc.edu. Research is partially supported by Simons Fellowship, NSF CAREER Grant DMS-0745185, Marie Curie FP7-PEOPLE-2012-IIF 327763 and Arnold O. Beckman Research Award (UIUC Campus Research Board 13039).

†Department of Mathematical Sciences, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA hliu36@illinois.edu.

‡Department of Mathematical Sciences, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA sharifz2@illinois.edu.

Theorem 1.1. *Let G be a C_6-free graph with average degree d. Then TK_ℓ is a subgraph of G with $\ell = \lfloor cd \rfloor$ for some small constant c independent of d.*

Similar proof gives the following result, whose proof is omitted.

Theorem 1.2. *Let G be a C_{2k}-free graph with $k \geq 3$ and average degree d. Then TK_ℓ is a subgraph of G with $\ell = \lfloor cd \rfloor$ for some small constant c independent of d.*

Our next result verifies the dense case of Mader's conjecture in a stronger sense.

Theorem 1.3. *For every c there is a $c' > 0$ such that the following holds. Let G be a C_4-free n-vertex graph with $cn^{3/2}$ edges. Then G contains a TK_ℓ with $\ell = \lfloor c'n^{1/2} \rfloor$, in which every edge of the K_ℓ is subdivided exactly 3 times.*

Notation: For a vertex v, denote by $S(v,i)$ the i-th sphere around v, i.e. the set of vertices of distance i from v and denote by $B(v,r)$ the ball of vertices of radius r around v, so $B(v,r) = \cup_{i \leq r} S(v,i)$. For a set $X \subseteq V(G)$, denote by $\Gamma(X)$ the external neighborhood of X, that is $\Gamma(X) := N(X) \setminus X$. Denote by $d(G)$ the average degree of G and for $S \subseteq V(G)$ denote by $d(S)$ the average degree of the induced subgraph $G[S]$. For a set of vertices S, denote by $N_i(S)$ the i-th common neighborhood of S, i.e. vertices of distance exactly i from every vertex in S. For a set $B \subseteq V(G)$, let $\Delta(B) := \max_{v \in B} d_G(v)$.

We will omit floors and ceilings signs when they are not crucial.

2 Preliminaries

For any graph G, there is a bipartite subgraph G' such that $e(G') \geq e(G)/2$. We shall use a result of Györi [3] which states that every bipartite C_6-free graph has a C_4-free subgraph with at least half of its edges. So having a loss of factor of 4 in the average degree, we may assume that our C_6-free graph is bipartite and also C_4-free. Following Komlós and Szemerédi, we introduce the following concept.

(ϵ, t)-expander: For $\epsilon_1 > 0$ and $t > 0$, let $\epsilon(x)$ be the function as follows:

$$\epsilon(x) = \epsilon(x, t, \epsilon_1) := \begin{cases} 0 & \text{if } x < t/5 \\ \epsilon_1 / \log^2(15x/t) & \text{if } x \geq t/5. \end{cases} \tag{1}$$

A graph G is an (ϵ, t)-*expander* if $|\Gamma(X)| \geq \epsilon(|X|) \cdot |X|$ for all subsets $X \subseteq V$ of size $t/2 \leq |X| \leq |V|/2$. Note that $\epsilon(x) \cdot x$ is increasing for $x \geq t/2$.

We will let $\epsilon_1 < \frac{1}{24000}$ be a sufficiently small constant so that $\int_1^\infty \frac{\epsilon(x)}{x} dx < \frac{1}{8}$ (note that ϵ_1 does not depend on t). Komlós and Szemerédi [8, 9] showed that every graph G contains an (ϵ, t)-expander that is almost as dense as G. For the sake of brevity, on $\epsilon(x)$ we do not write the dependency of ϵ_1 and t.

Theorem 2.1. *Let $t > 0$, and choose $\epsilon_1 > 0$ sufficiently small (independent of d) so that $\epsilon = \epsilon(x)$ defined in (1) satisfies $\int_1^\infty \frac{\epsilon(x)}{x} dx < \frac{1}{8}$. Then every graph G has an (ϵ, t)-expander subgraph H with $d(H) \geq d(G)/2$ and $\delta(H) \geq d(H)/2$.*

Remark: The subgraph H might be much smaller than G. For example if G is a vertex-disjoint collection of K_{d+1}'s, then H will be just one of the K_{d+1}'s.

Every (ϵ, t)-expander graph has the following robust "small diameter" property (see Corollary 2.3 in [9]):

Corollary 2.2. *If G is an (ϵ, t)-expander, then any two vertex sets, each of size at least $x \geq t$, are of distance at most*

$$diam := diam(n, t, \epsilon_1) = \frac{2}{\epsilon_1} \log^3(15n/t),$$

and this remains true even after deleting $x\epsilon(x)/4$ arbitrary vertices from G.

By Theorems 2.1, we may assume, when proving Theorem 1.1, that G is a bipartite, $\{C_4, C_6\}$-free, (ϵ, t)-expander graph with average degree d, $\delta(G) \geq d/2$ and $t = d^2/200$, as instead of G we might work in a still dense subgraph of it, having the properties listed before. The next lemma finds in G a "nice" subgraph with "bounded" maximum degree.

Lemma 2.3. *Let G be an n-vertex bipartite, C_4-free, $(\epsilon, d^2/200)$-expander graph with average degree d and $\delta(G) \geq d/2$. Then either G contains a subdivision of a clique of order linear in d, or G has a C_4-free subgraph G' with average degree $d(G') \geq d/2$ and $\delta(G') \geq d(G')/4$, that is $(\epsilon', d(G')^2/50)$-expander with $\epsilon'(x) = \epsilon(x)/2$. Furthermore, G' has at least $n/2$ vertices and $\Delta(G') \leq d(G') \log^8(|V(G')|/d(G')^2)$.*

Note that we do not use the C_6-freeness in Lemma 2.3. Using Lemma 2.3, to prove Theorem 1.1, it will be sufficient to show Theorem 2.4 below.

