1 Lectures 26 and 27: Proof of k-uniform hypergraph container lemma (4/26/19 and 4/29/19)

The exposition for these lectures follow closely the proofs in the original hypergraph container method paper of Balogh, Morris and Samotij [1]. Recall the main hypergraph container theorem which roughly states that if $\mathcal{H}$ is a uniform hypergraph that is $(\mathcal{F}, \epsilon)$-dense for some family $\mathcal{F}$ and whose edge distribution satisfies certain natural boundedness conditions, then the collection $\mathcal{I}(\mathcal{H})$ of all independent sets in $\mathcal{H}$ admits a partition into relatively few classes such that all independent sets in one class are essentially contained in a single set $A \notin \mathcal{F}$.

Recall that if $\mathcal{H}$ is a uniform hypergraph with vertex set V and $\mathcal{F}$ is an increasing family of subsets of V with $\epsilon \in (0, 1]$, we say that $\mathcal{H}$ is $(\mathcal{F}, \epsilon)$-*dense* if

$$e(\mathcal{H}[A]) \geqslant \epsilon \cdot e(\mathcal{H})$$

for every $A \in \mathcal{F}$, and recall that we define

$$\Delta_\ell(\mathcal{H}) = \max\big\{ \deg_{\mathcal{H}}(T) : T \subseteq V(\mathcal{H}) \text{ and } |T| = \ell \big\},$$

where $\deg_{\mathcal{H}}(T) = |\{e \in \mathcal{H} : T \subseteq e\}$.

Theorem 1.1. *For every $k \in \mathbb{N}$ and all positive c, c' and ϵ, there exists a positive constant C such that the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph and let $\mathcal{F} \subseteq \mathcal{P}(V(\mathcal{H}))$ be an increasing family of sets such that $|A| \geqslant \epsilon \cdot v(\mathcal{H})$ for all $A \in \mathcal{F}$. Suppose that $\mathcal{H}$ is $(\mathcal{F}, \epsilon)$-dense and $p \in (0, 1)$ is such that $p^{k-1} e(\mathcal{H}) \geqslant c' v(\mathcal{H})$ and for every $\ell \in [k-1]$,*

$$\Delta_\ell(\mathcal{H}) \leqslant c \cdot \min\left\{ p^{\ell-k}, p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})} \right\}.$$

Then there exists a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leqslant Cp \cdot v(\mathcal{H})}$ and functions $f : \mathcal{S} \to \overline{\mathcal{F}}$ and $g : \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that for every $I \in \mathcal{I}(\mathcal{H})$,

$$g(I) \subseteq I \qquad \text{and} \qquad I \setminus g(I) \subseteq f(g(I)).$$

In this section, we shall prove the following proposition, which (roughly) says that the main theorem holds in the special case when $\mathcal{F}$ is the family of all subsets of $V(\mathcal{H})$ with at least $(1-\delta)v(\mathcal{H})$ elements. The main theorem follows by applying this proposition a constant number of times.

Proposition 1.2. *For every integer k and positive c and c', there exists a positive δ such that the following holds. Let $p \in (0,1)$ and suppose that $\mathcal{H}$ is a k-uniform hypergraph such $p^{k-1}e(\mathcal{H}) \geqslant c'v(\mathcal{H})$ and for every $\ell \in [k-1]$,*

$$\Delta_\ell(\mathcal{H}) \leqslant c \cdot \min\left\{p^{\ell-k}, p^{\ell-1}\frac{e(\mathcal{H})}{v(\mathcal{H}))}\right\}.$$

Then there exists a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leqslant (k-1)p\cdot v(\mathcal{H})}$ and functions $f_0 : \mathcal{S} \to \mathcal{P}(V(\mathcal{H}))$ and $g_0 : \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that for every $I \in \mathcal{I}(\mathcal{H})$,

$$g_0(I) \subseteq I \subseteq f_0(g_0(I)) \cup g_0(I) \quad \textit{and} \quad |f_0(g_0(I))| \leqslant (1-\delta)v(\mathcal{H}).$$

Moreover, if for some $I, I' \in \mathcal{I}(\mathcal{H})$, $g_0(I) \subseteq I'$ and $g_0(I') \subseteq I$, then $g_0(I) = g_0(I')$.

The final line of Proposition 1.2 states that the labeling function g_0 exhibits a certain consistency. This property of g_0, which may look somewhat puzzling, will be crucial in the proof of the main theorem.

In order to prove Proposition 1.2, given an independent set $I \in \mathcal{I}(\mathcal{H})$, we shall construct a sequence $(B_{k-1}, \ldots, B_q)$ of subsets of I with $|B_{k-1}|, \ldots, |B_q| \leqslant pv(\mathcal{H})$, for some $q \in [k-1]$, and use it to define a sequence $(\mathcal{H}_{k-1}, \ldots, \mathcal{H}_r)$, where $r \in \{q, q+1\}$, of hypergraphs such that the following holds for each $i \in \{r, \ldots, k-1\}$:

(a) $\mathcal{H}_i$ is an i-uniform hypergraph on the vertex set $V(\mathcal{H})$,

(b) I is an independent set in $\mathcal{H}_i$,

(c) $\Delta_1(\mathcal{H}_i) = O\left(\frac{e(\mathcal{H})}{v(\mathcal{H})}\right)$, and

(d) $e(\mathcal{H}_i) \geqslant \Omega(p^{k-i}e(\mathcal{H}))$.

