1 Lecture 15: Volume Computing

1.1 "Weak"-containers

For this section, see [4] for reference. Consider a K_r-free graph G with n vertices. For all $\epsilon > 0$, Szemerédi's Regularity Lemma [9] yields a a constant t such that there is an ϵ- regular partition of $V(G)$ into sets $V_1, ..., V_t$, i.e., for all $i, j \in [t]$, $||V_i| - |V_j|| \leqslant 1$ and all but ϵt^2 pairs of (V_i, V_j) with are ϵ-regular (see [4] for definition).

Then we can define the cluster graph G_C as following,

$$V(G_C) = \{V_1, \ldots, V_t\}$$
$$V_iV_j \in E(G_C) \Leftrightarrow (V_i, V_j) \text{ is } \epsilon\text{-regular and } d(V_i, V_j) > \delta \gg \epsilon > 0.$$

The cluster graph G_C is also K_r-free since otherwise $K_r \subseteq G$ by the Counting Lemma. From G_C we can define another graph G' on the same vertex set as in G. Let $V(G') = V_1 \cup \cdots \cup V_t$, where each V_i is an independent set, and $G'[V_i \cup V_j]$ is complete bipartite if $V_iV_j \in E(G_C)$.

Then we define another graph H as follows,

$$V(H) = V(G); \quad E(H) = E(G) - E(G').$$

As $\{V_1, \ldots, V_t\}$ is an ϵ-regular partition of $V(G)$, we know $|E(H)| \leqslant (\delta + 2\epsilon + 1/t)n^2$.

The graph G' is an approximate container for G, as it may be the case that $G \not\subseteq G'$. The number of approximate containers is small, since the number of vertices in the cluster graph is a constant t, assigning each vertex of G to the cluster graph yields t^n choices for G'. Also we have G' is K_r-free.

We know $H \cup G'$ is another container for G, and it is proper. We have

$$\begin{aligned} \#\text{ of choices for } H \cup G' &\leqslant (\#\text{ of choices for } G') \cdot (\#\text{ of choices for } H) \\ &= t^n \cdot \binom{\binom{n}{2}}{d \cdot n^2}, \end{aligned}$$

where $d = f(\epsilon, \delta, t)$ is small. This number is too big for applications in random graphs, as $\mathbb{E}[e(G(n,p))] = p \cdot \binom{n}{2}$ while the number of containers is $2^{\omega(n^2)}$.

Also the number of K_r's is bounded above by $d \cdot n^r = dn^2 \cdot n^{r-2}$. To see this, as G' is K_r-free, we need an edge from H to form K_r. This explains the $d \cdot n^2$ term. Having chosen an edge in K_r, we need to choose the rest $r-2$ vertices, which accounts for the n^{r-2} term.

1.2 Volume Computing

Let $d : E(K_n) \to [0,2]$ be a distance function from the edge set of a complete graph to the interval $[0,2]$. We say that d is *metric* if for every triangle uvw in K_n, the triangle inequality $d(uv) \leqslant d(uw) + d(vw)$ holds. We know $d \in [0,2]^{\binom{n}{2}}$. The set of metric functions d forms a convex polytope in the cube $[0,2]^{\binom{n}{2}}$, because for each triangle uvw, the linear equation $d(uv) = d(uw) + d(vw)$ is a hyperplane in $\mathbb{R}^{\binom{n}{2}}$, and intersection of finite half-spaces is a convex polytope. If $d \in [1,2]^{\binom{n}{2}}$, then d is metric, so this convex polytope contains the cube $[1,2]^{\binom{n}{2}}$, so we have $1 \leqslant vol \leqslant 2^{\binom{n}{2}}$ where vol denotes the volume of the polytope.

We approximate d by scaling it up: instead of mapping the edges to points in the interval $[0,2]$, we consider $d : E(K_n) \to \{1, ..., M\}$, where $M \in \mathbb{N}$. For bigger M, the approximation is better. Define $\mathcal{M}_n^M = \{d : E(K_n) \to [M] \text{ with triangle inequality}\}$, where the triangle inequality means if three edges $e, f, g \in E(K_n)$ form a triangle, then $d(e) + d(f) \geqslant d(g)$. We are interested in how many such functions exist for each n and M. In other words, what is the volume of the polytope $\mathcal{M}_n^M$ in $[M]^{\binom{n}{2}}$? A trivial lower bound would be $(\lfloor \frac{M}{2} \rfloor + 1)^{\binom{n}{2}}$, as limiting the range of d to $\{\lceil \frac{M}{2} \rceil, \dots, M\}$ makes triangle inequality always true. A better lower bound [6] is $\left[\left(\frac{1}{2} + \frac{c}{\sqrt{n}}\right) M\right]^{\binom{n}{2}}$ where $M \gg \sqrt{n}$.

