1 Lecture 6: The number of C_4-free graphs (1/28/2019)

It is well known that $ex(n, C_4) = (\frac{1}{2}+o(1))n^{3/2}$, see Lecture 5 for more details. In 1982, in an influential paper, Kleitman and Winston [1] gave a nontrivial upper bound on the number of C_4-free graphs, which resolved a longstanding open question posed by Erdős.

Theorem 1.1 (Kleitman-Winston [1])**.** *The number of C_4-free graphs on $[n]$ is at most $2^{1.08n^{3/2}}$.*

Definition 1.2 (Min-degree ordering)**.** *For a graph G on $[n]$, a min-degree ordering is an ordering $v_n < v_{n-1} < \ldots < v_1$ such that v_i is a vertex of minimum degree in the graph $G_i = G[v_i, \ldots, v_1]$, for every $i \in [n]$ (if there are multiple vertices of minimum degree, choose the one with the largest label).*

The main idea in proving Theorem 1.1 is to estimate the number of ways to choose $N_{G_i}(v_i)$ under a fixed min-degree ordering (using the graph container method). Ideally, if we can show that for each v_i, there are $2^{O(\sqrt{n})}$ ways to select $N_{G_i}(v_i)$, then the number of C_4-free graphs is $2^{O(\sqrt{n})\cdot n} = 2^{O(n^{3/2})}$ since a graph can be uniquely determined by a min-degree ordering $v_n < v_{n-1} < \ldots < v_1$ and the set of neighborhoods $N_{G_i}(v_i)$.

Fix an arbitrary C_4-free graph G, and let $v_n < v_{n-1} < \ldots < v_1$ be its min-degree ordering. Let $d_i = d_{G_i}(v_i)$. If $d_i < \frac{\sqrt{n}}{\log n}$, then the number of ways to choose $N_{G_i}(v_i)$ is at most $\binom{n}{d_i} \leqslant \binom{n}{\sqrt{n}\log n} \leqslant 2^{\sqrt{n}}$. Therefore, it suffices to consider the case when $d_i \geqslant \frac{\sqrt{n}}{\log n}$.

For a graph F, denote by F^2 the multigraph defined on $V(F)$ such that for every distinct $u, v \in V(F^2)$, the multiplicity of uv in F^2 is the number of (u, v)-paths of length 2 in F. From the definition and the C_4-freeness of G, we immediately obtain the following proposition.

Proposition 1.3. *$N_{G_i}(v_i)$ is an independent set in G^2_{i-1}.*

Lemma 1.4 (Supersaturation)**.** *For integers $n > m \geqslant d$, let F be an m-vertex graph with $\delta(F) \geqslant d$ and $H = F^2$. Then for every $J \subseteq V(H)$ of size at least $4n/d$, we have*

$$e(H[J]) \geqslant \frac{d^2|J|^2}{4n}.$$

Furthermore, if F is C_4-free, then we have

$$d(H[J]) \geqslant \frac{d^2|J|}{2n}.$$

Proof. Write $V(F) = \{v_1, \ldots, v_m\}$. For every $j \in [m]$, let $b_j = d_F(v_j, J)$. Then we have $\sum_{j=1}^m b_j = \sum_{v \in J} d_F(v) \geqslant |J|d \geqslant 4n > 4m$. Therefore, we obtain that

$$e(H[J]) = \sum_{j=1}^{m} \binom{b_j}{2} \geqslant m\binom{\frac{\sum b_j}{m}}{2} \geqslant m\binom{\frac{|J|d}{m}}{2} \geqslant \frac{d^2|J|^2}{4m} \geqslant \frac{d^2|J|^2}{4n}.$$

The second part follows by the observation that F^2 is a simple graph when F is C_4-free. □

For every vertex v_i, we run the graph container method on the graph G_{i-1}^2 and count its independent sets of size d_i. For a given independent set I, the container algorithm outputs a certificate $\{T, C\}$ such that

- $I \subseteq T \cup C$;
- $|T| \leqslant \log^2 n$;
- $|C| \leqslant \frac{4n}{d_i}$.

Therefore, the number of independent sets of size d_i in G_{i-1}^2 is at most

$$\binom{n}{\leqslant \log^2 n}\binom{4n/d_i}{d_i} = 2^{O(\sqrt{n})}.$$

Sketch of the proof of Theorem 1.1. A graph can be uniquely determined by its min-degree ordering and their neighborhoods. To estimate the number of C_4-free graphs, we first fix a min-degree ordering $v_n < v_{n-1} < \ldots < v_1$ and their degrees d_i. The number of ways to choose them are $n!$ and n^n respectively, which are small. From the above discussion, for each v_i, the number of ways to choose its neighborhood $N_{G_i}(v_i)$ is $2^{O(\sqrt{n})}$. Hence, we obtain that the number of C_4-free graphs is at most

$$n!n^n(2^{O(\sqrt{n})})^n = 2^{O(n\log n)}2^{O(n^{3/2})} = 2^{O(n^{3/2})}.$$

References

[1] D. J. Kleitman and K. J. Winston, *On the number of graphs without 4-cycles*, Discrete Mathematics, 41 (1982), pp. 167–172.