We shall be able to do it in such a way that in the end, there will be a set $A \subseteq V(\mathcal{H})$ of size at most $(1-\delta)v(\mathcal{H})$ such that the remaining elements of I, the set $I \setminus S$ (where $S = B_k \cup \ldots \cup B_q$), must all lie inside A. If $r = 1$, then we will simply let A be the set of non-edges of the 1-uniform hypergraph $\mathcal{H}_1$; in this case, the upper bound on $|A|$ will follow from (d) and our assumption that $p^{k-1}e(\mathcal{H}) \geqslant c'v(\mathcal{H})$. If $r > 1$, then we will obtain an appropriate A while trying (and failing) to construct the hypergraph $\mathcal{H}_{r-1}$ using the hypergraph $\mathcal{H}_r$ and the set B_r. Crucially, this set A will depend solely on S, that is, if for some pair $I, I' \in \mathcal{I}(\mathcal{H})$ our procedure generates (S, A) and (S', A'), respectively, and $S = S'$, then also $A = A'$. This will allow us to set $g_0(I) = S$ and $f_0(S) = A$.

For the rest of this section, we fix k, c', c, p and $\mathcal{H}$ as in the statement of Proposition 1.2. Without loss of generality, we may assume that $c \geqslant 1$. Let I be an independent set of $\mathcal{H}$. We shall describe a procedure of choosing the sets $B_i \subseteq I$ and constructing the hypergraphs $\mathcal{H}_i$ as above. This procedure, which we shall call the **Scythe Algorithm** lies at the heart of the proof of the proposition.

At each step of the Scythe Algorithm, we shall order the vertices of a certain subhypergraph of $\mathcal{H}$ with respect to their degrees in that subhypergraph. For the sake of brevity and clarity of the presentation, let us make the following definition.

Definition 1.3. *(Max-degree order). Given a hypergraph $\mathcal{G}$, we define the* ***max-degree order*** *on $V(\mathcal{G})$ as follows:*

(1) Fix an arbitrary total ordering of $V(\mathcal{G})$.

(2) For each $j \in \{1, \ldots, v(G)\}$, let u_j be the maximum-degree vertex in the hypergraph $\mathcal{G}[V(\mathcal{G}) \setminus \{u_1, \ldots, u_{j-1}\}]$; ties are broken by giving preference to vertices which come earlier in the order chosen in (1).

(3) The max-degree order on $V(\mathcal{G})$ is $(u_1, \ldots, u_{v(\mathcal{G})})$.

Finally, we write $W(u)$ to denote the initial segment of the max-degree order on $V(\mathcal{G})$ that ends with u, i.e., for every j, we let $W(u_j) = \{u_1, \ldots, u_j\}$.

We remark here that the only property of the max-degree order that will be important for us is that for every $j \in \{1, \ldots, v(\mathcal{G})\}$, the degree of the vertex u_j in the hypergraph $\mathcal{G}[V(\mathcal{G}) \setminus W(u_{j-1})]$ is at least as large as the average degree of this hypergraph.

We next define the numbers Δ_ℓ^i, where $1 \leqslant \ell < i \leqslant k$, which will play a crucial role in the description and the analysis of the algorithm.

Definition 1.4. *For every $\ell \in [k-1]$, let $\Delta_\ell^k = \Delta_\ell(\mathcal{H})$ and for all $i \in [k-1]$ and $\ell \in [i-1]$, let*

$$\Delta_\ell^i = \max\left\{2 \cdot \Delta_{\ell+1}^{i+1},\, p \cdot \Delta_\ell^{i+1}\right\}, \tag{1}$$

where i is still the uniformity of the hypergraph $\mathcal{H}_i$ in our case.

The aim is to have Δ_ℓ^i as an upper bound for the codegree of $\mathcal{H}_i$. We use the numbers Δ_ℓ^i to define the following families of sets with high degree.

Definition 1.5. *For some $i \in [k]$, given an i-uniform hypergraph $\mathcal{G}$ and an $\ell \in [i-1]$, let*

$$M_\ell^i(\mathcal{G}) = \left\{ T \in \binom{V(\mathcal{G})}{\ell} \colon \deg_{\mathcal{G}}(T) \geqslant \frac{\Delta_\ell^i}{2} \right\}.$$

Moreover, let $M_i^i(\mathcal{G}) = E(\mathcal{G})$.

We think of this set of hypergraphs as those with "too many edges" that we will need to remove in our algorithm.

Now let $b = p \cdot v(\mathcal{H})$ and for each $i \in [k]$, let $c_i = (ck2^{k+1})^{i-k}$.

Observation 1.6. *The key properties that we would like the hypergraph $\mathcal{H}_i$ constructed from $\mathcal{H}_{i+1}$ to possess are the following.*

(P1) $\mathcal{H}_i$ is i-uniform and $V(\mathcal{H}_i) = V(\mathcal{H})$,

(P2) I is an independent set in $\mathcal{H}_i$,

(P3) $\Delta_\ell(\mathcal{H}_i) \leqslant \Delta_\ell^i$ for each $\ell \in [i-1]$,

(P4) $e(\mathcal{H}_i) \geqslant c_i p^{k-i} e(\mathcal{H})$.