Theorem 2.4. *Let G be an n-vertex bipartite, $\{C_4, C_6\}$-free, $(\epsilon, d^2/50)$-expander graph with average degree d, $\delta(G) \geq d/4$ and $\Delta(G) \leq d \log^8 n$. Then G contains a $TK_{\ell/2}$ for $\ell = cd$ for some constant $c > 0$ independent of d.*

The rest of the paper will be organized as follows: The proof of Lemma 2.3 will be given in Section 3 as well as the reduction of Theorem 1.1 to Theorem 2.4. The proof of Theorem 2.4 will be divided into two parts according to the range of d: the dense case when $d \geq \log^{14} n$ will be handled in Section 4, and the sparse case when $d < \log^{14} n$ in Section 5. The proof of Theorem 1.3 will be given in Section 6. In Section 7, we will give some concluding remarks as well as the sketch of the proof for Theorem 1.2.

3 Reduction to "bounded" maximum degree

Let G be an n-vertex bipartite C_4-free $(\epsilon, d^2/200)$-expander graph with average degree d and $\delta(G) \geq d/2$, where ϵ is defined in (1).

In this section, we will show that we can transform G into a subgraph G' with $\Delta(G') \leq d(G') \log^8(|V(G')|/d(G')^2)$ and $\delta(G') \geq d(G')/4$, where G' is an $(\epsilon', d(G')^2/50)$-expander graph with $\epsilon'(x) = \epsilon(x)/2$. To prove Lemma 2.3, we shall use the following two lemmas: Lemmas 3.1 and 3.2.

Choose a constant c such that $c \ll \epsilon_1 < \frac{1}{24000}$. Set the parameters as follows:

$$\ell = cd, \quad m = \log \frac{3000n}{d^2}, \quad \Delta = \frac{dm^8}{600}, \quad \Delta' = dm^4, \quad \epsilon(n) = \frac{\epsilon_1}{m^2}, \quad diam = \frac{2m^3}{\epsilon_1}.$$

Note that d has to be sufficiently large (say $d > 1/c$) so that $\ell \geq 1$.

If $m \leq 1/c^2$, then $d \geq e^{-1/2c^2} n^{1/2}$, and we can apply Theorem 1.3 to get a subdivision of a clique of order linear in d. Thus we may assume that $1/m \ll c \ll \epsilon_1$. By the same argument, we may also assume that $d\Delta \leq n$.

Let $L \subseteq V(G)$ be the set of all vertices of degree at least Δ.

Lemma 3.1. *We can find in G either a $TK_{\ell/2}$, or $|L| \leq \ell$ and $G' := G[V \setminus L]$ has maximum degree at most Δ.*

Proof. Indeed, if $|L| \geq \ell$, then we can choose $v_1, \ldots, v_\ell \in L$. We shall build a copy of $TK_{\ell/2}$ using a subset of these high-degree vertices as core vertices.

First we choose for each vertex v_i, $S_1(v_i) \subseteq S(v_i, 1)$ and $S_2(v_i) \subseteq S(v_i, 2)$ such that:

(i) all $S_1(v_i)$'s are pairwise disjoint, and each $S_1(v_i)$ is of size exactly $\Delta/2$;

(ii) every $S_2(v_i)$ is disjoint from $\bigcup_{j=1}^{\ell} S_1(v_j)$, and each $S_2(v_i)$ is of size $d\Delta/5$;

(iii) for every $1 \leq i \leq \ell$, each vertex in $S_1(v_i)$ has at most $d/2$ neighbors in $S_2(v_i)$.

We can indeed select such sets:

For (i), since G is C_4-free, for any v_i, every other v_j with $j \neq i$ has at most one neighbor in $S(v_i, 1)$. Since $|S(v_i, 1)| - \ell \geq \Delta - \ell \geq \Delta/2$, we can remove these neighbors of v_j's from $S(v_i, 1)$ and then choose exactly $\Delta/2$ vertices for $S_1(v_i)$.

For (ii) and (iii), recall that G is bipartite and $\delta(G) \geq d/2$. Thus we can choose, for each vertex in $S_1(v_i)$, exactly $d/2 - 1$ vertices in $S(v_i, 2)$. Since G is C_4-free, for a given v_i, all chosen vertices should be distinct. Thus we have chosen at least $(d/2 - 1)(\Delta/2) \geq 100\ell\Delta \geq 100 \left| \bigcup_{j=1}^{\ell} S_1(v_j) \right|$ vertices, simply discard those vertices which are in $\bigcup_{j=1}^{\ell} S_1(v_j)$ and then choose $d\Delta/5$ vertices for $S_2(v_i)$. Clearly $S_2(v_i)$ satisfies both (ii) and (iii).

We now describe the greedy algorithm that we use to connect the vertices v_i's. Denote by $B_1(v_i) := S_1(v_i) \cup \{v_i\}$ and by $B_2(v_i) := B_1(v_i) \cup S_2(v_i)$.

Greedy Algorithm: We try to connect these ℓ core vertices pair by pair in an arbitrary order. For the current pair of core vertices v_i, v_j, we try to connect $B_2(v_i)$ and $B_2(v_j)$ using a shortest path of length at most *diam* and then exclude all the internal vertices in this path from further connections. We need to justify that such a short path exists.

Suppose we have already connected some pairs using paths of length at most *diam*. We will exclude all previously used vertices from $B_1(v_i) \cup B_1(v_j)$ and also those vertices from $S_2(v_i)$, $S_2(v_j)$ adjacent to removed vertices from $S_1(v_i)$ or $S_1(v_j)$. Formally, let U be the set of vertices used in previous connections and denote by $U_i := U \cap S_1(v_i)$ and by $U_j := U \cap S_1(v_j)$. Define $N := (\Gamma(U_i) \cap S_2(v_i)) \cup (\Gamma(U_j) \cap S_2(v_j))$. Then the set of vertices excluded is $U \cup N$.

First we bound the size of U, it is at most

$$\ell^2 \cdot diam \le c^2 d^2 \cdot \frac{2m^3}{\epsilon_1} \le cd^2 m^3,$$

as there are at most ℓ^2 pairs of core vertices and for each connection, the length of a path is bounded by $diam$.

Call a core vertex v_i bad, if more than Δ' vertices from $S_1(v_i)$ are used in previous connections. During the connections, we discard a core vertex when it becomes bad. We discard in total at most $\ell/2$ core vertices. Indeed, we have used at most $\ell^2 \cdot diam$ vertices. Since by (i), $S_1(v_i)$'s are pairwise disjoint, each bad core vertex, by definition, uses at least Δ' of them. Thus the number of discarded bad core vertices is at most

$$\frac{\ell^2 \cdot diam}{\Delta'} \le \frac{cd^2 m^3}{dm^4} = \frac{cd}{m} \ll \frac{\ell}{2}.$$

Hence there are at least $\ell/2$ core vertices survive the entire process.

