1 Lecture 1 (1/14/2019)

In these lecture notes we describe a recently-developed technique - the hypergraph container method - a method which can be used to bound the number of finite objects with forbidden substructures, see for details [3], [4], [10]. It has applications in various fields such as extremal graph theory, Ramsey theory, additive combinatorics, number theory and discrete geometry. The hypergraph container method was discovered 2015 by Balogh, Morris and Samotij and independently by Saxton and Thomason. However, several theorems of the same flavour appeared earlier in the literature. The earliest container-type argument was used 1980 by Kleitman and Winston for bounding the number of lattices [8] and of C_4-free graphs [9]. We will keep using the following notation throughout the notes.

Let $\mathcal{H}_N$ be a k-uniform hypergraph with vertex set $[N] = \{1, \ldots, N\}$, $k \ll N$ (usually k is a fixed constant, N is big). We say $I \subseteq [N]$ is an independent set if $\mathcal{H}[I] = \emptyset$.

The general concept behind the container method is that the family of independent sets have some structure. For example, in $K_{n,n}$, the independent sets are the subsets of two sets, the two classes of the bipartitiion; in $\bar{K}_n$, every subset the entire vertex set is an independent set.

Omitting formulaes, the hypergraph container theorem says the following.

Theorem 1.1 (**Container Theorem - vague version**). *Given a hypergraph $\mathcal{H}_N$, there exists a family of containers $C_1, \ldots, C_t \subseteq [N]$ s.t.*

(i) $\forall I \subseteq [N]$ an independent set, $\exists C_i$ s.t. $I \subseteq C_i$;

(ii) t is "small";

(iii) $e(\mathcal{H}[C_i])$ is "small".

One should think about the sets C_i's as containers. Then, condition (i) says that every independent set is covered by one of the containers. Condition (ii) says that not too many containers are needed. This restriction is important as otherwise we could just declare every independent set to be a container; condition (iii) is needed so that we do not just take

$C_1 = [N]$. In some sense, conditions (ii) and (iii) are at odds with each other; if one wants to get a better bound in (ii), one will get a worse bound in (iii) and vice versa.

An application of Theorem 1.1 is giving upper bounds on the number of independent sets. We have that the number of independent sets is bounded above by $\sum_i 2^{|C_i|} \leqslant t \cdot 2^{\max |C_i|}$. If we can describe C_i, we can get more information about the structure of the independent sets.[1] In many cases, the "typical" structure of independent sets is given thusly: if $|C_i|$ is small, then $2^{|C_i|}$ is small, and if $|C_i|$ is big, then there exists some structure.

Hypergraphs can model many combinatorial structures, which is why the container method has a variety of applications. Now, we will take a look at two standard examples; one from number theory and one from extremal graph theory.

1.1 Example 1: arithmetic progressions

Let $V(\mathcal{H}_k) = [N]$ and the edge set $E(\mathcal{H})$ be the set of k-term arithmetic progressions, where k is fixed and $N \to \infty$. Thus, an independent set is a subset of integers containing no k-A.P..

In a breakthrough paper Szemerédi [12] proved that $\alpha(\mathcal{H}_k) = o(N)$. Cameron and Erdős [5] asked what the number of k-A.P.-free subsets of $[N]$ is. The trivial answer is $2^{o(N)}$ by Szemerédi's result; the real question is if it is $2^{(1+o(1))\alpha(\mathcal{H}_k)}$. We know

$$2^{\alpha(\mathcal{H}_k)} \ll \#\text{of } k\text{-A.P.-free subsets of } [N] \leqslant t \cdot 2^{\max |C_i|},$$

where t is "small" and $\max |C_i| = O(\alpha(\mathcal{H}_k))$ for infinitely many N, see below. The lower bound is from [5], and the upper bound is derived from Theorem 1.1. If the number of k-A.P.'s in C_i is small, it implies that $|C_i|$ is small. In particular, the implication

$$\#\text{ of } k\text{-A.P.'s in } C_i \leqslant N \log^{100} N \quad \Longrightarrow \quad |C_i| = (1 + o(1))\alpha(\mathcal{H}) \tag{1}$$

would solve the conjecture. However, only $|C_i| = O(\alpha(\mathcal{H}))$ is known only for infinitely many n [2]. It would be nice to prove (1), an even harder *fun* problem is below.

Open Problem 1.2. *Is it true that if $|C_i| = \alpha(\mathcal{H})+1$ then C_i contains at least two k-A.P.'s?*

1.2 Example 2: K_r-free graphs

A classical result in graph theory is Turán's Theorem on the extremal number of K_r-free graphs.

[1] The phrase "structure of independent sets" will be clear after looking some of the examples.