Therefore we need to show that if $\mathcal{H}_{i+1}$ satisfies the above properties, then the $\mathcal{H}_i$ we obtain from the Scythe Algorithm satisfies these properties as well. Set $\mathcal{H}_k := \mathcal{H}$ and note that (P1)–(P4) are vacuously satisfied for $i = k$. The main step of the Scythe Algorithm will be a procedure that, given $\mathcal{H}_{i+1}$ and I satisfying (P1)–(P4), outputs a set $B_i \subseteq I$ of cardinality b, a set $A_i \subseteq V(\mathcal{H})$ with the property that $I \setminus B_i \subseteq A_i$, and a hypergraph $\mathcal{H}_i$ satisfying (P1)–(P3). Moreover, if the constructed $\mathcal{H}_i$ does not satisfy (P4), then we have $|A_i| \leqslant (1-c_i)v(\mathcal{H})$. Crucially, these A_i and $\mathcal{H}_i$ depend solely on B_i and $\mathcal{H}_{i+1}$, that is, if on two inputs $(\mathcal{H}_{i+1}, I)$ and $(\mathcal{H}_{i+1}, I')$, the procedure outputs the same set B_i, it also outputs the same A_i and $\mathcal{H}_i$. The details of how we construct $\mathcal{H}_i$ from $\mathcal{H}_{i+1}$ are as follows.

The Scythe Algorithm. *Given an $(i+1)$-uniform hypergraph $\mathcal{H}_{i+1}$ and an independent set $I \in \mathcal{I}(\mathcal{H}_{i+1})$, set $\mathcal{A}_{i+1}^{(0)} = \mathcal{H}_{i+1}$ and let $\mathcal{H}_i^{(0)}$ be the empty hypergraph on the vertex set $V(\mathcal{H})$. For $j = 0, \ldots, b-1$, do the following:*

(1) If $I \cap V\big(\mathcal{A}_{i+1}^{(j)}\big) = \emptyset$, then set $\mathcal{H}_i = \mathcal{H}_i^{(0)}$, $A_i = \emptyset$, and $B_i = \{u_0, \ldots, u_{j-1}\}$ and STOP.

(2) Let u_j be the first vertex of I in the max-degree order on $V\big(\mathcal{A}_{i+1}^{(j)}\big)$.

(3) Let $\mathcal{H}_i^{(j+1)}$ be the hypergraph on the vertex set $V(\mathcal{H})$ defined by:

$$\mathcal{H}_i^{(j+1)} = \mathcal{H}_i^{(j)} \cup \left\{ D \in \binom{V(\mathcal{H})}{i} : D \cup \{u_j\} \in \mathcal{A}_{i+1}^{(j)} \right\}.$$

(4) Let $\mathcal{A}_{i+1}^{(j+1)}$ be the hypergraph on the vertex set $V\big(\mathcal{A}_{i+1}^{(j)}\big) \setminus W(u_j)$ defined by:

$$\mathcal{A}_{i+1}^{(j+1)} = \left\{ D \in \mathcal{A}_{i+1}^{(j)} : D \cap W(u_j) = \emptyset \text{ and } T \not\subseteq D \text{ for every } T \in \bigcup_{\ell=1}^{i} M_\ell^i\big(\mathcal{H}_i^{(j+1)}\big) \right\}.$$

Finally, set $\mathcal{H}_i = \mathcal{H}_i^{(b)}$, $A_i = V\big(\mathcal{A}_{i+1}^{(b)}\big)$, and $B_i = \{u_0, \ldots, u_{b-1}\}$.

We check that this constructed $\mathcal{H}_i$ statisfies the properties above. We begin by making some basic (but key) observations.

Lemma 1.7. *The following hold for every $i \in [k-1]$:*

(a) $\mathcal{H}_i$ is i-uniform and $V(\mathcal{H}_i) = V(\mathcal{H})$.

(b) If $I \in \mathcal{I}(\mathcal{H}_{i+1})$, then $I \in \mathcal{I}(\mathcal{H}_i)$.

(c) $B_i \subseteq I \subseteq A_i \cup B_i$.

(d) The hypergraph $\mathcal{H}_i$ and the set A_i depend only on $\mathcal{H}_{i+1}$ and the set B_i.

Proof. Property (a) is trivial. To see (b), simply observe that each edge of $\mathcal{H}_i$ is of the form $D \setminus \{u\}$ for some $D \in \mathcal{H}_{i+1}$ and $u \in I$. Thus, if I contains an edge of $\mathcal{H}_i$, it must also contain an edge of $\mathcal{H}_{i+1}$. To see (c), observe that for each j, u_j is the first vertex of I in the max-degree order on $V\big(\mathcal{A}_{i+1}^{(j)}\big)$ and hence $W(u_j) \cap I = \{u_j\}$. It follows that $B_i \subseteq I$ and that $I \setminus A_i = B_i$. Note in particular that if $A_i = \emptyset$, then $I \cap V\big(\mathcal{A}_{i+1}^{(j)}\big) = \emptyset$ for some $j \in \{0, \ldots, b\}$, which implies that $B_i = I$. Finally, to prove (d), observe that all steps of the Scythe Algorithm are deterministic and that every element of I that we need to observe in order to define A_i and $\mathcal{H}_i$ is placed in B_i. More precisely, note that while choosing the vertex u_j, we only need to know the first vertex of I in the max-degree order on $V\big(\mathcal{A}_{i+1}^{(j)}\big)$; the remaining vertices remain unobserved. Since we have $W(u_j) \cap B_i = W(u_j) \cap I = \{u_j\}$, this information can be recovered from B_i. Thus, at each step, the hypergraph $\mathcal{H}_i^{(j+1)}$ can be recovered from $\mathcal{H}_i^{(j)}$ and B_i, and the hypergraph $\mathcal{A}_{i+1}^{(j+1)}$ can be recovered from $\mathcal{A}_{i+1}^{(j)}$, $\mathcal{H}_i^{(j+1)}$ and B_i. Hence, a trivial inductive argument proves that, if the algorithm does not stop in step (1), for each $j \in \{0, \ldots, b\}$, the hypergraphs $\mathcal{H}_i^{(j)}$ and $\mathcal{A}_{i+1}^{(j)}$ are determined by $\mathcal{H}_{i+1}$ and the set B_i, as required. Finally, the algorithm stops in step (1) if and only if $|B_i| < b$. If this happens, then $\mathcal{H}_i$ and A_i are empty. □

