1 The number of maximal sum-free subsets of integers

A fundamental notion in combinatorial number theory is that of a sum-free set: A set S of integers is *sum-free* if $x + y \notin S$ for every $x, y \in S$ (note x and y are not necessarily distinct here).

Note that both the set of odd numbers in $[n]$ and the set $\{\lfloor n/2 \rfloor + 1, \ldots, n\}$ are maximal sum-free sets. (A sum-free subset of $[n]$ is *maximal* if it is not properly contained in another sum-free subset of $[n]$.) By considering all possible subsets of one of these maximal sum-free sets, we see that $[n]$ contains at least $2^{\lceil n/2 \rceil}$ sum-free sets. Cameron and Erdős [3] conjectured that in fact $[n]$ contains only $O(2^{n/2})$ sum-free sets. The conjecture was proven independently by Green [6] and Sapozhenko [10]. Recently, a refinement of the Cameron–Erdős conjecture was proven in [1], giving an upper bound on the number of sum-free sets in $[n]$ of size m (for each $1 \leq m \leq \lceil n/2 \rceil$).

Let $f(n)$ denote the number of sum-free subsets of $[n]$ and $f_{\max}(n)$ denote the number of maximal sum-free subsets of $[n]$. Recall that the sum-free subsets of $[n]$ described above lie in just two maximal sum-free sets. This led Cameron and Erdős [4] to ask whether the number of maximal sum-free subsets of $[n]$ is "substantially smaller" than the total number of sum-free sets. In particular, they asked whether $f_{\max}(n) = o(f(n))$ or even $f_{\max}(n) \leq f(n)/2^{\varepsilon n}$ for some constant $\varepsilon > 0$. Łuczak and Schoen [8] answered this question, showing that $f_{\max}(n) \leq 2^{n/2 - 2^{-28} n}$ for sufficiently large n. More recently, Wolfovitz [11] proved that $f_{\max}(n) \leq 2^{3n/8 + o(n)}$.

In the other direction, Cameron and Erdős [4] observed that $f_{\max}(n) \geq 2^{\lfloor n/4 \rfloor}$. Indeed, let $m = n$ or $m = n - 1$, whichever is even. Let S consist of m together with precisely one number from each pair $\{x, m - x\}$ for odd $x < m/2$. Then S is sum-free. Moreover, although S may not be maximal, no further odd numbers less than m can be added, so distinct S lie in distinct maximal sum-free subsets of $[n]$.

Balogh, Liu, Sharifzadeh, and Treglown proved that this lower bound is in fact, 'asymptotically', the correct bound on $f_{\max}(n)$. The proof will be given in Section 3.

Theorem 1.1 ([12]). *There are at most $2^{(1/4+o(1))n}$ maximal sum-free sets in $[n]$. That is,*

$$f_{\max}(n) = 2^{(1/4+o(1))n}.$$

The proof of Theorem 1.1 makes use of 'container' and 'removal' lemmas of Green [6, 7] as well as a result of Deshouillers, Freiman, Sós and Temkin [5] on the structure of sum-free sets.

The following theorem was also proved by Balogh, Liu, Sharifzadeh, and Treglown.

Theorem 1.2 ([2]). *For each $1 \leq i \leq 4$, there is a constant C_i such that, given any $n \equiv i$ mod 4, $[n]$ contains $(C_i + o(1))2^{n/4}$ maximal sum-free sets.*

Notation: Given a set $A \subseteq [n]$, denote by $f_{\max}(A)$ the number of maximal sum-free subsets of $[n]$ that lie in A and by $\min(A)$ the minimum element of A. Let $1 \leq p < q \leq n$ be integers, denote $[p, q] := \{p, p+1, \ldots, q\}$. Denote by E the set of all even numbers in $[n]$ and by O the set of all odd numbers in $[n]$. A triple $x, y, z \in [n]$ is called a *Schur triple* if $x + y = z$ (here $x = y$ is allowed).

Throughout, all graphs considered are simple unless stated otherwise. We say that a graph G is a *graph possibly with loops* if G can be obtained from a simple graph by adding at most one loop at each vertex. Given a vertex x in G, we write $\deg_G(x)$ for the *degree of x in G*. Note that a loop at x contributes two to the degree of x. We write $\delta(G)$ for the *minimum degree of* G and $\Delta(G)$ for the *maximum degree of* G. Given a graph G, denote by $\mathrm{MIS}(G)$ the number of maximal independent sets in G. Given $T \subseteq V(G)$, denote by $\Gamma(T)$ the external neighbourhood of T, i.e. $\Gamma(T) := \{v \in V(G) \setminus T : \exists u \in T, uv \in E(G)\}$. Denote by $G[T]$ the induced subgraph of G on the vertex set T and let $G \setminus T$ denote the induced subgraph of G on the vertex set $V(G) \setminus T$. Denote by $E(T)$ the set of edges in G spanned by T and by $E(T, V(G) \setminus T)$ the set of edges in G with exactly one vertex in T.