Recall that by (iii), each vertex in U_i (or U_j resp.) has at most $d/2$ neighbors in $S_2(v_i)$ (or $S_2(v_j)$ resp.). Note that every survived core vertex is not bad, namely $|U_i| \le \Delta'$. Thus $|N| \le \Delta' \cdot d/2 = d\Delta'/2 = d^2 m^4/2$. Hence the total number of vertices we exclude from $B_2(v_i)$ (or $B_2(v_j)$ resp.) is at most

$$\ell^2 \cdot diam + |N| \le cd^2 m^3 + \frac{1}{2} d^2 m^4 \le d^2 m^4.$$

After excluding these vertices, we still have at least

$$|S_2(v_i)| - \ell^2 \cdot diam - |N| \ge \frac{d\Delta}{5} - d^2 m^4 \ge \frac{d\Delta}{10}$$

vertices left in $S_2(v_i)$, the same holds for $S_2(v_j)$. Recall that, when $x \ge \frac{1}{2}\left(\frac{d^2}{200}\right)$, $\epsilon(x)$ is decreasing and $x\epsilon(x)$ is increasing, we have that the number of vertices we are allowed to exclude, by Corollary 2.2, is at least

$$\frac{1}{4} \cdot \frac{d\Delta}{10} \cdot \epsilon\left(\frac{d\Delta}{10}\right) \ge \frac{d\Delta}{40} \cdot \epsilon(n) \ge \frac{d^2 m^8}{24000} \cdot \frac{\epsilon_1}{m^2} \ge \frac{\epsilon_1 d^2 m^6}{24000} \gg d^2 m^4,$$

where the last inequality follows from $1/m \ll c \ll \epsilon_1 < \frac{1}{24000}$. Thus the exclusion of these vertices will not affect the robust small diameter property between $B_2(v_i)$'s. So the $\ell/2$ remaining core vertices can be connected to form a $TK_{\ell/2}$. □

Given that c is sufficiently small and now we can assume $|L| \le \ell$, we have that $|V(G')| \ge n - \ell \ge n/2$, $d(G') \ge \frac{2(dn/2 - \ell n)}{n} = d - 2\ell \ge d/2$ and $\delta(G') \ge \delta(G) - \ell \ge d/2 - \ell \ge d(G')/4$.

Lemma 3.2. *The obtained graph G' is an $(\epsilon', d(G')^2/50)$-expander graph with $\epsilon'(x) = \epsilon(x)/2$.*

Proof. Indeed, by definition, for any set X in G' of size $x \geq \frac{1}{2}\left(\frac{d(G')^2}{50}\right) \geq \frac{1}{2}\left(\frac{d^2}{200}\right)$, since $c \ll \epsilon_1$ we have

$$|\Gamma_G(X)| \geq x\epsilon(x) \geq \frac{d^2}{400} \cdot \epsilon\left(\frac{d^2}{400}\right) = \frac{\epsilon_1 d^2}{400\log^2(7.5)} \gg \ell = |L|.$$

Hence $|\Gamma_{G'}(X)| \geq |\Gamma_G(X)| - |L| \geq x\epsilon(x) - \ell \geq x\epsilon'(x)$. □

Since $24000 \ll n/d^2 \leq 2|V(G')|/d(G')^2$, the maximum degree of G' is at most

$$\Delta = \frac{dm^8}{600} \leq \frac{d(G')}{300} \cdot \log^8 \frac{6000|V(G')|}{d(G')^2} \leq \frac{d(G')}{300}\left(2\log\frac{|V(G')|}{d(G')^2}\right)^8 \leq d(G')\log^8\frac{|V(G')|}{d(G')^2}.$$

Slightly abusing the notation, we work in the future only with G', we will rename G' as G, relabelling $n = |V(G')|$ and $d = d(G')$, and by changing ϵ_1 to $\epsilon_1/2$, we assume that G is $(\epsilon, d^2/50)$-expander and its maximum degree is at most $d\log^8(n/d^2)$. This completes the reduction step, i.e. to prove Theorem 1.1 it is sufficient to prove Theorem 2.4.

4 Dense case of Theorem 2.4

Let G be a graph satisfying the conditions in Theorem 2.4, i.e. an n-vertex bipartite $\{C_4, C_6\}$-free $(\epsilon, d^2/50)$-expander graph, with average degree d, $\delta(G) \geq d/4$ and $\Delta(G) \leq d\log^8 n$.

Recall that the constants are chosen such that $c \ll \epsilon_1$ so that $\ell = cd$. In addition, set the parameters in this section as follows:

$$d \geq \log^{14} n, \quad \Delta = d\log^8 n, \quad \Delta'' = d\log^{13} n, \quad b = \frac{d}{\log^9 n} diam = \frac{2}{\epsilon_1}\log^3\left(\frac{750n}{d^2}\right) \leq \frac{1}{c}\log^3 n.$$

Note that $\Delta \gg d \gg b$, $\Delta'' = o(d^2)$, and $\ell/b \leq d/b = \log^9 n$.

We will first find ℓ vertices, $v_1, \ldots, v_\ell$ serving as core vertices, along with some sets $B_3(v_i) \subseteq B(v_i, 3)$. We then connect all core vertices by linking $B_3(v_i)$'s using a greedy algorithm. Similarly to the proof in Section 3, we might discard few core vertices during the process.

4.1 Choosing core vertices and building $B_3(v_i)$.

We will select ℓ vertices $v_1, \ldots, v_\ell$ in ℓ/b steps to serve as core vertices.
Stage 1: choose core vertices $v_1, \ldots, v_\ell$ and $B_2(v_i)$'s.

In each step, we choose a block of vertices consisting of: b core vertices and for each core vertex v_i a set $B_2(v_i) := S_1(v_i) \cup S_2(v_i) \cup \{v_i\}$, where $S_1(v_i) \subseteq S(v_i, 1)$ and $S_2(v_i) \subseteq S(v_i, 2)$ with the following properties.