We next show that the Scythe Algorithm exhibits a certain 'consistency' while generating its output. This property will be very important in the proof of Proposition 1.2.

Lemma 1.8. *Suppose that on inputs $(\mathcal{H}_{i+1}, I)$ and $(\mathcal{H}_{i+1}, I')$, the Scythe Algorithm outputs $(A_i, B_i, \mathcal{H}_i)$ and $(A_i', B_i', \mathcal{H}_i')$, respectively. If $B_i \subseteq I'$ and $B_i' \subseteq I$, then $(A_i, B_i, \mathcal{H}_i) = (A_i', B_i', \mathcal{H}_i')$.*

Proof. By Lemma 1.7, it suffices to show that $B_i = B_i'$. Suppose that $B_i \neq B_i'$. Let us first consider the (degenerate) case when $\min\{|B_i|, |B_i'|\} < b$. Without loss of generality, we may assume that $|B_i| < b$. This means that, while running on $(\mathcal{H}_{i+1}, I)$, the Scythe Algorithm stopped in step (1). By Lemma 1.7, it follows that $B_i = I$ and hence $B_i' \subseteq B_i$, which means that $|B_i'| < b$ and therefore $B_i' = I'$. Hence, $B_i = B_i'$, as claimed. On the other hand, if $|B_i| = |B_i'| = b$, then there must exist some j such that $u_j \neq u_j'$. Let j be the smallest such index. Note that by the minimality of j, we have $\mathcal{A}_{i+1}^{(j)} = \big(\mathcal{A}_{i+1}^{(j)}\big)' = \mathcal{A}$. Since $u_j \neq u_j'$, one of these vertices comes earlier in the max-degree order on $V(\mathcal{A})$; without loss of generality, we may suppose that it is u_j. Since $B_i \subseteq I'$, it follows that $u_j \in I'$ and hence the Algorithm, while running on the input $(\mathcal{H}_{i+1}, I')$, would not pick u_j' in step j, a contradiction. This shows that in fact $B_i = B_i'$, as required. □

The next lemma motivates the definition of $M_i^i(\mathcal{G})$; it will be an important tool in the proof of Lemma 1.12, below.

Lemma 1.9. *For every $D \in \mathcal{H}_i$, there is a unique $j \in \{0, \ldots, b-1\}$ such that $D \cup \{u_j\} \in \mathcal{A}_{i+1}^{(j)}$. In other words, no edge is added to $\mathcal{H}_i$ more than once.*

Proof. If an edge D is added to $\mathcal{H}_i$ in step j, i.e., if $D \cup \{u_j\} \in \mathcal{A}_{i+1}^{(j)}$, then $D \in M_i^i\big(\mathcal{H}_i^{(j+1)}\big)$ and consequently all edges containing D are deleted from $\mathcal{A}_{i+1}^{(j+1)}$. It follows that $D \cup \{u_{j'}\} \notin \mathcal{A}_{i+1}^{(j')}$ for every $j' \in \{j+1, \ldots, b-1\}$. □

The next lemma shows that if $\mathcal{H}_{i+1}$ satisfies (P3), then so does $\mathcal{H}_i$. The lemma follows easily from the definitions of Δ_ℓ^i and $M_\ell^i(\mathcal{G})$.

Lemma 1.10. *If $\Delta_{\ell+1}(\mathcal{H}_{i+1}) \leqslant \Delta_{\ell+1}^{i+1}$ for some $\ell \in [i-1]$, then $\Delta_\ell(\mathcal{H}_i) \leqslant \Delta_\ell^i$.*

Proof. The crucial observation is that if

$$\deg_{\mathcal{H}_i^{(j)}}(T) \geqslant \frac{\Delta_\ell^i}{2}$$

for some $T \in \binom{V(\mathcal{H})}{\ell}$ and $j \in [b]$, then all edges containing T are removed from $\mathcal{A}_{i+1}^{(j)}$ and hence no more such edges are added to $\mathcal{H}_i$. It follows that $\deg_{\mathcal{H}_i}(T) = \deg_{\mathcal{H}_i^{(j)}}(T)$. Moreover, when we extend $\mathcal{H}_i^{(j-1)}$ to $\mathcal{H}_i^{(j)}$, then we only add to it sets D such that $D \cup \{u_j\} \in \mathcal{A}_{i+1}^{(j-1)} \subseteq \mathcal{H}_{i+1}$ and hence

$$\deg_{\mathcal{H}_i^{(j)}}(T) - \deg_{\mathcal{H}_i^{(j-1)}}(T) \leqslant \deg_{\mathcal{H}_{i+1}}(T \cup \{u_j\}) \leqslant \Delta_{|T|+1}(\mathcal{H}_{i+1}).$$

It follows that

$$\Delta_\ell(\mathcal{H}_i) \leqslant \frac{\Delta_\ell^i}{2} + \Delta_{\ell+1}(\mathcal{H}_{i+1}) \leqslant \frac{\Delta_\ell^i}{2} + \Delta_{\ell+1}^{i+1} \leqslant \Delta_\ell^i$$

where the last inequality follows from (1). □

Next, let us establish some simple properties of the numbers Δ_ℓ^i.

