0.1 List Coloring

We first define the following:

Definition 0.1. *Let G be a graph with vertex set $V(G)$. A function $f : V(G) \longrightarrow [k]$ is called a k-**coloring**. Such a coloring is called **proper** if $f(u) \neq f(v)$ whenever uv is an edge in G. The **chromatic number** $\chi(G)$ is the minimum k such that a proper k-coloring of G exists.*

Definition 0.2. *Let G be a graph with vertex set $V(G)$. Let $\mathbb{Z}$ be the set of all available colors. For each vertex $v \in V(G)$, assign a list $L(v) \subseteq \mathbb{Z}$ of colors. A **list coloring** is a function $f : V(G) \longrightarrow \mathbb{Z}$ such that $f(v) \in L(v)$ for each v and $f(u) \neq f(v)$ whenever uv is an edge in G.*

Definition 0.3. *A graph is k-**choosable** if it has a proper list coloring no matter how one assigns a list of k colors to each vertex. The **list chromatic number** $\chi_l(G)$ (also called the **choice number**) is the minimum k such that G is k-choosable.*

Note that $\chi_l(G) \geqslant \chi(G)$ for any graph G, since we can take $L(v) = [k]$ for each v.

The idea of choosability for graphs was first considered by Vizing [1] and Erdős, Rubin and Taylor [2]. One of the main discoveries of [2] is that $\chi_l(G)$ can be much larger than $\chi(G)$. In the case of $K_{d,d}$, the complete bipartite graph with d vertices in each part, [2] shows that $\chi_l(K_{d,d}) = (1 + o(1)) \log_2 d$, whereas $\chi(K_{d,d}) = 2$.

Alon showed in [3] that unlike $\chi(G)$, $\chi_l(G)$ must grow with the minimum degree of the graph G. In fact, if G is d-regular, then $\chi_l(G)$ is "big". Others asked whether the result extends to r-uniform hypergraphs.

Definition 0.4. *A hypergraph is said to be **simple** (or **linear**) if any pair of edges have at most one vertex in common.*

In 2012, Saxton and Thomason [4] showed that for a simple d-regular r-uniform hypergraph G, $\chi_l(G)$ must be of order at least $\log d$.

Theorem 0.5. *Let G be a simple d-regular r-uniform hypergraph. Then*

$$\chi_l(G) \geqslant \left(\frac{1}{2r \log(2r^2)} + o(1) \right) \log d$$

where the $o(1)$ term is as $d \to \infty$.

The aim of this lecture is to prove Theorem 0.5.

0.2 Small Covers for Independent Sets

Here we extend the concept of list colorings to hypergraphs. In this section, we assume that the hypergraph $\mathcal{H}$ has n vertices with vertex set $[n]$. A vertex coloring of a hypergraph is a partition $A_1, \dots, A_t$ of its vertex set. Each vertex v is given the color i, where $v \in A_i$. For the coloring to be proper, we require that each A_i is an independent set.

Definition 0.6. *Let $A_1, \dots, A_t$ be a partition of the vertex set of a hypergraph $\mathcal{H}$. A collection of lists $\{L_v : v \in [n]\}$* ***admits*** *$A_1, \dots, A_t$ if the color given to each $v \in [n]$ belongs to L_v, i.e. for each $i \in [t]$, $A_i \subseteq \{v : i \in L_v\}$.*

Definition 0.7. *Let $\mathcal{C} \subseteq 2^{[n]}$ be a family of containers and let $A_1, \dots, A_t$ be a partition of $[n]$. Say that the partition is* ***$\mathcal{C}$-compatible*** *if for $1 \leqslant i \leqslant t$, $A_i \subseteq C$ for some $C \in \mathcal{C}$.*

The following result says, roughly, that if $\mathcal{C}$ is not "large" and no set in $\mathcal{C}$ is too close to $[n]$, then there exists a collection of "large" lists that do not admit a $\mathcal{C}$-compatible coloring.

Theorem 0.8. *[4] For $c > 0$ and $k < n$, let $\mathcal{C} \subseteq 2^{[n]}$ satisfy*

(i) $|\mathcal{C}| \leqslant 2^{n/k}$,

(ii) $|C| \leqslant (1-c)n$ for all $C \in \mathcal{C}$.

Then there exists a collection of lists $\{L_v : v \in [n]\}$ each of size

$$|L_v| \geqslant (1 + o(1)) \log k / \log(1/c)$$

($o(1) \to 0$ as $k \to \infty$ with c fixed) which does not admit a $\mathcal{C}$-compatible partition.