(i) $S_1(v_i)$'s are pairwise disjoint for all $1 \leq i \leq \ell$ and $|S_1(v_i)| = d/2$.
(ii) For every i, $|S_2(v_i)| = d^2/10$.
(iii) Every vertex $w \in S_1(v_i)$ has at most $d/4$ neighbors in $S_2(v_i)$.

(iv) Inside each block, $B_2(v_i)$'s are pairwise disjoint.

(v) Every $S_2(v_i)$ is disjoint from $\cup_{j=1}^{\ell} S_1(v_j)$.

This can be done as follows: suppose we have chosen some core vertices $v_1, v_2, \ldots, v_{i-1}$ and surrounding $B_2(v_j)$'s for $j \le i-1$. Denote D be the current block. To choose the next core vertex v_i, we will exclude $\{\bigcup_{j\le i-1} S_1(v_j)\} \cup \{\bigcup_{v_k \in D} B_2(v_k)\}$. The number of excluded vertices is at most

$$\sum_{j\le i} |S_1(v_j)| + b \cdot \max_{v_k\in D} |B_2(v_k)| \le \ell d + b \cdot d^2/2 \le b \cdot d^2.$$

The number of the edges incident to the excluded vertices is at most

$$\Delta \cdot b \cdot d^2 = \frac{d^4}{\log n} \ll dn/2 = e(G),$$

where the last inequality holds since G is C_6-free and $d = O(n^{1/3})$. Thus, we can easily find in G, excluding these vertices, a subgraph G' with average degree at least $d/2$ and minimum degree at least $d/4$. We then choose v_i to be any vertex in G' of degree at least $d/2$. Choose $d/2$ neighbors of v_i to be $S_1(v_i)$. Let $B_1(v_i) := S_1(v_i) \cup \{v_i\}$. Since G is bipartite, for each vertex $u \in S_1(v_i)$, we can choose $d/4-1$ neighbors of u not in $S_1(v_i)$. Again, by C_4-freeness, we have chosen $d^2/8 - d/2$ different vertices. Denote the resulting set $S_2'(v_i)$.

Note that $S_2'(v_i)$ might and probably will intersect $S_1(v_j)$ for $j \ge i+1$ chosen after v_i. Delete from $S_2'(v_i)$ vertices that are in $\bigcup_{j\ge i+1} S_1(v_j)$. We will delete at most $\ell \cdot d$ vertices from $S_2'(v_i)$. Since $|S_2'(v_i)| - \ell \cdot d \ge d^2/10$, we can choose exactly $d^2/10$ vertices to be $S_2(v_i)$.

Stage 2: For each $1 \le i \le \ell$, choose $S_3(v_i)$ and $B_3(v_i)$ with $|B_3(v_i)| \ge d^3/50$.

For each vertex in $S_2(v_i)$, since G is bipartite and C_4-free, we can choose $d/4-1$ of its neighbors not in $S_1(v_i) \cup S_2(v_i)$ and denote the resulting set $S_3'(v_i)$. Since G is C_6-free, $|S_3'(v_i)| = |S_2(v_i)| \cdot (d/4-1) = d^3/40 - d^2/10$. Delete from $S_3'(v_i)$ any vertex in $\bigcup_{1\le j\le \ell} S_1(v_j)$. Since we delete at most d^2 vertices, we can choose a subset of size $d^3/50$ to be $S_3(v_i)$. Let $B_3(v_i) := B_2(v_i) \cup S_3(v_i)$.

4.2 Connecting core vertices

Greedy Algorithm: Now we will connect these ℓ core vertices pair by pair in arbitrary order. For each pair v_i and v_j, we will connect them with a path of length at most *diam* avoiding $\bigcup_{p\ne i,j} B_1(v_p)$.

(I) Discard bad core vertices:

Call a core vertex v_i bad, if we use more than Δ'' vertices from $S_2(v_i)$. Discard a core vertex as soon as it becomes bad. During the entire process, we use at most $\ell^2 \cdot diam$ vertices from previous connections. Since $B_2(v_i)$'s are pairwise disjoint inside each block, each of the excluded vertices can appear in at most ℓ/b many $S_2(v_i)$'s. Hence, the number of bad core vertices is at most:

$$\frac{\ell^2 \cdot diam \cdot (\ell/b)}{\Delta''} \le \frac{d^2 \cdot diam \cdot (\ell/b)}{d \log^{13} n} = \frac{d \log^3 n \cdot \ell}{cb \log^{13} n} = \frac{\ell}{c \log n} \ll \ell/2.$$

(II) Cleaning before connection:

Assume that we have already connected some pairs of core vertices, and now we want to connect v_i and v_j. Before we start connecting them, clean $B_3(v_i)$ (do the same for $B_3(v_j)$) in the following way. Notice that we have used in previous connections at most ℓ vertices in $S_1(v_i)$, at most Δ'' vertices in $S_2(v_i)$ and at most $\ell^2 \cdot diam$ vertices in $S_3(v_i)$, since vertices in $S_1(v_i)$ were only used when connecting v_i to other core vertices and v_i is not bad. Also, delete those vertices that are no longer available, i.e. those adjacent to used ones. Call the resulting set $B_3'(v_i)$. Since every vertex in $S_k(v_i)$, $k = 1, 2$, has at most $d/4$ neighbors in $S_{k+1}(v_i)$, we have deleted at most $\ell(1 + d/4 + d^2/16) + \Delta''(1 + d/4) + \ell^2 \cdot diam \ll d^3/100$ vertices. Thus $|B_3'(v_i)| \geq |B_3(v_i)| - d^3/100 \geq d^3/100$.

(III) Connecting core vertices:

We will connect v_i and v_j by a shortest path from $B_3'(v_i)$ to $B_3'(v_j)$ avoiding $\bigcup_{p\neq i,j} B_1(v_p)$ which is of size at most d^2. This path has length at most $diam$ if we do not break the robust diameter property. We then exclude this path for further connections. The number of excluded vertices from previous paths and from $\bigcup_{p\neq i,j} B_1(v_p)$ is at most $\ell^2 \cdot diam + d^2 \leq d^2 \log^3 n$. On the other hand, the number of vertices we are allowed to exclude without breaking the robust small diameter among $B_3'(v_i)$'s is

$$\frac{1}{4}|B_3'(v_i)|\epsilon(|B_3'(v_i)|) \geq \frac{d^3}{400}\epsilon(n) \geq \frac{\epsilon_1 d^3}{400 \log^2 n} \gg d^2 \log^3 n.$$

Thus the robust diameter property is guaranteed during the entire process.