Lemma 1.11. *The following inequalities hold:*

(a) $\Delta_i^{i+1} \leqslant c2^k p^{-1}$ *for every* $i \in [k-1]$ *and*

(b) $\Delta_1^i \leqslant c2^k p^{k-i} \frac{e(\mathcal{H})}{v(\mathcal{H})}$ *for every* $i \in \{2, \ldots, k\}$.

Proof. To prove the lemma, simply note that, by the definition of Δ_ℓ^i, for every $i \in [k]$ and every $\ell \in [i-1]$,

$$\Delta_\ell^i = 2^d p^{k-i-d} \Delta_{d+\ell}(\mathcal{H}) \quad \text{for some } d \in \{0, \ldots, k-i\}. \tag{2}$$

One easily proves (2) by induction on $k - i$. Intuitively, d in (2) is the number of times that the first term in the maximum in (1) is larger than the second term when following the recursive definition of Δ_ℓ^i back to $\Delta_{d+\ell}^k$.

Since $\Delta_\ell(\mathcal{H}) \leqslant c \cdot \min\left\{p^{\ell-k}, p^{\ell-1}\frac{e(\mathcal{H})}{v(\mathcal{H})}\right\}$, as in the statement of Proposition 1.2, it follows from (2) that

$$\Delta_i^{i+1} = \max_{0 \leqslant d \leqslant k-i} \left\{2^d p^{k-(i+1)-d} \Delta_{d+i}(\mathcal{H})\right\} \leqslant \max_{0 \leqslant d \leqslant k-i} \left\{2^d p^{k-i-1-d} \cdot cp^{d+i-k}\right\} \leqslant c \cdot 2^k p^{-1},$$

and

$$\Delta_1^i = \max_{0 \leqslant d \leqslant k-i} \left\{2^d p^{k-i-d} \Delta_{d+1}(\mathcal{H})\right\} \leqslant \max_{0 \leqslant d \leqslant k-i} \left\{2^d p^{k-i-d} \cdot cp^d \cdot \frac{e(\mathcal{H})}{v(\mathcal{H})}\right\} \leqslant c \cdot 2^k p^{k-i} \frac{e(\mathcal{H})}{v(\mathcal{H})},$$

as required. □

Finally, we show that if $\mathcal{H}_{i+1}$ satisfies (P3) and (P4), then either $\mathcal{H}_{i+1}$ also satisfies (P4) or we have $|A_i| \leqslant (1-c_i)v(\mathcal{H})$. Recall that $c_i = (ck2^{k+1})^{i-k}$.

Lemma 1.12. *Let $i \in [k-1]$ and suppose that $e(\mathcal{H}_{i+1}) \geqslant c_{i+1}p^{k-(i+1)}e(\mathcal{H})$ and that $\Delta_\ell(\mathcal{H}_{i+1}) \leqslant \Delta_\ell^{i+1}$ for every $\ell \in [i]$. Then either*

$$e(\mathcal{H}_i) \geqslant \frac{p}{c \cdot 2^{k+1}k} e(\mathcal{H}_{i+1}) \geqslant c_i p^{k-i} e(\mathcal{H}) \tag{3}$$

or $|A_i| \leqslant (1-c_i)v(\mathcal{H})$.

Proof. If the Scythe Algorithm stops in step (1), then $|A_i| = 0$ and there is nothing to prove. Hence, we may assume that steps (2)–(4) are executed b times. Since no edge is added to $\mathcal{H}_i$ more than once, see Lemma 1.9, then for each $j \in \{0, \ldots, b-1\}$,

$$e\big(\mathcal{H}_i^{(j+1)}\big) - e\big(\mathcal{H}_i^{(j)}\big) = \deg_{\mathcal{A}_{i+1}^{(j)}}(u_j). \tag{4}$$

By the definition of the max-deg order, the right-hand side of (4) is at least the average degree of the hypergraph $\tilde{\mathcal{A}}_{i+1}^{(j)}$, the subhypergraph of $\mathcal{A}_{i+1}^{(j)}$ induced by the set $\big(V\big(\mathcal{A}_{i+1}^{(j)}\big) \setminus W(u_j)\big) \cup \{u_j\}$. Therefore, by the definition of $\mathcal{A}_{i+1}^{(j+1)}$, we have

$$e\big(\mathcal{H}_i^{(j+1)}\big) - e\big(\mathcal{H}_i^{(j)}\big) \geqslant \frac{(i+1)e\big(\tilde{\mathcal{A}}_{i+1}^{(j)}\big)}{v\big(\tilde{\mathcal{A}}_{i+1}^{(j)}\big)} \geqslant \frac{(i+1)e\big(\mathcal{A}_{i+1}^{(j+1)}\big)}{v\big(\mathcal{H}\big)}.$$

