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The maximum triangle-free graph has $n^2/4$ edges [8]. Hence, the number of triangle-free graphs is at least $2^{n^2/4}$. This was shown to be the correct order of magnitude by Erdős, Kleitman and Rothschild [5] (see Balogh-Morris-Samotij [2] or Saxton-Thomason [13] for recent proofs). Moreover, almost every triangle-free graph is bipartite [5], even if there is a restriction on the number of edges (first shown by Osthus-Prömel-Taraz [11], extended by Balogh-Morris-Samotij-Warnke [3]; see [2] and [3] for a more detailed history of the problem). This suggests that most of those graphs are bipartite, and subgraphs of a complete bipartite graph, therefore most of them are not maximal. Erdős suggested the following problem (as stated in Simonovits [14]):

**Problem 0.1.** *Determine or estimate the number of maximal triangle-free graphs on $n$ vertices.*

Note that every complete-bipartite graph (roughly $2^{n-1}$ graphs) is maximal triangle-free, but we can also find more.

We give a construction of a family of $2^{n^2/8}$ maximal triangle-free graphs. Let $M = \{a_1b_1, \dots a_{n/4}b_{n/4}\}$ be a matching on $n/2$ vertices (and $n/4$ edges), and let $C = \{c_1, \dots, c_{n/2}\}$ be an independent set of size $n/2$.

Our family of graphs can be built from the above construction by adding edges as follows: for every $i \leqslant n/4, j \leqslant n/2$, we put exactly one edge from $\{a_ic_j, b_ic_j\}$ into our graph. There are $(n/4)(n/2) = n^2/8$ choices to make, and hence $2^{n^2/8}$ possible graphs obtained from this construction. It is easy to see that any such graph is triangle-free, but not necessarily maximally triangle-free. Thus if applicable, we add edges to each non-maximal graph until the resulting graph is maximally triangle-free (note that any such edge must be contained in either $M$ or $C$).

Next we present an asymptotically-matching upper bound due to Balogh and Petříčková [4].

**Theorem 0.2.** *The number of maximal triangle-free graphs with vertex set $[n]$ is at most $2^{n^2/8+o(n^2)}$.*

Later, Balogh, Liu, Petříčková, and Sharifzadeh [1] showed that the typical structure of a maximal triangle-free graph is that of the above construction.

**Theorem 0.3.** *For almost every maximal triangle-free graph $G$ on $[n]$, there is a vertex partition $X \cup Y$ such that $G[X]$ is a perfect matching and $Y$ is an independent set.*

A main tool for proving Theorem 0.2 will be the Ruzsa-Szemerédi triangle-removal lemma [12].

**Theorem 0.4.** *For every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that any graph $G$ on $n$ vertices with at most $\delta(\varepsilon)n^3$ triangles can be made triangle-free by removing at most $\varepsilon n^2$ edges.*

An independent set $I$ is a *maximal independent set* if $I \cup \{v\}$ contains an edge for every $v \in V(G) - I$. Moon and Moser [10] proved that the number of maximal independent sets in an $n$-vertex graph is at most $3^{n/3}$. Moreover, Hujter and Tuza [7] proved a smaller upper bound for triangle-free graphs.

**Theorem 0.5.** *Every triangle-free graph $G$ has at most $2^{|V(G)|/2}$ maximal independent sets.*

We use the container method due to Balogh-Morris-Samotij [2, Theorem 2.2.], and Saxton-Thomason [13].

**Theorem 0.6.** *For each $\delta > 0$ there is $t < 2^{O(\log n \cdot n^{3/2})}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $\delta n^3$ triangles, such that for every triangle-free graph $H$ there is $i \in [t]$ such that $H \subseteq G_i$, where $n$ is sufficiently large.*

By the Erdős-Simonovits supersaturation theorem [6], each $G_i$ has at most about $n^2/4$ edges.

**Theorem 0.7.** *For every $\gamma > 0$ there is $\delta(\gamma) > 0$ such that every $n$-vertex graph with $(1/4 + \gamma)n^2$ edges contains as least $\delta(\gamma)n^3$ triangles.*

We now give a proof for Theorem 0.2.

*Proof of Theorem 0.2.* We show that for every $\varepsilon > 0$ and for every $\gamma > 0$, the number of maximal triangle-free graphs with vertex set $[n]$ is $2^{(1/8+2\varepsilon+\gamma)n^2}$ for sufficiently large $n$. Fix arbitrarily small constants $\varepsilon, \gamma > 0$, and apply Theorem 0.4 with $\varepsilon$, and Theorem 0.7 with $\gamma$. This provides us with $\delta(\varepsilon)$ and $\delta(\gamma)$.

Define $\delta := \min\{\delta(\varepsilon), \delta(\gamma)\}$ and apply the containers theorem (Theorem 0.6) with $\delta$. We obtain a set of containers $G_1, \ldots, G_t$ such that each maximal triangle-free graph $H$ is contained in some $G_i$. For every $i \in [t]$, we will count the number of such graphs $H$ that $H_i$ contains. Denote $\mathcal{H}$ the set of maximal triangle-free graphs with vertex set $[n]$, and let $\mathcal{H}_i = \{H \in \mathcal{H} : H \subseteq G_i\}$.

