1 Lectures 10-11: The (3,4)-Problem (2/11/19-2/15/19)

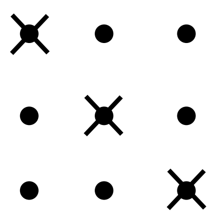
Given $P \subset \mathbb{R}^2$ with $|P| = n$, define

$$f_3(P) = \max\{|S| : S \subseteq P, S \text{ in general position}\}$$

and

$$f_{3,4}(n) = \min\{f_3(P) : |P| = n, P \text{ has no 4 points in a line}\}.$$

For example, consider the set P of nine points shown below. Since the set S of points which have not been crossed out from P are in general position, we have $f_3(P) \geqslant 6$, and hence $f_{3,4}(9) \leqslant 6$.



Conjecture 1.1 (Erdős [5]). $f_{3,4}(n) = o(n)$.

1.1 Lower Bound on $f_{3,4}(n)$

We can use a greedy algorithm to obtain a lower bound on $f_{3,4}(n)$ as follows: Given a set P with no collinear 4-tuple, choose points one by one from P to build a set S in general position. After step i, let $S = \{s_1, \ldots, s_{i-1}\}$. When we add the i^{th} point s_i to S, we exclude up to $i-1$ others (one per previous line including s_i, as we want to avoid creating a collinear triple with s_i and some s_j for $1 \leqslant j \leqslant i-1$). We will not run out of points to add to S before $1 + \cdots + i < n$, which gives $i < \sqrt{2n}$. Therefore, $\sqrt{2n} \leqslant f_{3,4}(n)$.

It is frequently the case that a lower bound given by a greedy algorithm can be improved by a logarithmic factor. In this case, Füredi observed that he could phrase the problem in terms of hypergraphs and use a previous result of Phelps and Rödl [15] to obtain an improvement.

Theorem 1.2 (Füredi [8]). *There is a positive constant c such that $c\sqrt{n \log n} < f_{3,4}(n)$ holds for all n.*

The idea of the proof is to define, given a point set P with no four points on a line, an appropriate hypergraph. Let $\mathcal{H}$ be the 3-uniform hypergraph with vertices $V(\mathcal{H}) = P$ and edges $E(\mathcal{H}) = \{\text{collinear triples in } P\}$. Then since P contains no four collinear points, we have $|A \cap B| \leqslant 1$ for all $A, B \in E(\mathcal{H})$, so $\mathcal{H}$ is a linear hypergraph. Any independent set in $\mathcal{H}$ corresponds to a subset of P in general position. Therefore, we can apply the following result of Phelps and Rödl (which was based on a more general result of Ajtai, Komló s, Pintz, Spencer, Szemeré'di [1].

Theorem 1.3 (Phelps–Rödl [15]). *For every 3-uniform linear hypergraph $\mathcal{H} = (V, E)$ with $|V| = n$, there exists a subset $I \subset V$ of size $|I| \geqslant c\sqrt{n \log n}$ containing no edges from E, where c is an absolute constant not depending on n.*

1.2 Upper Bound on $f_{3,4}(n)$

To prove an upper bound on $f_{3,4}(n)$, we must construct n points in the plane with no 4 in a line such that every "big" subset contains 3 collinear points.

Füredi [8] confirmed the conjectured upper bound on $f_{3,4}(n)$ by giving a construction which uses the Density Hales-Jewett Theorem of Furstenberg and Katznelson [9, 10].

Theorem 1.4 (Füredi [8]). *When n tends to infinity,* $\lim f_{3,4}(n)/n = 0$.

To state the Density Hales-Jewett Theorem, we need some additional definitions. We denote the cube $\{1, 2, \ldots, k\}^m$ by $[k]^m$. In this cube, a *combinatorial line* is a set of k points formed by taking a string with one or more wildcards $*$ and replacing those wildcards by $1, 2, \ldots, k$. For example, the strings of the form $(1, 3, *, 2, *, 3, *)$ form a combinatorial line in $[4]^6$ containing these points:

$$\begin{gathered}(1,3,1,2,1,3,1)\\(1,3,2,2,2,3,2)\\(1,3,3,2,3,3,3)\\(1,3,4,2,4,3,4)\end{gathered}$$

Theorem 1.5 (Furstenberg–Katzenslon [9, 10]). *For every $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists m_0 such that for all $m > m_0$, any subset $A \subseteq [k]^m$ with $|A| \geqslant \varepsilon k^m$ contains a combinatorial line.*

The current best bound on m is a tower function $m > 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ of height $1/\varepsilon^2$. It is a major open problem to find a better lower bound on m.

