Almost all C_4-free graphs have less than $(1-\varepsilon)\,\mathrm{ex}(n, C_4)$ edges

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Abstract

A graph is called *H-free* if it contains no copy of H. An old result of Kleitman and Winston [12] states that there are $2^{\Theta(n^{3/2})}$ C_4-free graphs on n vertices. Füredi [8] showed that almost all C_4-free graphs of order n have at least $c \cdot \mathrm{ex}(n, C_4)$ edges for some positive constant $c > 0$. We prove that there is an $\varepsilon > 0$ such that almost all C_4-free graphs have at most $(1-\varepsilon) \cdot \mathrm{ex}(n, C_4)$ edges. This resolves a conjecture of Balogh, Bollobás and Simonovits [4] for the 4-cycle.

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1 Introduction

Let H be a fixed graph. A graph is called *H-free* if it does not contain a copy of H as a (not necessarily induced) subgraph. We will denote the family of all labeled H-free graphs on the vertex set $[n] = \{1, \ldots, n\}$ by $\mathcal{F}_n(H)$. Let $\mathrm{ex}(n, H)$ be the Turán number of H, i.e. the maximum number of edges that an H-free graph on n vertices may have. The problem of estimating $|\mathcal{F}_n(H)|$ has been an object of intense study. Erdős conjectured (see [5]) that if H contains a cycle, then $|\mathcal{F}_n(H)| = 2^{(1+o(1))\,\mathrm{ex}(n,H)}$. The conjecture was resolved in the affirmative by Erdős, Frankl and Rödl [7], under the additional assumption that $\chi(H) \geq 3$. More precise estimates and structural results can be found in Balogh, Bollobás and Simonovits [2]. The case of bipartite H is still wide open. For some partial results see [11] and [12].

The following statement can be proved using the methods from [2] and [3]. Let H be a fixed non-bipartite graph. Then for every positive $\varepsilon > 0$, almost all H-free graphs of order n have at least $(\frac{1}{2} - \varepsilon)\,\mathrm{ex}(n, H)$ and at most $(\frac{1}{2} + \varepsilon)\,\mathrm{ex}(n, H)$ edges. It is not unreasonable to claim a similar behavior when $\chi(H) = 2$, though it seems that the case when H is bipartite

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is much harder to handle. The main reasons might be that if $\chi(H) = 2$, all H-free graphs are sparse (i.e. $\mathrm{ex}(n, H) = o(n^2)$) and no Erdős-Simonovits type stability results are known. There are natural settings where a similar concentration around the half does not occur. For example, recently Gerke and McDiarmid [9] proved that the expected number of edges in a random n-vertex planar graph is at least $(13/7 + o(1))n > (3n-6)/2$. Still one should expect that the number of edges in a "typical" H-free graph is bounded away from the extremal values, $\mathrm{ex}(n, H)$ and 0. Balogh, Bollobás and Simonovits [4] formalized this intuition in the following conjecture.

Conjecture 1. *For every bipartite graph H that contains a cycle, there is a constant $c = c(H) > 0$ such that almost all H-free graphs on n vertices have at least $c \cdot \mathrm{ex}(n, H)$ and at most $(1 - c) \cdot \mathrm{ex}(n, H)$ edges.*

The lower bound is known only for $H = C_4$ (see [8]) and C_6 (the argument from [8] can be quite easily extended using methods from [11]), whereas the upper bound has not been proved for any bipartite H. In this note we estimate the upper bound for $H = C_4$, consequently resolving Conjecture 1 for the 4-cycle.

For a fixed real number $\varepsilon \in (0, 1)$, let $\mathcal{F}_n^{\varepsilon}(H)$ denote the subset of $\mathcal{F}_n(H)$ consisting only of graphs that have at least $(1 - \varepsilon) \cdot \mathrm{ex}(n, H)$ edges. Our main result is the following.