2 Preliminary results

We prove Theorem 1.1 in Section 3. A key tool in the proof is the following container lemma of Green [6] for sum-free sets.

Lemma 2.1 ([6]). *There exists a family $\mathcal{F}$ of subsets of $[n]$ with the following properties.*

(i) Every member of $\mathcal{F}$ has at most $o(n^2)$ Schur triples.

(ii) If $S \subseteq [n]$ is sum-free, then S is contained in some member of $\mathcal{F}$.

(iii) $|\mathcal{F}| = 2^{o(n)}$.

(iv) Every member of $\mathcal{F}$ has size at most $(1/2 + o(1))n$.

We refer to the elements of $\mathcal{F}$ from Lemma 2.1 as *containers*.

Note that conditions (ii) and (iii) in Lemma 2.1 imply that, to prove Theorem 1.1, it suffices to show that every member of $\mathcal{F}$ contains at most $2^{n/4+o(n)}$ maximal sum-free subsets of $[n]$. For this purpose, we need to get a handle on the structure of the containers. The following theorem of Deshouillers, Freiman, Sós and Temkin [5] provides a structural characterisation of the sum-free sets in $[n]$.

Theorem 2.2 (Stability Theorem [5]). *Every sum-free set S in $[n]$ satisfies at least one of the following conditions:*

(i) $|S| \leq 2n/5 + 1$;

(ii) S consists of odd numbers;

(iii) $|S| \leq \min(S)$.

We also need the following removal lemma of Green [7] for sum-free sets.

Lemma 2.3 (Removal Lemma [7]). *Suppose that $A \subseteq [n]$ is a set containing $o(n^2)$ Schur triples. Then, there exist B and C such that $A = B \cup C$ where B is sum-free and $|C| = o(n)$.*

Moon and Moser [9] showed that for any graph G, $\mathrm{MIS}(G) \leq 3^{|G|/3}$. We will need a looped version of this statement. Since any vertex with a loop cannot be in an independent set, the following statement is an immediate consequence of Moon and Moser's result.

Proposition 2.4. *Let G be a graph possibly with loops. Then*

$$\mathrm{MIS}(G) \leq 3^{|G|/3}.$$

When a graph has "few" triangles, the bound in Proposition 2.4 can be improved.

Lemma 2.5. *Let G be a graph possibly with loops. If there exists a set T such that $G \setminus T$ is triangle-free, then*

$$\mathrm{MIS}(G) \leq 2^{|G|/2+|T|/2}.$$

The following lemma gives an improvement on Proposition 2.4 for graphs that are 'not too sparse and almost regular'.

Lemma 2.6. *Let $k \geq 1$ and let G be a graph on n vertices possibly with loops. Suppose that $\Delta(G) \leq k\delta(G)$ where $\delta(G) \geq f(n)$ for some real valued function f with $f(n) \to \infty$ as $n \to \infty$. Then*

$$\mathrm{MIS}(G) \leq 3^{\left(\frac{k}{k+1}\right)\frac{n}{3}+o(n)}.$$

Proof. Fix a maximal independent set I in G and set $b := \delta(G)^{1/2}$. We will repeat the following process as many times as possible. Let $V_1 := V(G)$. At the i-th step, for $i \geq 1$, choose $v_i \in V_i \cap I$ such that $\deg_{G[V_i]}(v_i) \geq b$ and set $V_{i+1} := V_i \setminus (\{v_i\} \cup \Gamma(v_i))$. This process is repeated $j \leq n/b$ times. Let $U := V_{j+1}$ be the resulting set. Define $Z := \{v \in U : \deg_{G[U]}(v) < b\}$. Notice that $\deg_{G[U]}(v) < b$ for all $v \in I \cap U$, hence $I \cap U \subseteq Z$. We have

$$\delta(G) \cdot |Z| \leq \sum_{v \in Z} \deg(v) = 2|E(Z)| + |E(Z, V \setminus Z)| \leq b|Z| + \Delta(G) \cdot (n - |Z|).$$

Hence,

$$|Z| \leq \frac{\Delta(G) \cdot n}{\delta(G) + \Delta(G) - b} \leq \frac{k}{k+1} n + \frac{2n}{b}. \tag{1}$$