In order to apply this theorem to the $(3,4)$-Problem, Füredi used the point set $[3]^m$. In this setting, a combinatorial line is a line with three points, and there are no collinear 4-tuples. By Theorem 1.5, if $A \subset [3]^m$ with $|A| \geqslant \varepsilon 3^m$, then A contains 3 points in a line. However, this theorem gives the result in an m-dimensional cube, not the plane. So, we must take a random projection to $\mathbb{R}^2$. The projection will preserve the properties we care about; for example, we can show that if there are no four collinear points in the cube, then the same is true in the plane.

Balogh and Solymosi [3] improved Füredi's bound on $f_{3,4}(n)$ using the container method.

Theorem 1.6 (Balogh–Solymosi [3]). *As n goes to infinity, $f_{3,4}(n) \leqslant n^{5/6+o(1)}$.*

In order to obtain this result, they considered subsets of $[m]^3$ instead of $[3]^m$. Given the cube $[m]^3$, let $p \approx 1/m$ and keep each point in the cube with probability p independently of one another. Then, the number of points is $n \approx pm^3 \approx m^2$. The idea of the proof is to use the Alteration Method to destroy any collinear 4-tuples. Since we still want a large set of points after the alteration, we would like the expected number of collinear 4-tuples to be much less than the expected number of points.

Lemma 1.7 (Balogh–Solymosi [3]).

1. *$\frac{m^6}{3^6} \leqslant \#$ of collinear 3-tuples in $[m]^3 \leqslant 32m^6$.*
2. *If $S \subseteq [m]^3$ with $|S| = m^{3-s}$, then the number of collinear 3-tuples in S is $\geqslant \frac{m^{6-4s}}{3 \cdot 10^9 \log m}$.*
3. *The number of collinear 4-tuples in $[m]^3$ is at most $\frac{5 \cdot 2^7}{24} \cdot m^6 \cdot \log m$.*

Therefore, we want $m^6 \log m \cdot p^4 \ll pm^3$, or $p^3 \ll \frac{m^{-3}}{\log m}$. So, let $p \approx \frac{1}{m \log^{1/2} m}$. Then, we can remove one point per collinear 4-tuple and obtain a point set with $\frac{m^2}{\log^{1/2} m}$ points and no four collinear points. When we project this point set to $\mathbb{R}^2$, we preserve collinearity. It remains to show that large subsets contain three collinear points.

To this end, fix a slope in $\mathbb{R}^3$. Consider the set of parallel lines L with this slope which cover $[m]^3$. The number of these lines varies with the slope, as does the average number of points in each of these parallel lines in $[m]^3$, which is $\frac{m^{3-s}}{L}$. So, the number of collinear 3-tuples is at least $L \cdot \binom{m^{3-s}/L}{3}$. More precisely, the number of collinear 3-tuples is $\sum_{\ell \in L} \binom{\ell}{3}$.

Note that while you may expect the set of vertical lines covering $[m]^3$ to give the best result, this is not the case; each vertical line contains m^3 triples, and there are m^2 vertical lines, giving a total of just m^5 collinear 3-tuples, while we want m^6.

Instead, we can view the points (a_1, a_2, a_3) with each coordinate modulo 2, giving eight equivalence classes. In one class, if a midpoint of two given points is a grid point of $[m]^3$, then we get a collinear triple. There are $\binom{m^3/8}{2} \approx m^6$ midpoints, and hence, we get more collinear 3-tuples by using these lines than by using vertical lines.

Now we are ready to see the proof using the container method.

1.3 Proof Sketch of Theorem 1.6

Proof sketch. Let $\mathcal{H}$ be the 3-uniform hypergraph with vertex set $[m]^3$ and edge set consisting of all collinear 3-tuples. In this setting, independent sets are sets of points in general position. Applying the container lemma to $\mathcal{H}$ yields a family of containers $\mathcal{C}$ so that each independent set I is contained in some $C_i \in \mathcal{C}$.

As described above, we take a p-random subset of $[m]^3$, keeping each point with probability $p = \frac{1}{m \log^{1/2} m}$ independently of the other points. Then, we remove a point from each collinear 4-tuple in this random subset. Since the expected number of random points is much bigger than the expected number of collinear 4-tuples ($pm^3 \gg 100p^4 \log m \cdot m^6$), most of the random points will stay. Call the set of these remaining points S.