Theorem 2. *There exist an $\varepsilon > 0$ such that*

$$\lim_{n\to\infty} \frac{|\mathcal{F}_n^{\varepsilon}(C_4)|}{|\mathcal{F}_n(C_4)|} = 0. \tag{1}$$

We would like to remark that our result seems to be closely related to another classical Turán-type problem of estimating $\mathrm{ex}(G(n, 1/2), C_4)$, proposed by Erdős and Spencer (see [6]). Trivially almost surely we have

$$(\tfrac{1}{2} + o(1))\,\mathrm{ex}(n, C_4) \leq \mathrm{ex}(G(n, 1/2), C_4) \leq \mathrm{ex}(n, C_4), \tag{2}$$

but no better estimates are known. Since actually the proof of Theorem 2 shows that for some $\varepsilon > 0$,

$$\left|\mathcal{F}_n^{\varepsilon}(C_4)\right| = 2^{-\Omega(n^{3/2})}\left|\mathcal{F}_n(C_4)\right|, \tag{3}$$

it follows that if Erdős' conjecture was true for C_4, i.e. $\left|\mathcal{F}_n(C_4)\right| = 2^{(1+o(1))\,\mathrm{ex}(n,C_4)}$, then (3) coupled with a simple first moment argument would improve the upper bound in (2) to $(1 - \varepsilon') \cdot \mathrm{ex}(n, C_4)$, for some positive constant $\varepsilon' > 0$.

The remainder of this paper is organized as follows. In Section 2 we introduce some notation and prove some technical lemmas. The proof of the main result, Theorem 2, is given in Section 3.

2 Notation and technical lemmas

Given an arbitrary set A, we will denote its power set (i.e. the set of all subsets of A) by $\mathcal{P}(A)$. For a positive integer k, the set of all functions from A to the set $\{0, \ldots, k-1\}$ will be abbreviated by k^A. Given a graph G and a subset of its vertices $A \subseteq V(G)$, the subgraph induced on $V(G) - A$ will be denoted by $G - A$. The number of edges in G is $e(G)$. Given a vertex $v \in V(G)$, its degree in G is denoted by $\deg_G(v)$ or simply $\deg(v)$, whenever G is clear from the context. The minimum degree of G is $\delta(G) = \min_{v \in V(G)} d_G(v)$. Finally, for any $A \subseteq V(G)$ and $v \in V(G)$, we let $\deg(v, A)$ denote the number of neighbors of v in A.

First, let us recall that $\mathrm{ex}(n, C_4) = \frac{1+o(1)}{2} n^{3/2}$. Suppose that G is a C_4-free graph on n vertices which has at least $(1-\varepsilon)\,\mathrm{ex}(n, C_4)$ edges. If the minimum degree of G was smaller than $\frac{3}{4}n^{1/2}$, then by removing a vertex of smallest degree from G, we would increase the *relative edge density* (the ratio $e(G)/v(G)^{3/2}$) of the resulting graph. Since removing vertices cannot create a copy of C_4 in our graph, the relative edge density will never exceed $\frac{1}{2} + o(1)$, and hence we cannot continue removing low degree vertices indefinitely. It follows that after removing a relatively small set of low degree vertices from our graph G, we will obtain a graph with large minimum degree. We formalize the above discussion in the following Lemma.

Lemma 3. *For every positive constant $\alpha > 0$, there is an $\varepsilon > 0$ such that the following holds. Let $G \in \mathcal{F}_n^{\varepsilon}(C_4)$, where n is large enough. Then there is a set $X \subseteq V(G)$ of size $|X| \le \alpha n$ such that $\delta(G - X) \ge (3/4 - \alpha)n^{1/2}$.*

Proof. Fix $\varepsilon = \alpha^2/3$. Define an ordering of the vertices of G as follows. Let v_1 be a vertex of minimum degree in G. Provided that $v_1, \ldots, v_i$ have already been defined, we let v_{i+1} to be a vertex of minimum degree in $G - \{v_1, \ldots, v_i\}$. Since every subgraph of G is C_4-free, and $\mathrm{ex}(n, C_4) \le \frac{2n^{3/2}+n}{4}$, the function

$$f(k) \quad = \quad \frac{2(n-k)^{3/2} + (n-k)}{4} - e(G - \{v_1, \ldots, v_k\})$$

is non-negative for all k.