This completes the proof in the dense case of Theorem 2.4.

5 Sparse case of Theorem 2.4

Let G be a graph satisfying the conditions in Theorem 2.4, i.e. an n-vertex bipartite C_4-free $(\epsilon, d^2/50)$-expander graph, with ϵ defined in (1), with average degree d, $\delta(G) \geq d/4$ and $\Delta(G) \leq d \log^8 n$.

In this section, we use an idea from [8] to deal with the sparse case, i.e. when $d \leq \log^{14} n$. Set $\ell = cd$ with $c \ll \epsilon_1$. Then we have

$$\Delta = d \log^8 n \leq \log^{22} n, \quad diam \leq \frac{1}{c} \log^3 n, \quad \ell^2 \cdot diam \leq d^2 \log^3 n \leq \log^{31} n.$$

We set some additional parameters as follows:

$$R = \frac{\log n}{(\log\log n)^2}, \qquad r = (\log\log n)^4, \qquad \mu = \frac{\epsilon_1^{1/3}}{6}. \tag{2}$$

The following lemma gives a robust expansion property of $B(v, k)$, which roughly says that the expansion property of the ball around any vertex remains even after deleting few vertices from each of its spheres.

Lemma 5.1. *Let $G = (V, E)$ be the graph defined at the beginning of this section and let r and μ defined in (2). Then for any vertex $v \in V$ and any integer k, $B(v, k)$ has the following robust expansion property:*

Let $D \subseteq V$ be a set of vertices such that:

(i) $|D \cap S(v, i)| \leq \ell$, for every $i \leq r$;

(ii) $|D| \leq e^{\mu r^{1/3}} \epsilon(e^{\mu r^{1/3}})/2$.

For every $i \geq 1$, let $D_i := D \cap B(v, i)$ and set $S(v, 0) = S'(v, 0) = \{v\}$ and for every $k \geq 0$, define $B'(v, k) := \cup_{j=0}^{k} S'(v, j)$ and $S'(v, k+1) := \Gamma(B'(v, k)) - D_{k+1}$. Then we have, for every $k \geq 3$, that

$$|B(v, k)| \geq |B'(v, k)| \geq \min\{\exp\{\mu k^{1/3}\}, n/2\}. \tag{3}$$

Proof. Fix a vertex $v \in V(G)$. Notice, crucially, that for every $k \geq 1$, the new sphere $S'(v, k)$ by definition uses only vertices from $B(v, k)$. Hence $B'(v, k) \subseteq B(v, k)$ for every $k \geq 1$. Note also that, for every $1 \leq i \leq r$, we have $|D_i| \leq i \cdot \ell$ by property (i) of D.

For every $k \geq 1$, let $x_k := |B(v, k)|$ and $x'_k := |B'(v, k)|$. Note that $x'_1 \geq \delta(G) - |D_1| \geq d/4 - \ell > d/5$ and since G is C_4-free, $x'_2 \geq x'_1 \delta(G)/2 \geq d^2/50$. Recall that G is $(\epsilon, d^2/50)$-expander, so $B'(v, k)$ expands for every $k \geq 2$.

We now show, by induction on k, that for every $k \geq 3$,

$$x'_k \geq x'_2 \cdot \prod_{i=2}^{k-1} (1 + \epsilon(x'_i)/2) \tag{4}$$

and

$$x'_k \geq \min\{\exp\{\mu k^{1/3}\}, n/2\}. \tag{5}$$

For the base case when $k = 3$, since $B'(v, 2)$ expands, for (4), we have

$$x'_3 \geq x'_2(1 + \epsilon(x'_2)) - |D_3| \geq x'_2(1 + \epsilon(x'_2)/2),$$

where the last inequality follows from $|D_3| \leq 3\ell < d \ll x'_2 \epsilon(x'_2)/2$. For (5), we have $x'_3 \geq x'_2 \geq d^2/50 \gg e^{\mu 3^{1/3}}$.

For the inductive step, we split the proof of (4) into two cases: $k \leq r - 1$ and $k \geq r$.

Case 1: $3 \leq k \leq r - 1$, and suppose (4) and (5) are true up to k. Notice that

$$\begin{aligned} x'_{k+1} &\geq |S'(v, k+1)| + |B'(v, k)| \geq |\Gamma(B'(v, k))| - |D_{k+1}| + |B'(v, k)| \\ &\geq x'_k \epsilon(x'_k) - (k+1)\ell + x'_k = x'_k(1 + \epsilon(x'_k)) - (k+1)\ell. \end{aligned}$$

By the induction hypothesis, $x'_k \geq x'_2 \cdot \prod_{i=2}^{k-1}(1 + \epsilon(x'_i)/2)$. Hence, it is sufficient to show that $(k+1)\ell \leq x'_k \epsilon(x'_k)/2$.

If $k \geq d$, then by the induction hypothesis, $x'_k \geq e^{\mu k^{1/3}} \gg d^2$ (here we assume that $d \gg (1/\epsilon_1)^2$). Recall that $x\epsilon(x)$ is increasing when $x \geq d^2$, thus we have

$$\frac{1}{2} x'_k \epsilon(x'_k) \geq \frac{1}{2} e^{\mu k^{1/3}} \epsilon\left(e^{\mu k^{1/3}}\right) \geq \frac{\epsilon_1 e^{\mu k^{1/3}}}{2\log^2(e^{\mu k^{1/3}})} \geq \frac{\epsilon_1 e^{\mu k^{1/3}}}{2\mu^2 k^{2/3}} \gg k^2 \geq (k+1)\ell.$$

If $k < d$, since $x'_2 \geq d^2/50$ and $c \ll \epsilon_1$, we have

$$\frac{1}{2}x'_k\epsilon(x'_k) \geq \frac{1}{2}x'_2\epsilon(x'_2) \geq \frac{d^2}{100} \cdot \epsilon\left(\frac{d^2}{50}\right) = \frac{\epsilon_1 d^2}{100\log^2 15} \geq (k+1)\ell.$$