Hence, if $(i+1)e\big(\mathcal{A}_{i+1}^{(j+1)}\big) \geqslant e\big(\mathcal{H}_{i+1}\big)$ for every $j \in \{0, \ldots, b-1\}$, then

$$e(\mathcal{H}_i) \geqslant \sum_{j=0}^{b-1} \frac{(i+1)e\big(\mathcal{A}_{i+1}^{(j+1)}\big)}{v\big(\mathcal{H}\big)} \geqslant b \cdot \frac{e\big(\mathcal{H}_{i+1}\big)}{v\big(\mathcal{H}\big)} = pe(\mathcal{H}_{i+1}),$$

as required. Thus, we may assume that for some j,

$$e\big(\mathcal{A}_{i+1}^{(b)}\big) \leqslant e\big(\mathcal{A}_{i+1}^{(j+1)}\big) < \frac{e\big(\mathcal{H}_{i+1}\big)}{i+1}. \tag{5}$$

Intuitively, (5) means that while running the Scythe Algorithm on $\mathcal{H}_{i+1}$ and I, many edges are removed from $\mathcal{A}_{i+1}$ (that is, $\mathcal{H}_{i+1}$) in step (4). This may happen for one of the following two reasons: either many of the initial segments $W(u_j)$ are long or one of the families $M_\ell^i(\mathcal{H}_i)$ of sets with high degree in $\mathcal{H}_i$ is large.

Claim. *Either*

$$\sum_{j=0}^{b-1} |W(u_j)| \geqslant \frac{1}{4\Delta_1^{i+1}} \cdot e(\mathcal{H}_{i+1})$$

or for some $\ell \in [i]$,

$$\big|M_\ell^i(\mathcal{H}_i)\big| \geqslant \frac{1}{2(i+1)\Delta_\ell^{i+1}} \cdot e(\mathcal{H}_{i+1}).$$

Proof of claim. Recall that $\mathcal{A}_{i+1}^{(0)} = \mathcal{H}_{i+1}$ and observe that for every $j \in \{0, \dots, b-1\}$,

$$e\big(\mathcal{A}_{i+1}^{(j)}\big) - e\big(\mathcal{A}_{i+1}^{(j+1)}\big) \leqslant |W(u_j)| \cdot \Delta_1(\mathcal{H}_{i+1}) + \sum_{\ell=1}^{i} \Big| M_\ell^i\big(\mathcal{H}_i^{(j+1)}\big) \setminus M_\ell^i\big(\mathcal{H}_i^{(j)}\big)\Big| \cdot \Delta_\ell(\mathcal{H}_{i+1}). \quad (6)$$

Inequality (6) follows since in step (4) of the Scythe Algorithm, we remove from $\mathcal{A}_{i+1}^{(j)}$ only the edges that contain either a vertex of $W(u_j)$ or a member of $M_\ell^i\big(\mathcal{H}_i^{(j+1)}\big)$ for some $\ell \in [i]$. Thus, since $\Delta_\ell(\mathcal{H}_{i+1}) \leqslant \Delta_\ell^{i+1}$ for every $\ell \in [i]$, summing (6) over all j, we get

$$e\big(\mathcal{H}_{i+1}\big) - e\big(\mathcal{A}_{i+1}^{(b)}\big) \leqslant \sum_{j=0}^{b-1} |W(u_j)| \cdot \Delta_1^{i+1} + \sum_{\ell=1}^{i} \Big| M_\ell^i\big(\mathcal{H}_i^{(b)}\big)\Big| \cdot \Delta_\ell^{i+1}.$$

Since we assumed that $e\big(\mathcal{A}_{i+1}^{(b)}\big) < e\big(\mathcal{H}_{i+1}\big)/(i+1)$, see (5), and $\mathcal{H}_i = \mathcal{H}_i^{(b)}$, it follows that if

$$\sum_{j=0}^{b-1} \big|W(u_j)\big| \cdot \Delta_1^{i+1} < \frac{e(\mathcal{H}_{i+1})}{4} \leqslant \frac{i}{2(i+1)} \cdot e(\mathcal{H}_{i+1}),$$

then

$$\big|M_\ell^i\big(\mathcal{H}_i\big)\big| \cdot \Delta_\ell^{i+1} \geqslant \frac{1}{2(i+1)} \cdot e(\mathcal{H}_{i+1}) \quad \text{for some } \ell \in [i],$$

as claimed. □

Finally, let us deal with the two cases implied by the claim. In the remainder of the proof, we will show that if $M_\ell^i\big(\mathcal{H}_i\big)$ is large for some $\ell \in [i]$, then $e(\mathcal{H}_i)$ is large and if $\sum_{j=0}^{b-1} |W(u_j)|$ is large, then $|A_i|$ is small.

Case 1: $\big|M_\ell^i\big(\mathcal{H}_i\big)\big| \geqslant \frac{1}{2(i+1)\Delta_\ell^{i+1}} \cdot e(\mathcal{H}_{i+1})$ for some $\ell \in [i]$.