Let $I \subseteq C_i \cap S =: (C_i)_p$. Then we can bound the expected size of the independent sets since $\mathbb{E}|(C_i)_p| = pm^{8/3} = \frac{m^{5/3}}{\log^{1/2} m}$, and hence $\mathbb{E}|I| \leqslant \frac{m^{5/3}}{\log^{1/2} m}$.

We will apply Chernoff and union bounds to get that for each C_i, $|(C_i)_p| \leqslant 2pm^{8/3}$. Applying the Chernoff bound gives for each C_i that $\mathbb{P}(|(C_i)_p| > 2pm^{8/3}) < e^{-(pm^{8/3})/8}$. Then, by applying the union bound, we have

$$\mathbb{P}(\text{some } C_i \text{ has } |(C_i)_p| > 2pm^{8/3}) < e^{-(pm^{8/3})/8} \cdot \#(\text{containers}) = o(1).$$

Therefore, there is a set S of size $|S| \approx \frac{m^2}{\log^{1/2} m}$ with all independent sets having size at most $m^{5/3+o(1)}$ and which does not contain 4 points in a line. This set S can be projected into the plane in such a way that collinear point tuples stay on a line, and no new collinear point tuples are created. Thus, setting $n = m^2$, we have n points in the plane for which every subset of size $m^{5/3+o(1)} = n^{5/6+o(1)}$ contains three points in a line. □

To conclude this subsection, we will include several remarks about the proof above and how it might be improved.

- Balogh and Solymosi believe that this construction should give $n^{3/4+o(1)}$. One way to fix the proof to yield this stronger result would be to improve the supersaturation statement in Lemma 1.7 (ii) from m^{6-4s} to m^{6-3s}. This would be "best possible" for this construction, because choosing points greedily gives $n^{3/4}$ points with no 3 points in a line.

- On the other hand, Solymosi believes that applying the proof of the Szemerédi-Trotter Theorem might improve the lower bound.

- Note that in $[m]^3$, $2m^2$ is an upper bound on the number of points in general position because we can cover the points with m^2 with vertical lines and get at most 2 points on each line before creating a collinear triple. If we intersect with a random set of points included with probability $p < 1/m$, then the expected number of points in general position is at most $2pm^2 < m$. So, in expectation, each point set of $[m]^3_p$ in general position has size bounded by m. This appears to be good news because $m^2 \approx n$, so $m \approx \sqrt{n}$, which coincides with the lower bound. However, while this is true in expectation, it is not always true. There are too many sets to use the Union Bound.

- Another way to improve the result would be to start with a different hypergraph and apply the same method. In fact, Balogh and Solymosi originally started with $[k]^m$ and looked at combinatorial lines. This only gave a bound of $\frac{n}{\log n}$ (still an improvement since Hales-Jewett only gave $\frac{n}{\log^* n}$).

- The container method was useful because Balogh and Solymosi applied the quantive version, in which the dependence of every parameter is computed.

- This problem is degenerate since the size of the container is much smaller than the number of vertices. In this case, we can get better results by applying the container method iteratively.

1.4 Using the Hypergraph Container Theorem

Now we will present a version of the proof which applies Saxton and Thomason's 3-uniform hypergraph container lemma iteratively.

Let $\mathcal{H}$ be a 3-uniform hypergraph with average degree $d = \frac{3e(\mathcal{H})}{v(\mathcal{H})}$. For every pair of vertices $S \subset V(\mathcal{H})$ let the *co-degree* of S, denoted $d(S)$, be the number of edges in $\mathcal{H}$ containing S, i.e.,

$$d(S) = |\{e \in E(\mathcal{H}) : S \subseteq e\}|.$$

Let $\Delta_2 = \max\{d(S) : S \subseteq V(\mathcal{H}), |S| = 2\}$. For $\tau \in (0, 1)$, define

$$\Delta(\mathcal{H}, \tau) := \frac{4\Delta_2}{d\tau} + \frac{2}{d\tau^2}.$$

Theorem 1.8 (Saxton–Thomason [18]). *Let $\mathcal{H}$ be a 3-uniform hypergraph on vertex set $[n]$. If $0 < \varepsilon < 1/2$, $0 < \tau < 10^{-4}$, and $\Delta(\mathcal{H}, \tau) \leqslant \varepsilon/100$, then there exists $c \leqslant 10^6$ and a collection $\mathcal{C}$ of containers such that*

(i) every independent set in $\mathcal{H}$ is a subset of some $A \in \mathcal{C}$;

(ii) for every $A \in \mathcal{C}$, $e(\mathcal{H}[A]) \leqslant \varepsilon \cdot e(\mathcal{H})$;

(iii) $\log |\mathcal{C}| \leqslant cn\tau \cdot \log(1/\varepsilon) \cdot \log(1/\tau)$.