In other words, for a d-regular r-uniform hypergraph with size at most $(1-c)n$ with $c \sim \frac{1}{r} - \epsilon$ for $\epsilon > 0$, a container C should contain only "few" hyperedge. If $|C| > \left(1 - \frac{1}{r} + \epsilon\right) n$, then C must contain "many" edges.

Proof. Let $\epsilon > 0$, $c > 0$, and k an integer less than n. Let $l = \lfloor (1-\epsilon) \log k / \log(1/c) \rfloor$, the size of the lists, and let $t = \lfloor 2l^2/c \rfloor$, the number of colors. For each vertex $v \in [n]$, choose $L_v \in [t]^{(l)}$, a subset of $[t]$ of size l, uniformly and independently at random. It suffices to show that for large enough k, depending on ϵ and c, the probability of the lists $\{L_v : v \in [n]\}$ admitting a $\mathcal{C}$-compatible coloring is less than one.

Suppose for a contradiction that the collection of lists $\{L_v : v \in [n]\}$ admits some $\mathcal{C}$-compatible partition $A_1, \dots, A_t$. That would mean that there exists tuple $(C_1, \dots, C_t) \in \mathcal{C}^t$

such that $A_i \subseteq C_i$ for each $i \in [t]$. Call the C_i's the covering sets. For each collection of covering sets, define

$$B_v = B_v(C_1, \ldots, C_t) = \{i \in [t] : v \in C_i\}.$$

Each B_v, then, is a subset of $[t]$ giving the indices of the covering sets containing a specific vertex v.

Since $\{L_v : v \in [n]\}$ admits the partition $A_1, \ldots, A_t$ and the partition is $\mathcal{C}$-compatible, it must be that, for each $v \in [n]$, there exists $i \in L_v$ with $v \in A_i \subseteq C_i$; that is, $B_v \cap L_v \neq \emptyset$. For the contradiction, we show that with positive probability, this does not happen for any tuple $(C_1, \ldots, C_t) \in \mathcal{C}^t$; that is, with positive probability, for every tuple $(C_1, \ldots, C_t)$ there is some $v \in [n]$ with $B_v \cap L_v = \emptyset$, which is equivalent to $L_v \subseteq [t] \setminus B_v$.

For a given tuple $(C_1, \ldots, C_t)$, let p_v be the probability that $B_v \cap L_v = \emptyset$. Let $z_v = \max\{l-1, t-|B_v|\}$. Then

$$p_v = \mathbb{P}(B_v \cap L_v = \emptyset) = \binom{z_v}{l}\binom{t}{l}^{-1},$$

since $\binom{z_v}{l}$ is the number of ways to choose a list L_v from $[t] \setminus B_v$ and $\binom{t}{l}$ is the number of ways to choose a list L_v from $[t]$.

Let z be the average of $\{z_v : v \in [n]\}$; then we have the following:

$$\begin{aligned}
nz &= \sum_{v \in [n]} z_v \\
&\geqslant \sum_{v \in [n]} t - |B_v| && \text{By the definition of } z_v \\
&= nt - \sum_{v \in [n]} |B_v| && \text{Split the sum} \\
&= nt - \sum_{i=1}^{t} |C_i| && \text{Double counting} \\
&\geqslant nct && |C| \leqslant (1-c)n \text{ for all } C \in \mathcal{C}.
\end{aligned}$$

Note that this implies that $z \geqslant ct$.

Since the function $\binom{z_v}{l}$ is convex for $z_v \geqslant l-1$, we have

$$\begin{aligned}
\sum_{v\in[n]} p_v &= \sum_{v\in[n]} \binom{z_v}{l}\binom{t}{l}^{-1} && \text{Definition of } p_v \\
&\geqslant n\binom{z}{l}\binom{t}{l}^{-1} && \text{Convexity of } \binom{z_v}{l} \\
&\geqslant n\binom{ct}{l}\binom{t}{l}^{-1} && z \geqslant ct \\
&\geqslant n\left(c - \frac{l-1}{t}\right)^l && l \text{ is large when } k \text{ is large} \\
&\geqslant n\left(c - \frac{c}{2l}\right)^l && \frac{l-1}{t} \leqslant \frac{c}{2l} \text{ when } k \text{ and } l \text{ are large} \\
&= n\left[c\left(1-\frac{1}{2l}\right)\right]^l \\
&\geqslant \frac{nc^l}{2} && \left(1-\frac{1}{2l}\right)^l \geqslant \frac{1}{2}
\end{aligned}$$

Thus we have

$$Pr(B_v \cap L_v \neq \emptyset \text{ for all } v \in [n]) = \prod_{v\in[n]}(1-p_v) \leqslant \exp\left\{-\sum_{v\in[n]} p_v\right\} \leqslant \exp\left\{-\frac{nc^l}{2}\right\}.$$