Case 2: $k \geq r$, and suppose (4) and (5) are true up to k. By property (ii) of D, the induction hypothesis that $x'_r \geq e^{\mu r^{1/3}} \geq d^2$, we have $|D| \leq x'_r\epsilon(x'_r)/2$. Therefore,

$$\begin{aligned} x'_{k+1} &\geq |S'(v,k+1)| + |B'(v,k)| \geq |\Gamma(B'(v,k))| - |D| + |B'(v,k)| \\ &\geq x'_k\epsilon(x'_k) - x'_r\epsilon(x'_r)/2 + x'_k \geq x'_k(1+\epsilon(x'_k)/2) \geq x'_2 \cdot \prod_{i=2}^{k}(1+\epsilon(x'_i)/2), \end{aligned}$$

where the second last inequality comes from the fact that $x\epsilon(x)$ is increasing when $x \geq d^2$ and $x'_k \geq x'_r \geq d^2$ and the last inequality follows from the induction hypothesis.

For the inductive step of (5), recall that a set stops to expand when it has size at least $n/2$, it thus suffices to show that, if $\exp\{\mu(k+1)^{1/3}\} < n/2$, then $x'_{k+1} \geq \exp\{\mu(k+1)^{1/3}\}$. By the induction hypothesis, we have that

$$\log x'_{k+1} \geq \log x'_2 + \sum_{i=2}^{k} \log\left(1 + \frac{\epsilon(x'_i)}{2}\right) \geq (k-1)\log\left(1+\frac{\epsilon(x'_k)}{2}\right) \geq \frac{1}{3}k\epsilon(x'_k).$$

We may assume that $\log x'_k \leq \mu(k+1)^{1/3}$, otherwise $x'_{k+1} \geq x'_k \geq \exp\{\mu(k+1)^{1/3}\}$ and we are done. Therefore,

$$\epsilon(x'_k) \geq \frac{\epsilon_1}{\log^2(x'_k)} \geq \frac{\epsilon_1}{\mu^2(k+1)^{2/3}} \geq \frac{\epsilon_1}{2\mu^2 k^{2/3}}.$$

Consequently, we have

$$\log x'_{k+1} \geq \frac{1}{3}k\epsilon(x'_k) \geq \frac{\epsilon_1 k}{6\mu^2 k^{2/3}} = 36\mu k^{1/3} \geq \mu(k+1)^{1/3}.$$

□

Notice that since $r \ll R \ll \log n$, the minimum in (3) is always taken to be $\exp\{\mu k^{1/3}\}$ for all the calculations that follows.

Lemma 5.2. *Let R and r be defined in (2). Given a set $D \subseteq V$, for every vertex v, define $S'(v,k)$ and $B'(v,k)$ as in Lemma 5.1. Then there exists ℓ vertices, $v_1, \ldots, v_\ell$, in G satisfying the following properties:*

(i) The balls $B(v_i, R)$ are pairwise disjoint;

(ii) $|B'(v_i,R)|\epsilon(|B'(v_i,R)|) \gg \ell \cdot \max_i |B(v_i,r)| + \ell^2 \cdot diam$, for every $1 \leq i \leq \ell$.

Proof. We will repeat the following process ℓ times to get $v_1, \ldots, v_\ell$: at $(i+1)$-th time, choose an arbitrary vertex v_{i+1} in $V(G) \setminus \cup_{j=1}^{i} B(v_j, 2R)$.

Define $b'_k := \min_i |B'(v_i, k)|$ and $B_k := \max_i |B(v_i, k)|$. By Lemma 5.1, we have $|b'_k| \geq \min\{\exp\{\mu k^{1/3}\}, n/2\}$. On the other hand, note that

$$B_k \leq \Delta^k = \exp\{k \log \Delta\} \leq \exp\{22k \log\log n\}.$$

Thus (i) is satisfied if we do not run out of vertices, i.e. if $\ell \cdot B_{2R} < n$. Indeed,

$$B_{2R} \leq \exp\{44R \log\log n\} \leq \exp\left\{\frac{44 \log n}{(\log\log n)^2} \cdot \log\log n\right\} = \exp\left\{\frac{44 \log n}{\log\log n}\right\} \ll n/\ell.$$

For (ii), we have for every $1 \leq i \leq \ell$,

$$\ell \cdot \max_i |B(v_i, r)| + \ell^2 \cdot diam \leq \ell \cdot B_r + \log^{31} n \leq \log^{14} n \cdot \exp\{22r \log\log n\} + \log^{31} n \ll \exp\{23(\log\log n)^5\}.$$

On the other hand, since $R^{1/3} = \left(\frac{\log n}{(\log\log n)^2}\right)^{1/3} \gg \log^{1/4} n$, we have by Lemma 5.1 that

$$|B'(v_i, R)|\epsilon(|B'(v_i, R)|) \geq b'_R \cdot \epsilon(n) \geq \frac{\epsilon_1 \exp\{\mu R^{1/3}\}}{\log^2 n} \gg \frac{\epsilon_1 \exp\left\{\mu \log^{1/4} n\right\}}{\log^2 n} \gg \exp\{23(\log\log n)^5\}.$$

□

Proof of Theorem 2.4 for $d \leq \log^{14} n$: Apply Lemma 5.2 to get $v_1, \ldots, v_\ell$ with those two properties. We will build a TK_ℓ using v_i's as core vertices. We connect pairs of core vertices in an arbitrary order. The idea is that before we connect two core vertices, we first re-grow the balls around them avoiding using any vertex that has been used in previous connections and vertices that are close to other core vertices. We want to always build short paths for connections, consequently we never exclude too many vertices while re-growing. Hence by Lemma 5.1, new balls are big enough to enjoy the robust diameter property in Corollary 2.2.

Greedy Algorithm:

(I) Connecting the first pair: For the first pair of vertices, say v_1, v_2, we will connect $B(v_1, R)$ with $B(v_2, R)$ using a shortest path avoiding all vertices in $\bigcup_{p \neq 1,2} B(v_p, r)$. By Lemma 5.2 (ii) and Corollary 2.2, this path is of length at most *diam*. We then exclude this path for further connections.