Let k_0 be the smallest number such that $\delta(G - \{v_1, \ldots, v_k\}) \ge (3/4 - \alpha)n^{1/2}$, or $k_0 = n$ if such number does not exist. If $k_0 \le \alpha n$, then $X = \{v_1, \ldots, v_{k_0}\}$ is good for our purposes. Otherwise for all $k \le \alpha n$ we have

$$\begin{aligned} f(k) - f(k-1) \quad &= \quad \frac{(n-k)^{3/2} - (n-k+1)^{3/2}}{2} - \frac{1}{4} + \deg(v_k, V(G) - \{v_1, \ldots, v_{k-1}\}) \\ &\le \quad -\frac{3}{4}(n-k)^{1/2} + \left(\frac{3}{4} - \alpha\right) n^{1/2} \\ &\le \quad -\frac{3}{4}\left(1 - \alpha\right) n^{1/2} + \left(\frac{3}{4} - \alpha\right) n^{1/2} \le -\frac{\alpha}{4} n^{1/2}. \end{aligned}$$

It follows that

$$-f(0) \le f(\alpha n) - f(0) = \sum_{k=1}^{\alpha n} \big(f(k) - f(k-1)\big) \le \alpha n \cdot \left(-\frac{\alpha}{4} n^{1/2}\right) = -\frac{\alpha^2}{4} n^{3/2},$$

and hence

$$e(G) = \frac{2n^{3/2} + n}{4} - f(0) \le \frac{2 - \alpha^2}{4} n^{3/2} + \frac{n}{4} < (1 - \frac{3}{2}\varepsilon + o(1)) \cdot \mathrm{ex}(n, C_4),$$

which is a contradiction. □

The next Lemma formalizes the following intuition. A random partition of the vertex set of a graph into two sets of sizes a and b splits neighborhood of each vertex roughly in proportion $a : b$.

Lemma 4. *For all positive reals $\beta, \rho \in (0,1)$ there exists an $n_0 = n_0(\beta, \rho)$ such that the following holds. Let $n \geq n_0$ and G be an n-vertex graph with minimum degree $\delta(G) \geq \log^2 n$. Then there exists an $A \subseteq V(G)$ with $|A| \in \big((1-\rho)\beta n, (1+\rho)\beta n\big)$ such that for all $v \in V(G)$,*

$$(1-\rho)\beta \deg(v) \leq \deg(v, A) \leq (1+\rho)\beta \deg(v).$$

Proof. Let us pick, randomly and independently, each vertex of G with probability β. Let A be the set of selected vertices. By Chernoff bound (see, e.g., Corollary A.1.14 in [1])

$$\Pr\left[\big||A| - \beta n\big| \geq \rho\beta n\right] \leq 2e^{-cn},$$

where c is some positive constant depending only on β and ρ. Similarly, for every vertex v we have

$$\Pr\left[\big|\deg(v, A) - \beta \deg(v)\big| \geq \rho\beta \deg(v)\right] \leq 2e^{-c\deg(v)} \leq 2e^{-c\delta(G)} \leq 2e^{-c\log^2 n}.$$

By the union bound the set A has all the required properties with probability tending to 1 as n goes to infinity. Hence, provided that n is large enough, there exists a set A satisfying all the required conditions. □

The key ingredient in the proof of Theorem 2 is the following Lemma, which builds on a slick counting argument of Kleitman and Winston from [12], which was later used in many papers, including [10], [11] and [13].

Lemma 5. *Let G be an n-vertex C_4-free graph with minimum degree $\delta(G) \geq \delta \geq \frac{1}{2}n^{1/2}$. Suppose we want to add to G a new vertex v of degree $d \geq \frac{1}{2}n^{1/2}$, so that the resulting graph remains C_4-free, and moreover we have already chosen pd neighbors of v, where $p \in [0,1]$. Then the number of ways we can select the remaining $(1-p)d$ neighbors of v is at most*

$$2^{o(\log^3 n)} \cdot \binom{n/\delta - pd}{(1-p)d}.$$

Proof. We slightly modify the argument from [12]. The key idea there was the following. We proceed in d steps, adding one edge between v and G at each step. At the first step all n vertices of G are *eligible* to become neighbors of v. At each step we list all eligible vertices in the following way. After i vertices have been put on the list, the $(i+1)^{\text{st}}$ vertex is a vertex that is connected by a path of length 2 to the greatest number of eligible vertices not yet on the list. We then locate the vertex with the least index (in the above ordering) that we want to connect to v and add the appropriate edge to our graph. All vertices preceeding the chosen vertex on our list, as well as all vertices connected to it by a path of length 2, become ineligible.