For every fixed even M,

$$|\mathcal{M}_n^M| \leqslant (1 + o(1)) \left\lceil \frac{M+1}{2} \right\rceil^{\binom{n}{2}} \quad \text{as } n \to \infty.$$

See [8] for further results in this direction. The idea of the proof is to consider d as an M-edge-coloring of the complete graph, and then apply the colored version of the Regularity Lemma. This method works when M is a constant, because the number of colors should be constant in the Regularity Lemma. In this case, the error term of the approximation will be a small constant.

1.3 Proof by Container Approach

The problem becomes more difficult if M grows with n. Upper bounds for $|\mathcal{M}_n^M|$ are studied in [3] and [4] using the container approach, where M can be a function of n.

Theorem 1.1 ([4])**.** *Fix an arbitrary small constant* $\varepsilon > 0$*. For*

$$M = O\left(\frac{n^{1/3}}{\log^{4/3+\varepsilon} n}\right),$$

we have

$$|\mathcal{M}_n^M| \leqslant \left\lceil \frac{M+1}{2} \right\rceil^{\binom{n}{2}+o(n^2)}.$$

To apply the container method, we define the following hypergraph. The vertex set of the hypergraph $\mathcal{H}$ is defined as follows:

$$V(\mathcal{H}) = \{(f, r) : f \in E(K_n), r \in [m]\},$$

i.e., $|V(\mathcal{H})| = \binom{n}{2}M$. The vertex set $V(\mathcal{H})$ can be arranged in M rows, one for each color, and $\binom{n}{2}$ columns, one for each edge of K_n.

To define the hyperedges, consider the independent sets: they should not contain any non-metric triangle. Let $E(\mathcal{H})$ be the set of all triples $\{(e_1, d_1), (e_2, d_2), (e_3, d_3)\}$ such that e_1, e_2, e_3 form a triangle in K_n, but $d_{\sigma(1)} + d_{\sigma(2)} < d_{\sigma(3)}$ for some permutation σ of $\{1, 2, 3\}$. Then every metric d corresponds to an independent set of $\mathcal{H}$.

	uv	uw	vw
1	•	•	•
2	•	•	•
3	•	•	•
4	•	•	•
5	•	•	•
6	•	•	•

For example, for $n = 3, r = 6$, we have $|V(\mathcal{H})| = 18$ as shown above, and $\{(uv, 2), (uw, 5), (vw, 1)\}$ forms a nonmetric triangle and is thus a hyperedge in $\mathcal{H}$.

A coloring of $E(K_n)$ satisfying triangle inequality is an independent set in $\mathcal{H}$, but not vice versa. A coloring has to choose one vertex from each column. In the example above, the number of colorings contained in the shaded vertex set is $3 \times 4 \times 4 = 48$.

To prove Theorem 1.1, we need a supersaturation result.

Lemma 1.2 ([4])**.** *Let* $\varepsilon > 0$ *and let* $S \subset V(\mathcal{H})$ *with no empty columns.*
1. If M *is even and* $|S| \geqslant (1+\varepsilon)\binom{n}{2}\lceil\frac{M+1}{2}\rceil$*, then* S *contains at least* $\frac{\varepsilon}{10}\binom{n}{3}$ *hyperedges.*
2. If M *is odd,* $n \geqslant n_\varepsilon$ *sufficiently large and* $|S| \geqslant (1+\varepsilon)\binom{n}{2}\lceil\frac{M+1}{2}\rceil$*, then* S *contains at least* $\frac{\varepsilon^4}{40000}\binom{n}{3}$ *hyperedges.*

We will use a version of the container theorem stated in [7].

Theorem 1.3. *There exists a positive integer c such that the following holds for every positive integer N. Let $\mathcal{H}$ be a 3-uniform hypergraph of order N. Let $0 \leqslant p \leqslant 1/(3^6 c)$ and $0 < \alpha < 1$ be such that $\Delta(\mathcal{H}, p) \leqslant \alpha/(27c)$, where*

$$\Delta(\mathcal{H}, p) = \frac{4\Delta_2}{\bar{d}p} + \frac{2\Delta_3}{\bar{d}p^2}.$$

Then there exists a collection of containers $\mathcal{C} \subset \mathcal{P}(V(\mathcal{H}))$ such that
(i) every independent set in $\mathcal{H}$ is contained in some $C \in \mathcal{C}$,
(ii) for all $C \in \mathcal{C}$ we have $e(\mathcal{H}[C]) \leqslant \alpha e(\mathcal{H})$, and
(iii) the number of containers satisfies

$$\log |\mathcal{C}| \leqslant 3^9(1 + \log(1/\alpha))Np\log(1/p).$$

Proof of Theorem 1.1 Let $\varepsilon, \delta > 0$ be arbitrarily small constants. Set $p = \frac{1}{M\log^{2+\delta} n}$, $\alpha = \frac{10^{10} c \log^{4+2\delta} n}{n}$. We have $\Delta_1(\mathcal{H}) \leqslant nM^2, \Delta_2(\mathcal{H}) \leqslant M, \Delta_3(\mathcal{H}) = 1, \bar{d} \geqslant M^2 n/64$, then

$$\Delta(\mathcal{H}, p) \leqslant 4\left(\frac{64M^2\log^{2+\delta} n}{M^2 n} + \frac{64M^2\log^{4+2\delta} n}{2M^2 n}\right) \leqslant \frac{\alpha}{27c}.$$