Since $t \leqslant 2^{\varepsilon n^2}$ for sufficiently large $n$, we have

$$|\mathcal{H}| \leqslant \sum_{i=1}^{t} |\mathcal{H}_i| \leqslant 2^{\varepsilon n^2} \max_{i \in [t]} |\mathcal{H}_i|.$$

Fix $i \in [t]$. Since $G_i$ contains at most $\delta(\varepsilon)n^3$ triangles, by Theorem 0.4, there is $F_i \subseteq E(G_i)$ such that $|F_i| \leqslant \varepsilon n^2$ and $G_i - F_i$ is triangle-free. For each $G_i$ we fix one such $F_i$. For every subset of edges $F^* \subseteq F_i$ define $\mathcal{H}_i(F^*) := \{H \in \mathcal{H}_i : E(H) \cap F_i = F^*\}$. Note that we can assume that $F^*$ is triangle-free since otherwise $\mathcal{H}_i(F^*) = \emptyset$.

Now we show that for every choice of $F^*$ we have $|\mathcal{H}_i(F^*)| \leqslant 2^{e(G_i)/2}$. Fix $F^*$, and let

$$G := G_i - (F_i - F^*) - \{e \in E(G_i) : \exists f, g \in F^* \text{ such that } e, f, g \text{ form a triangle}\}.$$

I.e., $G$ is obtained from $G_i$ by removing edges that are contained in none of $H \in \mathcal{H}_i(F^*)$. We now count the number of ways to add edges of $E(G) - F^*$ to $F^*$ such that the resulting graph is maximal triangle-free. This will give us an upper bound for $|\mathcal{H}_i(F^*)|$.

We construct an auxiliary *link graph* $T$ as follows:

$$V(T) := E(G) - F^* \quad \text{and} \quad E(T) := \{ef \mid \exists d \in F^* \colon \{d, e, f\} \text{ spans a triangle in } G\}.$$

**Claim 1.** $T$ is triangle-free.

*Proof.* Suppose not. Let $e, f, g$ be vertices of a triangle in $T$. Then $e, f, g \in E(G) - F^*$ and there are $d_1, d_2, d_3 \in F^*$ such that the 3-sets $\{d_1, e, f\}$, $\{d_2, e, g\}$, and $\{d_3, f, g\}$ span triangles in $G$. As $G_i - F_i$ is triangle-free and $G - F^* \subseteq G_i - F_i$, it follows that the edges $e, f, g$ share a common endpoint in $G$, and that the set $\{d_1, d_2, d_3\}$ spans a triangle in $F^*$. This is a contradiction since $F^*$ is triangle-free. □

**Claim 2.** If $H \in \mathcal{H}_i(F^*)$, then $E(H) - F^*$ spans a maximal independent set in $T$.

*Proof.* Let $H \in \mathcal{H}_i(F^*)$. We first show that $E(H) - F^*$ spans an independent set in $T$. If not, then there is an edge $ef$ in $E(T)$ with $e, f \in E(H) - F^*$. By the definition of $E(T)$, there is $d \in F^*$ such that the edges $d, e, f$ form a triangle in $G$, which is clearly in $H$, contradicting that $H$ is triangle-free.

Suppose now that $E(H) - F^*$ is an independent set in $T$ that is not maximal. For simplicity, let $I$ denote the corresponding vertices of $E(H) - F^*$ in $T$. Then there exists some vertex $x \in V(T) \setminus I$ such that $I \cup \{x\}$ is also an independent set in $T$. By the definition of $T$, for every $y \in I$, there does not exist a $z \in F^*$ such that the edges $x, y, z$ form a triangle in $G$. This implies that $H \cup \{x\}$ is triangle free, contradicting the maximality of $H$.

□

By Theorem 0.5, the number of maximal independent sets in $T$ is at most $2^{|T|/2}$. Thus

$$|\mathcal{H}_i(F^*)| \leqslant 2^{|V(T)|/2} \leqslant 2^{e(G_i)/2} \leqslant 2^{(n^2/4+\gamma)/2},$$

where the last inequality follows from Theorem 0.7.

The number of ways to choose $F^* \subseteq F_i$ for a given $F_i$ is at most $2^{|F_i|} \leqslant 2^{\varepsilon n^2}$, so we can conclude that for sufficiently large $n$,

$$\begin{aligned}|\mathcal{H}| \leqslant 2^{\varepsilon n^2} \max_{i\in[t]} |\mathcal{H}_i| \leqslant 2^{\varepsilon n^2} \sum_{F^*\subseteq F_i} |\mathcal{H}_i(F^*)| \leqslant 2^{\varepsilon n^2} 2^{\varepsilon n^2} \max_{F^*\subseteq F} |\mathcal{H}_i(F^*)| \\ \leqslant 2^{2\varepsilon n^2} 2^{(n^2/4+\gamma n^2)/2} \leqslant 2^{(1/8+2\varepsilon+\gamma)n^2}.\end{aligned}$$

□

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