If $\ell < i$, then $\deg_{\mathcal{H}_i}(T) \geqslant \Delta_\ell^i/2$ for every $T \in M_\ell^i\big(\mathcal{H}_i\big)$, so by the handshaking lemma,

$$e(\mathcal{H}_i) = \binom{i}{\ell}^{-1} \sum_{T \in \binom{V(\mathcal{H})}{\ell}} \deg_{\mathcal{H}_i}(T) \geqslant \frac{\big|M_\ell^i(\mathcal{H}_i)\big| \cdot \Delta_\ell^i}{2\binom{i}{\ell}}. \quad (7)$$

Recalling that $\Delta_\ell^i \geqslant p\Delta_\ell^{i+1}$, see (1), we have

$$e(\mathcal{H}_i) \geqslant \frac{e\big(\mathcal{H}_{i+1}\big)}{4(i+1)\binom{i}{\ell}} \cdot \frac{\Delta_\ell^i}{\Delta_\ell^{i+1}} \geqslant \frac{p}{2^{i+2}(i+1)} \cdot e(\mathcal{H}_{i+1}) \geqslant \frac{p}{2^{k+1}k} \cdot e(\mathcal{H}_{i+1}),$$

as required. On the other hand, if $\ell = i$, then recalling that $\Delta_i^{i+1} \leqslant c2^k p^{-1}$, see Lemma 1.11, we have

$$e(\mathcal{H}_i) = \big|M_i^i(\mathcal{H}_i)\big| \geqslant \frac{e\big(\mathcal{H}_{i+1}\big)}{2(i+1)\Delta_i^{i+1}} \geqslant \frac{p}{c2^{k+1}(i+1)} \cdot e(\mathcal{H}_{i+1}) \geqslant \frac{p}{c2^{k+1}k} \cdot e(\mathcal{H}_{i+1}),$$

as required.

Case 2: $\sum_{j=0}^{b-1} |W(u_j)| \geqslant \frac{1}{4\Delta_1^{i+1}} \cdot e(\mathcal{H}_{i+1})$.

We claim that in this case, $|A_i| \leqslant (1-c_i)v(\mathcal{H})$. Indeed, we have

$$v(\mathcal{H}) - |A_i| = v\big(\mathcal{A}_{i+1}^{(0)}\big) - v\big(\mathcal{A}_{i+1}^{(b)}\big) = \sum_{j=0}^{b-1} |W(u_j)| \geqslant \frac{e(\mathcal{H}_{i+1})}{4\Delta_1^{i+1}}.$$

Recall that $\Delta_1^{i+1} \leqslant c2^k p^{k-i-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}$ by Lemma 1.11. Thus,

$$v(\mathcal{H}) - |A_i| \geqslant \frac{p^{i+1-k}}{c2^{k+2}} \cdot \frac{v(\mathcal{H})}{e(\mathcal{H})} \cdot e(\mathcal{H}_{i+1}) \geqslant c_i v(\mathcal{H}),$$

since $e(\mathcal{H}_{i+1}) \geqslant c_{i+1} p^{k-(i+1)} e(\mathcal{H})$ and $c_{i+1}/(c2^{k+2}) \geqslant c_i$. □

1.1 The proof of Proposition 1.2

Proof of Proposition 1.2. Let k be an integer and let c and c' be two positive constants. Furthermore, let $p \in (0,1)$ and let $\mathcal{H}$ be a k-uniform hypergraph that satisfy the assumptions of Proposition 1.2. Let $\delta = (ck2^{k+1})^{1-k}c'$ and let $b = pv(\mathcal{H})$. We shall use the Scythe Algorithm to construct a family $\mathcal{S}$ and functions f_0 and g_0 as in the statement of Proposition 1.2. We shall obtain them by running the following algorithm (with $\mathcal{H}_k = \mathcal{H}$) on every independent set $I \in \mathcal{I}(\mathcal{H})$. We shall define f_0 somewhat implicitly by defining a function $f_0^* \colon \mathcal{I}(\mathcal{H}) \to \mathcal{P}(V(\mathcal{H}))$ that is constant on the set $g_0^{-1}(S)$ for every $S \in \mathcal{S}$.

Constructing g_0 and f_0^*. *Given an $I \in \mathcal{I}(\mathcal{H})$, set $i = k-1$ and repeat the following:*

(1) Apply the Scythe Algorithm to $\mathcal{H}_{i+1}$ and I. Suppose that it outputs $\mathcal{H}_i$, A_i and B_i.

(2) If $|A_i| \leqslant (1-\delta)v(\mathcal{H})$, then set $q = i$, $r = i+1$ and STOP.

(3) If $i > 1$, then set $i = i-1$. Otherwise, set $q = r = 1$ and STOP.

Let I be an independent set and let us execute the above procedure (with $\mathcal{H}_k = \mathcal{H}$) on I. We claim that for every $i \in \{r, \ldots, k\}$, the hypergraph $\mathcal{H}_i$ satisfies properties (P1)–(P4) defined in Observation 1.6. This follows by induction on $k - i$. The base of the induction, the case $i = k$, follows vacuously from the definitions of c_k and Δ_ℓ^k for $\ell \in [k-1]$. The inductive step follows from Lemmas 1.7, 1.10 and 1.12. To see this, note that since $|A_i| > (1-\delta)v(\mathcal{H}) \geqslant (1-c_i)v(\mathcal{H})$ for all $i \in \{r, \ldots, k-1\}$, then (3) in Lemma 1.12 always holds.