(II) Re-growing balls: Let v_i, v_j be the current pair to be connected. Let $D \subseteq V$ be the set of vertices that have been used in previous connections. Notice that for every core vertex v_i, $1 \leq i \leq \ell$, any vertex $u \in B(v_i, r)$ can only be used in paths connecting v_i to other core vertices. Therefore there are at most ℓ vertices used in each sphere inside $B(v_i, r)$, i.e. $|D \cap S(v_i, k)| \leq \ell$ for every $1 \leq k \leq r$. Since all paths built so far have length at most *diam* and there are at most ℓ^2 paths, we have

$$\begin{aligned} |D| &\leq \ell^2 \cdot diam \leq \log^{31} n \ll \frac{\epsilon_1 \exp\{\mu(\log\log n)^{4/3}\}}{2\mu^2(\log\log n)^{8/3}} \\ &\leq \frac{\epsilon_1 \exp\{\mu r^{1/3}\}}{2\mu^2 r^{2/3}} = \frac{\epsilon_1 \exp\{\mu r^{1/3}\}}{2\log^2(\exp\{\mu r^{1/3}\})} \leq \frac{1}{2}\exp\{\mu r^{1/3}\}\epsilon(\exp\{\mu r^{1/3}\}). \end{aligned}$$

Thus we can apply Lemma 5.1 and re-grow balls around v_i, v_j avoiding D as in Lemma 5.1 to get $B'(v_i, R)$ and $B'(v_j, R)$ satisfying Lemma 5.2 (ii).

(III) Connecting the current pair: We connect $B'(v_i, R)$ and $B'(v_j, R)$ in G avoiding vertices in $\bigcup_{p\neq i,j} B(v_p, r) \cup D$. The number of vertices we exclude here is at most $\ell \cdot \max_i |B(v_i, r)| + \ell^2 \cdot diam$. Hence by Lemma 5.2 (ii) and Corollary 2.2, we can connect $B'(v_i, R)$ and $B'(v_j, R)$ using a path of length at most $diam$. We then exclude this path for further connection.

This finishes the sparse case of Theorem 2.4. □

6 Proof of Theorem 1.3

In this section, we will prove Theorem 1.3 using a variation of the Dependent Random Choice Lemma (see survey [4] for more details on the method of dependent random choice). The following lemma roughly says that in a dense C_4-free graph one can find a set in which every small subset has a large second common neighborhood.

Lemma 6.1. *Let $G = (A \cup B, E)$ be a C_4-free bipartite graph on n vertices with $cn^{3/2}$ edges and $|A| = |B| = \frac{n}{2}$, where $n > 1/c^{20}$. If there exist positive integers a, m, r and t such that*

$$c^{2t} n - \binom{n}{r} \left(\frac{m}{n}\right)^t \geq a, \tag{6}$$

then there exists $U \subseteq A$ with at least a vertices such that for every r-subset $S \subseteq U$, $|N_2(S)| \geq m$.

Proof. First notice that

$$\begin{aligned}\sum_{v\in A} |N_2(v)| &= \sum_{v\in B} (d(v)-1)d(v) = \sum_{v\in B} d(v)^2 - \sum_{v\in B} d(v) \geq \frac{n}{2}\left(\frac{\sum_{v\in B} d(v)}{n/2}\right)^2 - e(G)\\ &= \frac{n}{2}(2cn^{1/2})^2 - cn^{3/2} \geq c^2 n^2.\end{aligned}$$

Pick a set $T \subseteq A$ of t vertices uniformly at random with repetition. Let $W := N_2(T) \subseteq A$ and put $X := |W|$. Then by the linearity of expectation we have

$$\begin{aligned}\mathbb{E}[X] &= \sum_{v\in A} \mathbb{P}(v \in N_2(T)) = \sum_{v\in A} \left(\frac{|N_2(v)|}{n/2}\right)^t = \left(\frac{2}{n}\right)^t \cdot \frac{n}{2} \cdot \left(\frac{1}{n/2} \sum_{v\in A} |N_2(v)|^t\right)\\ &\geq \left(\frac{n}{2}\right)^{1-t} \cdot \left(\frac{\sum_{v\in A} |N_2(v)|}{n/2}\right)^t = \left(\frac{n}{2}\right)^{1-t} \cdot (2c^2 n)^t \geq c^{2t} n.\end{aligned}$$

Let Y be the random variable counting the number of r-sets in W that has fewer than m common second neighbors. The probability for a fixed such r-set S to be in W is at most $\left(\frac{m}{n}\right)^t$. There are at most $\binom{n}{r}$ r-sets, hence

$$\mathbb{E}[X - Y] \geq c^{2t} n - \binom{n}{r} \left(\frac{m}{n}\right)^t \geq a.$$

Thus there exists a choice of T, such that $X - Y \geq a$. Delete one vertex from X for each such "bad" r-set from W, and the resulting set U has the desired property. □

Claim 6.2. *When proving Theorem 1.3, we may assume that G is bipartite on $A \cup B$ with $|A| = |B| = n/2$, $d(G) = d$ and all vertices in B has degree smaller than $30d$.*

Proof. We may assume that for any $H \subseteq G$, $d(H) \leq d$, otherwise we can work in H instead. Let $X \subseteq V$ be the set of vertices of degree at least $10d$, thus $|X| \leq n/10$. Let $Y = V \setminus X$. Since $d(G[X]) \leq d$, we have $e(G[X]) \leq d|X|/2 \leq e(G)/10$. Take an $\frac{n}{2}$-subset B of Y uniformly at random and call $V \setminus B = A$. Then we have,

$$\mathbb{E}(e(G[A,B])) \geq 0.4[e(G[Y]) + e(G[X,Y])] = 0.4[e(G) - e(G[X])] \geq 0.36e(G).$$

Therefore there exists a choice of A, B such that $e(G[A,B]) \geq 0.36e(G)$. Hence we can replace G by $G' := G[A,B]$, and every vertex in B has degree less than $10d \leq 10 \cdot (d(G')/0.36) < 30d(G')$. □

Proof of Theorem 1.3. Assume G satisfies the conditions of Claim 6.2 and apply Lemma 6.1 to G with the following parameters:

$$a = \frac{c^6 n^{1/2}}{240}, \quad r = 2, \quad t = \frac{\log n}{4\log(1/c)}, \quad m = c^6 n.$$

