1 Lecture 17: Applications (2/25/19)

1.1 Folkman Numbers (Rödl, Ruciński, Schacht 2017)

Given graphs G and F and integer $r \geqslant 2$, we write $G \to (F)_r$ if every r-edge-coloring of G admits a monochromatic copy of F. Let $R(k;r)$ denote the *r-color Ramsey number* of K_k, which is the minimum N such that $K_N \to (K_k)_r$. It is known that $3^{rk/6} \leqslant R(k;r) \leqslant r^{rk}$.

Question 1.1 (Erdős–Hajnal [1]). *Is there a graph G such that $G \to (K_k)_r$ but G does not contain a copy of K_{k+1}?*

Folkman [2] proved the existence of such a graph for all $k \geqslant 3$ in the 2-color case, and Nešetřil-Rödl [4] later extended the result to all r.

For integers k and r, we say a graph G is $(k;r)$-*Folkman* if $G \to (K_k)_r$ but $G \not\supseteq K_{k+1}$. The goal is to find, for a given k and r, the minimum number of vertices in such a graph. The *r-color Folkman number* for K_k is

$$f(k,r) = \min\{n \in \mathbb{N} : \exists G \text{ such that } |V(G)| = n \text{ and } G \text{ is } (k;r) - \text{Folkman}\}.$$

Theorem 1.2 (Rödl–Ruciński–Schacht [5]). *For all integers $r \geqslant 2$ and $k \geqslant 3$,*

$$f(k;r) \leqslant k^{400k^4} R(k;r)^{40k^2} \leqslant 2^{c(k^4 \log k + k^3 r \log r)}$$

for some $c > 0$ independent of r and k.

Proof. Throughout the proof, we let $R := R(k;r)$.

Consider $G(n,p)$ with $p = Cn^{-2/(k+1)}$, where $C = C(n)$. We have two goals:

1. Estimate $\mathbb{P}(G(n,p) \to (K_k)_r)$ and $\mathbb{P}(G(n,p) \not\supseteq K_{k+1})$.
2. Show that the sum of these two probabilities is greater than 1 so we can conclude that there exists a $(k;r)$-Folkman graph on n vertices.

We start by giving a lower bound on $\mathbb{P}(G(n,p) \not\supseteq K_{k+1})$. Note that if subsets A and B of size $k+1$ overlap, then one of A and B inducing a copy of K_{k+1} only makes the other more likely to induce a copy of K_{k+1} as well. Therefore, since these events are positively correlated, the FKG Inequality implies the following lower bound:

$$\mathbb{P}(G(n,p) \not\supseteq K_{k+1}) \geqslant (1 - p^{\binom{k+1}{2}})^{\binom{n}{k+1}} > \exp(-C^{\binom{k+1}{2}}).$$

Lemma 1.3. *For all integers $r \geqslant 2$, $k \geqslant 3$, and $n \geqslant k^{400k^4} R^{40k^2}$, the following holds. Set $b = \frac{1}{2R^2}$, $C = 2^{5\sqrt{\log n \log k}} R^{16}$, and $p = Cn^{-2/(k+1)}$. Then*

$$\mathbb{P}(G(n,p) \to (K_k)_r) > 1 - \exp\left(-bp\binom{n}{2}\right).$$

Proof sketch. Consider the hypergraph $\mathcal{H}$ whose vertices are the edges of K_n and whose edges are the edge sets of each copy of K_k in $G(n,p)$.

We need a supersaturation result: *Set $\alpha\binom{R}{k}^{-1}$, $n \geqslant R$. Then for every $(r+1)$-edge-coloring of $E(K_n)$, either there are more than $\frac{\alpha}{2}\binom{n}{k}$ monochromatic copies of K_k in the first r colors or there are at least $\binom{n}{k}/R^2$ edges in color $r+1$.*

To prove this supersaturation result, note that for each R-set, either there is an edge of color $r+1$ or there is a monochromatic K_k in the first r-colors (by definition of $R = R(k;r)$). Taking an average and using a double-counting argument give the result.