By Theorem 1.3 we have a collection of containers $\mathcal{C}$ with

$$\log |\mathcal{C}| \leqslant \frac{c3^{10}Mn^2 \cdot \log n \cdot \log M \cdot \log\log n}{M\log^{2+\delta} n} = o(n^2),$$

and for every $C \in \mathcal{C}$,

$$e(\mathcal{H}[C]) \leqslant \alpha e(\mathcal{H}) \leqslant 10^4 cM^3 n^2 \log^{4+2\delta} n.$$

Now assume

$$M = o\left(\frac{n^{1/3}}{\log^{4/3+\varepsilon} n}\right).$$

Then the maximum number of edges in a container is $o(n^3)$. A useful container has no empty column. Therefore by Lemma 1.2, when n is large enough,

$$|V(C)| < (1+\varepsilon)\binom{n}{2}\left\lceil\frac{M+1}{2}\right\rceil.$$

So the number of good colorings in a container is at most

$$(1+\varepsilon)^{\binom{n}{2}}\lceil\frac{M+1}{2}\rceil^{\binom{n}{2}} = \lceil\frac{M+1}{2}\rceil^{\binom{n}{2}+o(n^2)}.$$

We have $\log|\mathcal{C}| \leqslant o(n^2)$. Therefore the total number of good colorings satisfies

$$|\mathcal{M}_n^M| \leqslant \lceil \frac{M+1}{2} \rceil^{\binom{n}{2}+o(n^2)}.$$

□

The following almost optimal estimate was obtained by Kozma, Meyerovitch, Peled, and Samotij [6].

Theorem 1.4 ([6])**.** *There exists a constant C such that*

$$|\mathcal{M}_n^M| \leqslant [(\frac{1}{2} + \frac{2}{M} + \frac{C}{\sqrt{n}})M]^{\binom{n}{2}}$$

for all n and M.

1.4 The $f(n, r, k)$ Problem

For $k = 2$ or 3, define

$$f(G, r, k) = \text{number of } k\text{-edge colorings of } G \text{ without monochromatic } K_r,$$
$$f(n, r, k) = \max_{V(G)=n} f(G, r, k).$$

A question of Erdős is to determine the value of $f(n, r, k)$.

For example, when $r = 3$, $K_{n/2,n/2}$ has no triangle, so every coloring is good. Therefore,

$$f(n, 3, k) \geqslant k^{\lfloor n^2/4 \rfloor}.$$

In general, by Turán's Theorem, there is a lower bound

$$f(n, r, k) \geqslant k^{(1-\frac{1}{r-1})\binom{n}{2}}. \tag{1}$$

If n is sufficiently large, then equality holds in (1). This was originally proved using Szemerédi's Regularity Lemma, for $k = 2$ or 3, when r is fixed and n is large. It was then proved by the Container Method in [5] where $r \leqslant (\log_2 n)^{1/4}$. Here n does not need to be as large as in the proof using the Regularity Lemma. A result along the same lines is the following.

Theorem 1.5 ([1])**.** *If $r \leqslant (\log_2 n)^{1/4}$, then almost all K_r-free graphs with n vertices are $(r-1)$-partite.*

Proof. To apply the container theorem, we define the following hypergraph:

$$V(\mathcal{H}) = E(K_n); \quad E(\mathcal{H}) = \text{edge sets of } K_r\text{'s in } K_n.$$

We have $e(\mathcal{H}) = \binom{n}{r}$, and the uniformity of $\mathcal{H}$ is $\binom{r}{2}$. For $1 \leqslant l \leqslant \binom{r}{2}$,

$$\Delta_l(\mathcal{H}) \leqslant \binom{n-t}{r-t},$$

where t is the smallest number with $\binom{t}{2} \geqslant l$. The codegrees of $\mathcal{H}$ then satisfies the assumptions of the container theorem stated in [2] with $q = n^{-2/(r+1)}$. By the container theorem, there exists a family $\mathcal{C}$ of containers $C_1, .., C_t$ such that every independent set in $\mathcal{H}$ is contained in some C_i. Then

$$\# \text{ of } K_r\text{-free graphs } \leqslant t \cdot 2^{\max |C_i|},$$

where $t = 2^{o(n^2)}$ and the number of K_r's in C_i is $o(\binom{n}{r})$. By supersaturation, we have $\max |C_i| \leqslant (1 - \frac{1}{r-1} + o(1))\binom{n}{2}$. Then for the number of K_r-free graphs we have

$$2^{(1-\frac{1}{r-1})\binom{n}{2}} \leqslant \# \text{ of } K_r\text{-free graphs } \leqslant 2^{(1-\frac{1}{r-1}+o(1))\binom{n}{2}}.$$

To prove the precise statement, a long, but somewhat standard arguments are needed. □

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