Summing this probability over all tuples $(C_1, \ldots, C_t)$, then the probability that some tuple $(C_1, \ldots, C_t)$ satisfies $B_v \cap L_v \neq \emptyset$ for all v is at most

$$\begin{aligned}
&|\mathcal{C}|^t \exp\left\{-\frac{nc^l}{2}\right\} \\
\leqslant &\exp\left\{\frac{nt\log 2}{k} - \frac{nc^l}{2}\right\} && |\mathcal{C}| \leqslant 2^{n/k} \\
\leqslant &\exp\left\{\frac{2nl^2\log 2}{ck} - \frac{nc^l}{2}\right\} && t = \left\lfloor \frac{2l^2}{c} \right\rfloor \\
= &\exp\left\{\frac{n}{2k}\left[\frac{4l^2\log 2}{c} - c^l k\right]\right\} \\
\leqslant &\exp\left\{\frac{n}{2k}\left[\frac{4\log 2}{c}\left(\frac{(1-\epsilon)\log k}{\log 1/c}\right)^2 - k^\epsilon\right]\right\} && l = \left\lfloor \frac{(1-\epsilon)\log k}{\log(1/c)} \right\rfloor,
\end{aligned}$$

for fixed c and ϵ, and is less than 1 for k sufficiently large. Thus the probability that the collection of lists does not admit a $\mathcal{C}$-compatible coloring. □

0.3 List Ramsey Numbers

For this next section, we begin with the following definition.

Definition 0.9. *Let G be a r-uniform hypergraph. The k-color (ordinary)* ***Ramsey number*** *of G is defined as*

$$R(G,k) := \min\{n : \forall k-\textit{coloring of } E(K_n^{(r)}), \exists \textit{ a monochromatic copy of } G\}.$$

The list coloring version of the Ramsey problem asks: when is it possible to assign lists of size k to the edges of $K_n^{(r)}$ such that for every list coloring, there exists a monochromatic copy of the given graph G. From this we have the following variant of the Ramsey number.

Definition 0.10. *The k-color* ***list Ramsey number*** *of an r-uniform hypergraph G is defined by*

$$R_l(G,k) := \min\{n : \exists L : E(K_n^{(r)}) \to \binom{\mathbb{N}}{k} \textit{ s.t. } \forall L\textit{-coloring of } E(K_n^{(r)}), \exists \textit{ a monochromatic copy of } G\}.$$

Note that from the definitions, for every G and every k, we have that

$$R_l(G,k) \leqslant R(G,k).$$

A question one may ask is under what conditions does the inequality become equality, which is explored by Alon, Bucić, Kalvari, Kuperwasser, and Szabó in [5]. More generally, they look at when the gap between the two numbers are "large" and when the gap is relatively small.

Theorem 0.11. *[5] For arbitrary positive integers $r \geqslant l$ and $k \in \mathbb{N}$, we have*

$$R_l(K_r^{(l)}, k) \leqslant 2^{4r^{3l-1}+4kr^{l-1}\log_2 r}.$$

In contrast, the ordinary Ramsey number is bounded below by a double exponential tower where the height of the tower depends on the uniformicity of the complete hypergraph.

In [5], the following is proved

Theorem 0.12. *[5] Let $H = K_r^{(l)}$ with $r > l$ and let $\delta > 0$. For any positive integer n there exists a collection of l-graphs $C_1, \dots, C_m$ on the vertex set $[n]$, such that*

(a) every H-free l-graph on the vertex set $[n]$ is contained within some C_i,

(b) $|E(C_i)| \leqslant \left(1 - \frac{2}{3}\binom{r-1}{l-1}^{-1}\right)\binom{n}{l}$ and

(c) $\log m \leqslant 2^{13\binom{r}{l}^2} n^{l-1/m(H)} \log n$.

Proof. (Sketch)
Here we give a sketch of a proof that's similar to the proof of Theorem 0.8. See [5] for a complete proof. Let $\epsilon > 0$ and let $t = \frac{\lceil k \rceil}{\epsilon}$. Let G be a H-free l-graph on the vertex set $[n]$. For each edge in G, we choose a k-list uniformly at random out of the universe of $k+t$ colors, where the list denotes the C_i's that contain the specified edge of G. Given our k-list, the number of possible container vectors is bounded by $2^{\binom{n}{l}(m-k)+[\binom{n}{l}-1]k} = 2^{\binom{n}{l}m-k}$. In this set-up, finding a monochromatic coloring of G is equivalent to finding a C_i containing G. Let B denote the event that there is a coloring from our lists having no monochromatic G. We show that $\mathbb{P}(B) < 1$. □

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