Now, let us define $g_0(I)$ and $f_0^*(I)$. Suppose first that $r > 1$ and note that in this case, the algorithm stopped in step (2), which means that $|A_q| \leqslant (1-\delta)v(\mathcal{H})$; we set

$$g_0(I) = B_{k-1} \cup \ldots \cup B_q \quad \text{and} \quad f_0^*(I) = A_q.$$

On the other hand, if $r = 1$, then we set

$$g_0(I) = B_{k-1} \cup \ldots \cup B_1 \quad \text{and} \quad f_0^*(I) = \big\{v \in V(\mathcal{H}_1) \colon \{v\} \notin \mathcal{H}_1\big\}.$$

Finally, we let

$$\mathcal{S} = \{g_0(I) \colon I \in \mathcal{I}(\mathcal{H})\}.$$

We will define f_0 by letting $f_0(S) = f_0^*(I)$ for some $I \in g_0^{-1}(S)$. We first show that this definition will not depend on the choice of I. In fact, we shall prove a slightly stronger statement, which also establishes the consistency property of g_0 stated in the final line of Proposition 1.2.

Claim. *Suppose that for some $I, I' \in \mathcal{I}(\mathcal{H})$, $g_0(I) \subseteq I'$ and $g_0(I') \subseteq I$. Then $g_0(I) = g_0(I')$ and $f_0^*(I) = f_0^*(I')$.*

Proof of claim. Suppose that while running the algorithm to construct g_0 and f_0^* on some I, we obtain a sequence $(B_{k-1}, \ldots, B_q)$. Since $g_0(I)$ depends solely on $(B_{k-1}, \ldots, B_q)$ and, by Lemma 1.7, for each i, the hypergraph $\mathcal{H}_i$ and the set A_i depend only on $(B_{k-1}, \ldots, B_i)$, then also $f_0^*(I)$ depends solely on $(B_{k-1}, \ldots, B_q)$. Hence, it suffices to show that if, while running the algorithm on some I' with $B_{k-1} \cup \ldots \cup B_q \subseteq I'$, we obtain a sequence $(B'_{k-1}, \ldots, B'_{q'})$ with $B'_{k-1} \cup \ldots \cup B'_{q'} \subseteq I$, then $(B'_{k-1}, \ldots, B'_{q'}) = (B_{k-1}, \ldots, B_q)$. To this end, let us first observe that, under the above assumptions, for every $i \in [k-1]$, if $\mathcal{H}_{i+1} = \mathcal{H}'_{i+1}$, then $B_i = B'_i$. Indeed, note that B_i and B'_i are the outputs of the Scythe Algorithm executed on the inputs $(\mathcal{H}_{i+1}, I)$ and $(\mathcal{H}'_{i+1}, I')$, respectively. Hence, if $\mathcal{H}_{i+1} = \mathcal{H}'_{i+1}$, then since

$$B_i \subseteq B_{k-1} \cup \ldots \cup B_q \subseteq I' \quad \text{and} \quad B'_i \subseteq B'_{k-1} \cup \ldots \cup B'_{q'} \subseteq I,$$

then Lemma 1.8 implies that $B_i = B'_i$. Since clearly $\mathcal{H}_k = \mathcal{H}'_k = \mathcal{H}$ and, as noted before, for each i, $\mathcal{H}_{i+1}$ depends only on $(B_{k-1}, \ldots, B_{i+1})$, it follows that $B_i = B'_i$ for all i, as required. □

By the above claim, we can define f_0 by letting, for every $S \in \mathcal{S}$, $f(S) = f_0^*(I)$ for any $I \in g_0^{-1}(S)$. Finally, let us show that the $\mathcal{S}$, g_0, and f_0, which we have just defined, satisfy the required conditions, that is, for all $I, I' \in \mathcal{I}(\mathcal{H})$,

(i) $|S| \leqslant (k-1)pv(\mathcal{H})$ for every $S \in \mathcal{S}$,

(ii) $g_0(I) \subseteq I \subseteq f_0(g_0(I)) \cup g_0(I)$,

(iii) $|f_0(g_0(I))| \leqslant (1-\delta)v(\mathcal{H})$,

(iv) $g_0(I) \subseteq I'$ and $g_0(I') \subseteq I$ imply that $g_0(I) = g_0(I')$.

To see (i), simply recall that $|B_i| \leqslant pv(\mathcal{H})$ for every $i \in [k-1]$. To see (ii), note that $B_i \subseteq I \subseteq A_i \cup B_i$ for every $i \in \{q, \ldots, k-1\}$, by Lemma 1.7, that I is an independent set in $\mathcal{H}_1$ (if $r = 1$) and, crucially, that $f_0(g_0(I)) = f_0^*(I)$. To see (iii), note that if $r > 1$, then $|A_q| \leqslant (1-\delta)v(\mathcal{H})$ – see step (2) of the algorithm to construct g_0 and f_0^*; if $r = 1$, then observe that $\big|\{v \in V(\mathcal{H}_1) \colon \{v\} \notin \mathcal{H}_1\}\big| \leqslant (1-\delta)v(\mathcal{H})$ since $\mathcal{H}_1$ satisfies property (P4) and hence

$$e(\mathcal{H}_1) \geqslant c_1 p^{k-1} e(\mathcal{H}) \geqslant c_1 c' \cdot v(\mathcal{H}) = \delta v(\mathcal{H}),$$

where the second inequality follows from our assumption that $p^{k-1}e(\mathcal{H}) \geqslant c' v(\mathcal{H})$. Finally, (iv) follows directly from the claim. □

References

[1] J. Balogh, R. Morris and W. Samotij, *Independent sets in hypergraphs*, J. Amer. Math. Soc. 28 (2015), 669-709.