The key observation in [12] is that after merely $z = n\delta^{-2}\log n \leq 4\log n$ steps the set of eligible vertices, from which the remaining $d - z$ neighbors of v must be chosen, has size at most n/δ. If z' out of the first z neighbors of v were the already chosen ones, we have to select the remaining $(1-p)d - z + z'$ vertices to complete the neighborhood of v from a set of size at most $n/\delta - pd + z'$. Hence the overall number of choices for the $(1-p)d$ new neighbors of v is bounded by

$$\sum_{z' \leq z} \binom{n}{z - z'}\binom{n/\delta - pd + z'}{(1-p)d - z + z'}. \tag{4}$$

By our assumptions on δ, we have $z \le 4 \log n$, and hence we can bound (4) by

$$(z+1)\binom{n}{z}\binom{n/\delta - pd + z}{(1-p)d} \le 2^{o(\log^3 n)} \cdot \binom{n/\delta - pd}{(1-p)d}.$$

$\square$

Finally, we need the following well-known estimate. Let

$$H(x) \quad = \quad -x \log_2 x - (1-x)\log_2(1-x)$$

be the binary entropy funtion.

Lemma 6. *For every* $0 \le \ell = \lambda n \le n$,

$$\frac{1}{n+1} 2^{nH(\lambda)} \le \binom{n}{\ell} \le 2^{nH(\lambda)}.$$

3 Proof of Theorem 2

Very vaguely, the idea of the proof can be summarized as follows. If G is a C_4-free graph with large minimum degree, then the number of ways in which we can remove a certain proportion of its edges is much larger than the number of ways we can add the same number of edges back, so that the resulting graph remains C_4-free. In other words, every $G \in \mathcal{F}_n^\varepsilon(C_4)$ has many more different subgraphs $F \in \mathcal{F}_n(C_4)$ than the number of supergraphs in $\mathcal{F}_n^\varepsilon(C_4)$ that any such F can possibly have. This implies that $|\mathcal{F}_n^\varepsilon(C_4)| = o(|\mathcal{F}_n(C_4)|)$. In the sequel we will formalize the above discussion.

Consider an arbitrary mapping

$$\varphi : \mathcal{F}_n^\varepsilon(C_4) \to \mathcal{P}\big(\mathcal{F}_n(C_4) \times 2^{[n]} \times n^{[n]}\big).$$

For a triple $T \in \mathcal{F}_n(C_4) \times 2^{[n]} \times n^{[n]}$, let

$$\psi(T) \quad = \quad \big\{G \in \mathcal{F}_n^\varepsilon(C_4) : T \in \varphi(G)\big\}.$$

Counting appearances of all triples T in the images $\varphi(G)$, where G ranges over $\mathcal{F}_n^\varepsilon(C_4)$ and T ranges over $\mathcal{F}_n(C_4) \times 2^{[n]} \times n^{[n]}$, yields

$$\sum_G |\varphi(G)| \quad = \quad \sum_T |\psi(T)|. \tag{5}$$

Equality (5) implies an obvious bound on the size of $\mathcal{F}_n^\varepsilon(C_4)$, namely

$$\big|\mathcal{F}_n^\varepsilon(C_4)\big| \quad \le \quad (2n)^n \cdot \frac{\sup_T |\psi(T)|}{\inf_G |\varphi(G)|} \cdot \big|\mathcal{F}_n(C_4)\big|. \tag{6}$$

Now, inequality (6) combined with any $o((2n)^{-n})$ bound on the $\sup_T |\psi(T)| / \inf_G |\varphi(G)|$ ratio (for a carefully chosen φ) will imply (1).