By construction of U, no vertex in $I \setminus U$ has a neighbour in U. So as $Z \subseteq U$, no vertex in Z is adjacent to $I \setminus U$. Together with the fact that I is maximal, this implies that $I \cap U$ is a maximal independent set in $G[Z]$. By the above process, every maximal independent set I in G is determined by a set $I \setminus U$ of at most n/b vertices and a maximal independent set in $G[Z]$. Note that $n/b = o(n)$. Thus, Proposition 2.4 and (1) imply that

$$\text{MIS}(G) \leq \sum_{0 \leq i \leq n/b} \binom{n}{i} 3^{\left(\frac{k}{k+1}\right)\frac{n}{3} + \frac{2n}{3b}} \leq 3^{\left(\frac{k}{k+1}\right)\frac{n}{3} + o(n)}. \tag{2}$$

□

3 Proof of Theorem 1.1

Let $\mathcal{F}$ be the family of containers obtained from Lemma 2.1. Recall that given a set $A \subseteq [n]$, $f_{\max}(A)$ denotes the number of maximal sum-free subsets of $[n]$ that lie in A. Since every sum-free subset of $[n]$ is contained in some member of $\mathcal{F}$ and $|\mathcal{F}| = 2^{o(n)}$, it suffices to show that $f_{\max}(A) \leq 2^{(1/4+o(1))n}$ for every container $A \in \mathcal{F}$.

Lemmas 2.1, 2.2 and 2.3 imply that for every container $A \in \mathcal{F}$, $A = B \cup C$, where B is sum-free, $|C| = o(n)$, and B satisfies at least one of the following conditions:

(a) $|B| \leq \frac{2}{5}n + 1 \leq 0.45n$

(b) B consists of odd numbers;

(c) $|B| \leq \min(B)$.

We deal with each of the three cases separately.

For any subsets $B, S \subseteq [n]$, let $L_S[B]$ be the *link graph of S on B* defined as follows. The vertex set of $L_S[B]$ is B. The edge set of $L_S[B]$ consists of the following two types of edges:

(i) Two vertices x and y are adjacent if there exists an element $z \in S$ such that $\{x, y, z\}$ forms a Schur triple;

(ii) There is a loop at a vertex x if $\{x, x, z\}$ forms a Schur triple for some $z \in S$ or if $\{x, z, z'\}$ forms a Schur triple for some $z, z' \in S$.

The following simple result will be applied in all three cases of our proof.

Lemma 3.1. *Suppose that B, S are both sum-free subsets of $[n]$. If $I \subseteq B$ is such that $S \cup I$ is a maximal sum-free subset of $[n]$, then I is a maximal independent set in $G := L_S[B]$.*

Proof. First notice that I is an independent set in G, since otherwise $S \cup I$ is not sum-free. Suppose to the contrary that there exists a vertex $v \notin I$ such that $I' := I \cup \{v\}$ is still an independent set in G. Then since $I' \subseteq B$ is sum-free, the definition of G implies that $S \cup I'$ is a sum-free set in $[n]$ containing $S \cup I$, a contradiction to the maximality of $S \cup I$. □

3.1 Small B

The following lemma deals with containers such that $|B| \leq .45n$.

Lemma 3.2. *If $A \in \mathcal{F}$ and $A = B \cup C$, where B is sum-free, $|C| = o(n)$, and $|B| \leq .45n$, then $f_{\max}(A) = o(2^{n/4})$.*

Proof. Notice crucially that every maximal sum-free subset of $[n]$ in A can be built in the following two steps:

(1) Choose a sum-free set S in C;

(2) Extend S in B to a maximal one.

(Note that it is not necessarily the case that given an arbitrary sum-free set $S \subseteq C$, there exists a set $R \subseteq B$ such that $R \cup S$ is a maximal sum-free set in $[n]$.)

The number of choices for S is at most $2^{|C|} = 2^{o(n)}$. For a fixed S, denote by $N(S, B)$ the number of extensions of S in B in Step (2). It suffices to show that for any given sum-free set $S \subseteq C$, $N(S, B) \leq 2^{0.249n}$. Let $G := L_S[B]$ be the link graph of S on B. Since $|B| \leq 0.45n$ and S and B are sum-free, Lemma 3.1 and Proposition 2.4 imply that

$$N(S, B) \leq \mathrm{MIS}(G) \leq 3^{|B|/3} \leq 3^{0.45n/3} \ll 2^{0.249n}.$$

□

3.2 Large B

We now turn our attention to containers such that $|B| > .45n$.

Lemma 3.3. *If $A \in \mathcal{F}$ and $A = B \cup C$, where B is sum-free, $|C| = o(n)$, $|B| > .45n$, and $|B| \leq \min(B)$, then $f_{\max}(A) \leq 2^{(1/4+o(1))n}$.*

Proof. Let A, B, and C be as in the statement of the lemma. Note $|B| > .45n$, so $|A| > .45n$. Lemma 2.1 implies $|A| \leq (1/2 + o(1))n$. Combining this with $|A| > .45n$, $|A| = (1/2 - \gamma + o(1))n$ for some $\gamma \leq 1/11$. Thus, $\min(B) \geq |B| \geq (1/2 - \gamma + o(1))n$ and $B \subseteq [(1/2 - \gamma + o(1))n, n]$. Therefore, all but $o(n)$ elements of A are contained in $[(1/2 - \gamma)n, n]$.