Now assume that $E(G(n,p))$ is r-colored with no monochromatic K_k. Say the color classes of this edge-coloring are $G_1, \ldots, G_r$. By the Container Lemma, there are containers $C_1 \supseteq G_1, \ldots, C_r \supseteq G_r$, and we can assign an $(r+1)$-edge-coloring of K_n as follows:

- If $e \in C_i$, then assign e color i. (If there are multiple choices for i, then pick its color arbitrarily.)
- If $f \notin \bigcup_{i=1}^{r} C_i$, assign f color $r+1$.

Since each container C_i contains few copies of K_k, the supersaturation result implies that color class $r+1$ must be big (i.e., must contain at least $\binom{n}{2}\frac{1}{R^2}$ edges of color $r+1$). But our construction guaranteed that there are no edges which are both in color class $r+1$ and in $E(G(n,p))$. Therefore, we have

$$\mathbb{P}(G(n,p) \to (K_k)_r) > (\#\text{containers})^r \cdot (1-p)^{\binom{n}{2}(\frac{1}{R^2})}.$$

So, any r-coloring $E(G(n,p))$ yields a container vector, and the complement of this vector (the $r+1$ color class) should be disjoint from $E(G(n,p))$, which is very unlikely using Chernoff. □

□

1.2 Random Ramsey Theory (Nenadov–Steger 2016)

Fix integers r and k and let $n \to \infty$. We want to find a monochromatic copy of K_k in every r-edge-coloring of $G(n,p)$. This is possible if p is small (~ 0) but not if p is big (~ 1). The probability that such a coloring exists is monotonically increasing in p, so there is a threshold where this change occurs.

Theorem 1.4. *Let $t = t(n, r, k) = n^{-2/(k+1)}$.*

1. *If $p \ll t$, then w.h.p. such a coloring exists.*

2. *If $p \gg t$, then w.h.p. such a color does not exist.*

Since $p = n^{-1/m_2(H)}$ is the density at which we expect every edge to be contained in roughly a constant number of copies of H, the threshold function can be intuitively understood. If $p \leqslant cn^{-1/m_2(H)}$ for small c, then the number of such copies is w.h.p. small enough that the copies of H are scattered too far throughout a coloring for a monochromatic copy of H to be found. Meanwhile, if $p \geqslant Cn^{-1/m_2(H)}$ for large C, then the many copies of H w.h.p. overlap so much that each coloring such induce at least one monochromatic copy of H.

Proof sketch of upper bound. For a graph H, let $m_2(H) := \dfrac{e(H) - 1}{v(H) - 2}$. Then $m_2(K_k) = \frac{k+1}{2}$, so we want to show that $t = n^{-1/m_2}$.

First, we consider the expected number of copies of K_k which contain the edge uv. This expectation is $\binom{n}{k-2} p^{\binom{k}{2}-1} \sim n^{k-2} p^{(k^2-k+2)/2}$. So, if we substitute $p = Cn^{-2/(k+1)}$, then we get some function of C: $(np^{(k+1)/2})^{k-2} \sim C^{k-2}$.

Next, we need a supersaturation statement. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that for n sufficiently large, the following holds. Assume there is a partial r-coloring of $E(K_n)$, say with color classes $E_1, \ldots, E_r$, such that each color class contains less than εn^k monochromatic copies of K_k. Then there are more than δn^2 uncolored edges.*

Consider an r-edge-coloring of $G(n, p)$. Say the color classes are $G_1, \ldots, G_r$. By the Container Lemma, there are containers $C_1, \ldots, C_r$ such that $C_i \supseteq G_i$ for each i. Then each container C_i must contain few monochromatic copies of K_k, so $|E(K_n) - \bigcup E(C_i)| > \delta n^2$ by the supersaturation result. These edges are not in $G(n, p)$, and we can show that $(1 - p)^{\delta n^2} \cdot (\#\text{containers})^r = o(1)$, which implies that such a coloring does not exist. □

References

[1] P. Erdős and A. Hajnal, *Research problems 2-5*, J. Combinatorial Theory 2 (1967), 104–105.

[2] J. Folkman, *Graphs with monochromatic complete subgraphs in every edge coloring*, SIAM J. Appl. Math. 18 (1970), 19–24.

[3] R. Nenadov and A. Steger, *A short proof of the random Ramsey theorem*, Comb. Prob. Comp. 25 (2016), 130–144.

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[5] V. Rödl, A. Ruciński, and M. Schacht, *An exponential-type upper bound for Folkman numbers*, Combinatorica 37(4) (2017) 767–784.