Since the remainder of the proof gets a tad technical, we will start by giving its short and informal outline. We start (Lemma 3) by noting that every C_4-free graph G with many edges, i.e. one with e(G) close to the extremal number $\mathrm{ex}(n, C_4)$, contains an almost spanning subgraph G_0 with minimum degree about $\frac{3}{4}n^{1/2}$. Next, using Lemma 4, we find a subset $A \subseteq V(G_0)$ of size about βn such that the minimum degree of $G' = G_0 - A$ is still almost $\frac{3}{4}n^{1/2}$ and all the vertices in A have (approximately) at least $\frac{3}{4}n^{1/2}$ neighbors in $V(G')$. Given such a set A we define $\varphi(G)$ to be the set of all graphs obtained from G by deleting $p = 90\%$ cross-edges incident to each vertex in A, together with all the necessary information to identify the set A and reconstruct all relevant degrees after such a deletion. Roughly speaking this proves that

$$\big|\varphi(G)\big| \geq \binom{\frac{3}{4}n^{1/2}}{p \cdot \frac{3}{4}n^{1/2}}^{\beta n} \approx 2^{0.3517\beta n^{3/2}}. \tag{7}$$

Finally, given a triple T consisting of a graph F, a set $A \subseteq V(F)$ and the list of degrees that the vertices in A had in the original C_4-free graph $G \supseteq F$, we prove, using Lemma 5, that the number of supergraphs $G \supseteq F$ with $T \in \varphi(G)$ is at most

$$\big|\psi(T)\big| \leq \left[\sup_d \binom{\frac{4}{3}n^{1/2} - pd}{(1-p)d}\right]^{\beta n} \approx 2^{0.3467\beta n^{3/2}}. \tag{8}$$

Combining (7) and (8) with (6), we conclude that $\big|\mathcal{F}_n^{\varepsilon}(C_4)\big| \leq 2^{-\Omega(n^{3/2})}\big|\mathcal{F}_n(C_4)\big|$.

Let us start by defining the mapping φ. Fix some very small constants α, β and ρ (we will specify them later), and let ε be as in the statement of Lemma 3. Recall that $p = 0.9$. Suppose $n \geq n_0/(1-\alpha)$, where n_0 is as in the statement of Lemma 4. Finally fix some $G \in \mathcal{F}_n^{\varepsilon}(C_4)$. By Lemma 3 there is a subset $X \subseteq V(G)$ of size $|X| \leq \alpha n$ such that $\delta(G - X) \geq (3/4 - \alpha)n^{1/2}$. Now, by Lemma 4, we can find an $A \subseteq V(G) - X$ of size $|A| \in \big((1-\rho)(1-\alpha)\beta n, (1+\rho)\beta n\big)$ such that, if we let $G' = G - X - A$,

- $\delta(G') \geq (1 - (1+\rho)\beta)\delta(G - X) \geq (1 - (1+\rho)\beta)(3/4 - \alpha)n^{1/2}$, and
- $\deg(v, V(G')) \geq (1-(1+\rho)\beta)\deg_{G-X}(v) \geq (1-(1+\rho)\beta)(3/4-\alpha)n^{1/2}$ for every $v \in A$.

We define $\varphi(G)$ to be the set of all triples $(F, X \cup A, f)$, where

$$f(v) \quad = \quad \begin{cases} \deg(v, V(G')), & v \in X \cup A, \\ 0, & \text{otherwise,} \end{cases}$$

and F is any subgraph of G obtained by deleting, for each vertex $v \in X \cup A$, a set of $(1-p)\deg(v, V(G'))$ edges connecting v to $V(G')$.

Claim 7. *For every $G \in \mathcal{F}_n^{\varepsilon}(C_4)$,*

$$\big|\varphi(G)\big| \geq 2^{H(p)(1-\rho)(1-\alpha)(1-(1+\rho)\beta)(3/4-\alpha)\beta n^{3/2} - O(n\log n)}.$$

Proof. It suffices to count the number of subgraphs F appearing in the definition of $\varphi(G)$. By our bounds on the size of A and the degrees of vertices in A, this is at least

$$\begin{aligned}\prod_{v\in A}\binom{\deg(v, V(G'))}{p\deg(v, V(G'))} \quad &\geq \quad (n+1)^{-|A|} \cdot 2^{H(p)\sum_{v\in A}\deg(v, V(G'))} \\ &\geq \quad (n+1)^{-(1+\rho)\beta n} \cdot 2^{H(p)(1-\rho)(1-\alpha)(1-(1+\rho)\beta)(3/4-\alpha)\beta n^{3/2}},\end{aligned}$$

where the first inequality follows from Lemma 6. □

Let $T = (F, S, f)$ be a triple from the image of φ. The way we defined φ guarantees that the set S has size at most $(1+\rho)\beta n + \alpha n$, and $F - S$ has minimum degree $\delta(F - S) \geq (3/4 - \alpha)n^{1/2}$. By Lemmas 5 and 6, we get the following bound on the size of $\psi(T)$:

$$\begin{aligned}
\left|\psi(T)\right| &\leq 2^{o(n\log^3 n)} \cdot \prod_{v\in S} \binom{(n-|S|)/((3/4-\alpha)n^{1/2}) - pf(v)}{(1-p)f(v)} \\
&\leq 2^{o(n\log^3 n)} \cdot \prod_{v\in S} \binom{n^{1/2}/(3/4-\alpha) - pf(v)}{(1-p)f(v)} \\
&\leq 2^{o(n\log^3 n)} \cdot \prod_{v\in S} 2^{(n^{1/2}/(3/4-\alpha)-pf(v))H\left(\frac{(1-p)f(v)}{n^{1/2}/(3/4-\alpha)-pf(v)}\right)} \\
&\leq 2^{o(n\log^3 n)} \cdot 2^{((1+\rho+\alpha)\beta+\alpha)sn^{3/2}},
\end{aligned}$$

where (we let $d = f(v)n^{-1/2}$)

$$s = \sup_d \left[[1/(3/4-\alpha) - pd] H\left(\frac{(1-p)d}{1/(3/4-\alpha) - pd}\right)\right].$$

If we set $\alpha = \rho = 0$, $\beta = 10^{-5}$ and $p = 0.9$, then $s \leq 0.3467$, and hence

$$\left|\psi(T)\right| \leq 2^{3467\cdot 10^{-9}n^{3/2}+o(n^{3/2})}.$$

On the other hand Claim 7 gives

$$\left|\varphi(G)\right| \geq 2^{3517\cdot 10^{-9}n^{3/2}-o(n^{3/2})}.$$

Since all the functions in question are continuous in α and ρ, if all these parameters are small enough, we will have

$$\frac{\sup_T |\psi(T)|}{\inf_G |\varphi(G)|} \leq 2^{-4\cdot 10^{-8}\cdot n^{3/2}},$$

and therefore by (6)

$$\left|\mathcal{F}_n^{\varepsilon}(C_4)\right| \leq 2^{-2\cdot 10^{-8}\cdot n^{3/2}} \cdot \left|\mathcal{F}_n(C_4)\right|,$$

provided that n is large enough.

References

[1] N. Alon and J. Spencer, *The probabilistic method. Third edition*, John Wiley & Sons, Inc., 2008.

[2] J. Balogh, B. Bollobás, and M. Simonovits, *The typical structure of graphs without given excluded subgraphs*, to appear in Random Structures and Algorithms.

[3] ———, *The number of graphs without forbidden subgraphs*, Journal of Combinatorial Theory B **91** (2004), 1–24.

[4] ———, *The fine structure of octahedron-free graphs*, submitted, 2009.

[5] F. Chung, *Open problems of Paul Erdős in graph theory*, Journal of Graph Theory **25** (1997), 3–36.

[6] P. Erdős, *Some of my old and new combinatorial problems*, Paths, Flows, and VLSI-Layout (Bonn), Springer-Verlag, 1990, pp. 35–45.

[7] P. Erdős, P. Frankl, and V. Rödl, *The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent*, Graphs and Combinatorics **2** (1986), 113–121.

[8] Z. Füredi, *Random Ramsey graphs for the four-cycle*, Discrete Mathematics **126** (1994), 407–410.

[9] S. Gerke and C. McDiarmid, *On the number of edges in random planar graphs*, Combinatorics, Probability and Computing **13** (2004), 165–183.

[10] P. Haxell, Y. Kohayakawa, and T. Łuczak, *Turán's extremal problem in random graphs: Forbidding even cycles*, Journal of Combinatorial Theory B **64** (1995), 273–287.

[11] D. Kleitman and D. Wilson, *On the number of graphs which lack small cycles*, manuscript, 1996.

[12] D. Kleitman and K. Winston, *On the number of graphs without* 4*-cycles*, Discrete Mathematics **41** (1982), 167–172.

[13] Y. Kohayakawa, B. Kreuter, and A. Steger, *An extremal problem for random graphs and the number of graphs with large even girth*, Combinatorica **18** (1998), 101–120.