Let $A_1 := A \cap [\lfloor n/2 \rfloor]$ and $A_2 := A \setminus A_1$. Since $|A \cap [(1/2 - \gamma)n]| = o(n)$, we have that $|A_1| \leq (\gamma + o(1))n$. Every maximal sum-free subset of $[n]$ in A can be built from choosing a sum-free set $S \subseteq A_1$ and extending S in A_2. The number of choices for S is at most $2^{|A_1|}$.

Let $G := L_S[A_2]$ be the link graph of S on vertex set A_2. Since S and A_2 are sum-free, Lemma 3.1 implies that $N(S, A_2) \leq \text{MIS}(G)$. Notice that G is triangle-free. Indeed, suppose to the contrary that $z > y > x > n/2$ form a triangle in G. Then there exists $a, b, c \in S$ such that $z - y = a, y - x = b$ and $z - x = c$, which implies $a + b = c$ with $a, b, c \in S$. This is a contradiction to S being sum-free. Thus by Lemma 2.5 we have $N(S, A_2) \leq \text{MIS}(G) \leq 2^{|A_2|/2}$. Then we have

$$f_{\max}(A) \leq 2^{|A_1|+|A_2|/2} = 2^{|A_1|+((1/2-\gamma+o(n))n-|A_1|)/2} = 2^{n/4+(|A_1|-\gamma n)/2+o(n)} \leq 2^{n/4+o(n)},$$

where the last inequality follows since $|A_1| \leq (\gamma + o(1))n$. □

Lemma 3.4. *If $A \in \mathcal{F}$ and $A = B \cup C$ where B is sum-free, $|C| = o(n)$, $|B| > .45n$, and $|B|$ consists of odd numbers, then $f_{\max}(A) \leq 2^{(1/4+o(1))n}$.*

Proof. Let A, B, and C be as in the statement of the lemma. Since B consists of odd numbers, $|A \setminus O| = o(n)$. Notice that if $S \subseteq T \subseteq [n]$ then $f_{\max}(S) \leq f_{\max}(T)$. Using this fact, we may assume that $A = O \cup C'$ with $C' \subseteq E$ and $|C'| = o(n)$. Similarly to before, every maximal sum-free subset of $[n]$ in A can be built from choosing a sum-free set $S \subseteq C'$ (at most $2^{|C'|} = 2^{o(n)}$ choices) and extending S in O to a maximal one. Fix an arbitrary sum-free set S in C' and let $G := L_S[O]$ be the link graph of S on vertex set O. Since O is sum-free, by Lemma 3.1 we have that $N(S, O) \leq \text{MIS}(G)$. It suffices to show that $\text{MIS}(G) \leq 2^{n/4+o(n)}$. We will achieve this in two cases depending on the size of S.
Case 1: $|S| \geq n^{1/4}$.

In this case, we will show that G is 'not too sparse and almost regular'. Then we apply Lemma 2.6.

We first show that $\delta(G) \geq |S|/2$ and $\Delta(G) \leq 2|S|+2$, thus $\Delta(G) \leq 6\delta(G)$. Let x be any vertex in O. If $s \in S$ such that $s < \max\{x, n-x\}$ then at least one of $x-s$ and $x+s$ is adjacent to x in G. If $s \in S$ such that $s \geq \max\{x, n-x\}$ then $s-x$ is adjacent to x in G. By considering all $s \in S$ this implies that $\deg_G(x) \geq |S|/2$ (we divide by 2 here as an edge xy may arise from two different elements of S). For the upper bound consider $x \in O$. If $xy \in E(G)$ then $y = x+s$, $x-s$ or $s-x$ for some $s \in S$ and only two of these terms are positive. Further, there may be a loop at x in G (contributing 2 to the degree of x in G). Thus, $\deg_G(x) \leq 2|S|+2$, as desired.

Since $\delta(G) \geq |S|/2 \geq n^{1/4}/2$ we can apply Lemma 2.6 to G with $k=6$. Hence,

$$\mathrm{MIS}(G) \leq 3^{\left(\frac{6}{7}\right)\frac{n/2}{3}+o(n)} \ll 2^{0.24n+o(n)} = o(2^{n/4}).$$

Case 2: $|S| \leq n^{1/4}$.

In this case, it suffices to show that G has very few, $o(n)$, triangles, since then by applying Lemma 2.5 with T being the vertex set of all triangles in G, we have $|T| = o(n)$ and then $\mathrm{MIS}(G) \leq 2^{n/4+o(n)}$. The proof that G has $o(n)$ triangles is not given due to the technicality of the proof. □

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