Note that in the statement of the theorem, τ and ε are not constants, but small functions of n, while c is an absolute constant.

Proof. We will apply Theorem 1.8 to the hypergraph $\mathcal{H}$ defined by $V(\mathcal{H}) = [m]^3$ and $E(\mathcal{H}) = \{\text{collinear 3-tuples}\}$. Recall the following observations about this hypergraph from Lemma 1.7:

- $\dfrac{m^6}{3^6} \leqslant e(\mathcal{H}) \leqslant 32m^6$, and hence $\dfrac{m^3}{3^5} \leqslant d \leqslant 3 \cdot 32m^3$.
- Super-saturation result: If $S \subseteq [m]^3$ and $|S| = m^{3-s}$, then $|\mathcal{H}(S)| \geqslant \dfrac{m^{6-4s}}{3 \cdot 10^9 \cdot \log m}$ and $d \geqslant \dfrac{m^{3-3s}}{10^9 \log m}$.

We would like to show that the number of containers is at most $2^{\tau 10^6 m^{5/3} \log m}$. Fix an arbitrarily small $f > 0$. Applying Theorem 1.8 to $\mathcal{H}$ gives a family of containers. We will check the size of each container C:

- If $|C| \leqslant m^{8/3+f}$, we add C to the final container family $\mathcal{C}$.
- Otherwise, $|C| > m^{8/3+f}$. Apply Theorem 1.8 to the hypergraph spanned by this container, $\mathcal{H}[C]$, and repeat the check.

A potential issue with this plan is that we may create too many containers in this process. Since each step creates $2^{10^6 m^{5/3} \log m}$ containers, we need to make sure that the number of rounds, t, is constant.

Pick s so that $|C| = m^{3-s}$, and say $|C| > m^{8/3+f}$. We know $e(\mathcal{H}[C]) \geqslant \frac{m^{6-4s}}{3 \cdot 10^9 \log m}$ and $d \geqslant \dfrac{m^{3-3s}}{10^9 \log m}$. Note that $\mathcal{H}$ (and hence all of the subhypergraphs of $\mathcal{H}$) satisfy that $\Delta_2 < m$. Set $\tau = m^{s-4/3}$. To apply the Hypergraph Container Theorem, we need

$$\Delta(\mathcal{H}[C], \tau) = \frac{4\Delta_2}{d\tau} + \frac{2}{d\tau^2} \leqslant \frac{4 \cdot 10^9 \log m}{m^{2-3s}\tau} + \frac{2 \cdot 10^9 \log m}{m^{3-3s}\tau^2}.$$

These fractions are positive until $s = \frac{1}{3} - f$, so pick $\varepsilon = m^{s-\frac{1}{3}+\frac{f}{2}}$. Now $\Delta(\mathcal{H}[C], \tau) \leqslant \frac{\varepsilon}{72}$. Then, by Theorem 1.8, there is a family of at most $2^{106 \cdot m^{3-s}\tau \log^2 m} = 2^{10^6 \cdot m^{5/3} \log^2 m}$ containers such that the number of edges in each container is at most ε times the number in the previous step. Since we started with m^6 edges, the number of iterations is at most $12/f$, which is constant, as desired.

In summary, there are less than $2^{O(m^{5/3} \log m)}$ containers, each of size less than $m^{8/3+f}$ for fixed $f > 0$. We pick $p \approx \frac{1}{m}$ and look at $[m]_p^3$. By the Alteration Method, we destroy all collinear 4-tuples. We showed that for each independent set $I \subseteq [m]_p^3$, if $|E| > m^{5/3+2f}$, then $I \subseteq$ (container) $\cap [m]_p^3$. But, the expected size of this intersection is at most $m^{5/3+f}$, and by Chernoff, we have concentration $\exp(m^{-5/3-f}/c)$. So, I is bounded by $m^{5/3+2f}$, as desired. $\square$

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