Theorem 1.3 (**Turán's Theorem** [13])**.** *Let G_n be a K_r-free graph on n vertices. Then,*

$$e(G_n) \leqslant \left(1 - \frac{1}{r-1}\right)\binom{n}{2} + O(n).$$

The complete $(r-1)$-partite graph achieves the upper bound. Theorem 1.3 is an extremal result; the supersaturation version of it is due to Erdős and Simonovits.

Theorem 1.4 (**Supersaturation** [7])**.** *Let $r \in \mathbb{N}$. $\forall \delta > 0, \exists \beta = \beta(\delta, r) > 0$ s.t. for all n sufficiently big and graphs G_n with*

$$e(G_n) > (1 - \frac{1}{r-1} + \delta)\binom{n}{2}$$

the following holds:

$$k_r(G_n) \geqslant \beta \cdot \binom{n}{r},$$

where $k_r(G_n)$ is the number of copies of K_r's in G_n.

Theorem 1.4 can be proved using the regularity lemma and Turán's theorem applied to the cluster graph; the original proof was based on a simple averaging argument.

One of the best-known extensions of Turán's theorem is it's stability version by Erdős and Simonovits. It says, in particular, that a K_r-free graph with slightly less edges than the extremal graph can be made $(r-1)$-partite by removing not too many edges.

Theorem 1.5 (**Erdős-Simonovits Stability Theorem** [6] [11])**.** *(i) $\forall \gamma, \exists \delta$ such that if*

$$e(G_n) > (1 - \frac{1}{r-1} - \delta)\binom{n}{2} \quad \text{and} \quad G_n \not\supseteq K_r,$$

then one can remove γn^2 edges from G_n to obtain an $(r-1)$-partite graph.

(ii) This can be strengthened to a version where we replace the condition $G_n \not\supseteq K_r$ with the condition $k_r(G_n) = o(n^r)$.

Next, we will use the Container Theorem (Theorem 1.1) combined with the supersaturation result (Theorem 1.4) to count the number of K_r-free graphs.

Theorem 1.6. *The number of K_r-free graphs on n vertices is $2^{(1-\frac{1}{r-1}+o(1))\binom{n}{2}}$.*

Proof. Define $\mathcal{H}_N^r$ to be a k-uniform hypergraph with $k = \binom{r}{2}$, $N = |E(K_n)|$ (it follows that $|N| = \binom{n}{2}$). We place a hyperedge on a set of edges when they form a K_r; then $e(\mathcal{H}_N^r) = \binom{n}{r}$. An independent set in $\mathcal{H}_N^r$ is the edge set of a K_r-free graph, so counting independent sets in $\mathcal{H}_N^r$ is the same as counting K_r-free graphs. For containers $C_1, \ldots, C_t$ (which are vertex sets of $\mathcal{H}_N^r$ and thus graphs on $[n]$), each K_r-free graph is a subgraph of some C_i. Then

$$2^{(1-\frac{1}{r-1})\binom{n}{2}} \leqslant \#\text{ independent sets} = \#K_r\text{-free graphs} \leqslant t \cdot 2^{\max|C_i|},$$

where $t = 2^{o(n^2)}$. By Theorem 1.1 (iii), the number of K_r's in C_i is $o(\binom{n}{r})$, and by the contrapositive of supersaturation, it follows that $|C_i| \leqslant (1 - \frac{1}{r-1} + o(1))\binom{n}{2}$. Then, the number of independent sets is at most $2^{(1-\frac{1}{r-1}+o(1))\binom{n}{2}}$. □

The container method can also be used to prove the following theorem [1].

Theorem 1.7. *Let $r = r(n) \in \mathbb{N}_0$ be a function satisfying $r \leqslant (\log n)^{1/4}$ for every $n \in \mathbb{N}$. Then almost all K_{r+1}-free graphs on n vertices are r-partite.*

Theorem 1.8 is an approximated version of Theorem 1.7. It was proven in [1] and is a straightforward application of the Container theorem. An additional cleaning argument gives the proof for Theorem 1.7.

Theorem 1.8. *Let $r = r(n) \in \mathbb{N}_0$ be a function satisfying $r \leqslant (\log n)^{1/4}$ for every $n \in \mathbb{N}$. Then almost all K_{r+1}-free graphs on n vertices are n^{1-1/r^2}-close to being r-partite.*

The proof of Theorem 1.8 relies on the stability method: Theorem 1.5. A container C_i is either γ-close to be $(r-1)$-partite, or γ-far which implies that the size of the container is small ($|C_i| < (1 - \frac{1}{r-1} - \delta))\binom{n}{2}$). This allows us to upper bound the number of γ-far graphs by the usual counting argument.

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