In order to prove that (6) is satisfied, we shall prove $2\binom{n}{2}\left(\frac{m}{n}\right)^t \leq c^{2t}n$ and $c^{2t}n \geq 2a$. Indeed,

$$2\binom{n}{2}\left(\frac{m}{n}\right)^t \leq c^{2t}n \ \Leftarrow\ n \leq \left(\frac{c^2 n}{m}\right)^t = \left(\frac{1}{c}\right)^{4t} \Leftarrow \log n \leq 4t \cdot \log\frac{1}{c} = \log n.$$

On the other hand, we have

$$c^{2t}n \geq 2a = \frac{c^6 n^{1/2}}{120} \Leftrightarrow \frac{120 n^{1/2}}{c^6} \geq \left(\frac{1}{c}\right)^{2t} \Leftarrow \log 120 + \frac{1}{2}\log n + 6\log\frac{1}{c} \geq 2t\log\frac{1}{c} = \frac{1}{2}\log n.$$

Thus there exists $U \subseteq A$ of size at least $a = \frac{c^6 n^{1/2}}{240}$ such that for every pair of vertices $S \subseteq U$, $|N_2(S)| \geq m = c^6 n$.

We embed a copy of TK_ℓ with $\ell = a = c^5 d/480$ greedily as follows: first embed all the core vertices arbitrarily to U. Then we connect all pairs of core vertices one by one, in an arbitrary order, with internally vertex-disjoint paths of length 4. Fix a pair of vertices $S \subseteq U$. For every vertex v in $N_2(S)$, call $C(v) := N(v) \cap \Gamma(S)$ its *connector set* and call v "bad" if $|C(v)| = 1$. Since G is C_4-free, $|N_1(S)| \leq 1$, so there are at most $\Delta(B) \leq 30d$ bad vertices in $N_2(S)$. Any vertex $v \in N_2(S)$ that is not bad has $|C(v)| = 2$. When connecting S, we will exclude from $N_2(S)$ the following vertices: (i) bad vertices (if they exist); (ii) vertices in U; (iii) vertices that were already used in previous connections; (iv) vertices whose connector set was used. It follows immediately that if there is a vertex left in $N_2(S)$, then together with its connector set, we can connect S.

For (i) and (ii), recall that there are at most $30d$ bad vertices and $|U| \le \ell$. For (iii), there are at most $\binom{\ell}{2}$ such vertices, one for each pair of core vertices. Thus there are at least $m - 30d - \ell - \binom{\ell}{2} \ge c^6 n - 60cn^{1/2} - \ell^2 \ge c^6 n/2$ many vertices left in $N_2(S)$.

For (iv), we say that two vertices in $N_2(S)$ have no *conflict* with each other if their connector sets are disjoint. Notice that every vertex v in $N_2(S)$ that is not bad can have conflict with at most $|C(v)| \cdot \Delta(B) = 2\Delta(B) \le 60d$ vertices. Thus we can find at least

$$\frac{c^6 n/2}{2\Delta(B)} \ge \frac{c^6 n}{120d} = \frac{c^6 n}{240cn^{1/2}} = \frac{c^5 n^{1/2}}{240} \ge 2\ell$$

not-previously-used vertices in $N_2(S)$ that are pairwise conflict-free. Again since G is C_4-free, any other core vertex in $U \setminus S$ can be adjacent to connector sets of at most 2 vertices in $N_2(S)$. Thus there are at least $2\ell - 2(\ell - 2) = 4$ vertices available in $N_2(S)$ to connect the pair of vertices in S. □

7 Concluding Remarks

- The proof of Theorem 1.2 is almost identical to the proof of Theorem 1.1. The only differences are the following: First we need a result of Kühn and Osthus [7], which finds a C_4-free subgraph G' in a C_{2k}-free graph G for $k \ge 4$ such that $d(G') = \Omega(d(G))$. Then after cleaning (as in Section 4.2), $|S_2'(v_i)| = \Omega(d^2)$. Recall that each vertex in $S_2'(v_i)$ sends $\Omega(d)$ edges to $S_3'(v_i)$, by a well-known result of Bondy and Simonovits [1], we have that the size of each $S_3'(v_i)$'s is at least $\Omega(d^{3-3/(k+1)})$. For $k \ge 4$, $d^{3-3/(k+1)}\epsilon(d^{3-3/(k+1)}) \gg \ell^2 \cdot diam + d^2$, thus the robust diameter property is guaranteed.

- With some additional ideas, we can show that Mader's conjecture is true for any C_4-free expander graph G with $d(G) = d < n^{1/5}$. Our main obstacle to prove Mader's conjecture is the case when the expander graph is fairly dense, $d \ge n^{1/5}$. With some more work we can find a copy of TK_ℓ in G with $\ell = \Omega(d/\log^{3/2} d)$, since the second sphere is large enough to have the robust diameter property: $|S_2| = \Omega(d^2)$ and $\ell^2 \cdot diam = O(d^2)$.

References

[1] J. Bondy, M. Simonovits, Cycles of even length in graphs. *J. Combin. Theory Ser. B*, 16, (1974), 97–105.

[2] B. Bollobás, A. Thomason, Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs. *European J. Combin.*, 19, (1998), 883–887.

[3] E. Györi, C_6-free bipartite graphs and product representation of squares. *Discrete Math.*, 165/166, (1997), 371–375.

[4] J. Fox, B. Sudakov, Dependent random choice. *Random Structures & Algorithms*, 38, (2011), 68–99.

[5] D. Kühn, D. Osthus, Topological minors in graphs of large girth. *J. Combin. Theory Ser. B*, 86, (2002), 364–380.

[6] D. Kühn, D. Osthus, Large topological cliques in graphs without a 4-cycle. *Combin. Probab. Comput.*, 13, (2004), 93–102.

[7] D. Kühn, D. Osthus, Four-cycles in graphs without a given even cycle. *J. Graph Theory*, 48, (2005), 147–156.

[8] J. Komlós, E. Szemerédi, Topological cliques in graphs. *Combin. Probab. Comput.*, 3, (1994), 247–256.

[9] J. Komlós, E. Szemerédi, Topological cliques in graphs II. *Combin. Probab. Comput.*, 5, (1996), 79–90.

[10] W. Mader, An extremal problem for subdivisions of K_5^-. *J. Graph Theory*, 30, (1999), 261–276.