Geodesics on Smooth Surfaces

Suppose M is a smooth surface in space. It could be the Euclidean plane $\mathbb{E}^2$, the sphere $\mathbb{S}^2$, a cylinder, or any other surface. We want to discuss some general properties of lines on M.

A curve on M which never turns left or right within the surface is called a geodesic (or line). We can make this precise in terms of a differential equation as follows.

Definition. Consider a curve $\gamma : \mathbb{R} \to M$ with constant speed $|\gamma'(t)| = c$. It is a *geodesic* if, at each point, the acceleration $\gamma''(t)$ is normal to the surface M.

The straightness of a ribbon means that a curve is a geodesic exactly if you can lay a ribbon flat along it.

Just as in the plane, a curve which is not straight could be shortened, by "cutting a corner". This argument can be made into a proof of the following.

Theorem. *If γ is the shortest path (in M) between its endpoints, then γ is a geodesic.*

It follows that a stretched rubber band or string in M must lie along a geodesic. A rubber band from x to y need not follow the globally shortest path between these points. For instance, it could wrap more than halfway around a cylinder. But its path will be shorter than any nearby paths, and thus is still a geodesic.

The converse of the statement "If A then B" is "If B then A". The converse of the theorem above would thus be: If γ is a geodesic, then γ is the shortest bath between its endpoints.

This statement, however, is not true in general. For instance, a path going more than halfway around a great circle on $\mathbb{S}^2$ fails to minimize distance. However, there is a partial converse to the theorem.

Theorem. *If γ is a geodesic, and x and y are nearby points along γ. Then the portion of γ connecting them is the shortest path between them.*

We have also discussed symmetries of lines on certain special surfaces. Remember that a reflection across a curve γ means an isometry which fixes points on γ, but locally exchanges points on the two sides of γ.

Theorem. *If there is a reflection across a curve γ, then γ is a geodesic.*

Proof. We prove this theorem by assuming that γ is not a geodesic and deriving a contradiction. If γ is not a geodesic, it must turn, say to the left, between some nearby points p and q. Then the shortest path between p and q lies to the left of γ, and not to the right. Thus there is not even a local reflection symmetry between the two sides. □

In general, there is no relationship between the intersection of a surface with a plane and geodesics.

Axioms. We have considered five axioms (each with several parts) for lines on surfaces; as we have seen, all are true on the Euclidean and hyperbolic planes $\mathbb{E}^2$ and $\mathbb{H}^2$, but some are false on other surfaces. We would now like to consider which axioms are true on a general smooth surface.

The ruler axiom says that you can always continue every geodesic infinitely in either direction. This is not true if the surface we are working on ends somewhere, like the halfplane.

Definition. A surface is *geodesically complete* if every geodesic can be extended indefinitely in both directions.

Of course, the sphere and the cylinder show that we can expect *closed geodesics*, which revisit the same points again and again, even on geodesically complete surfaces.

The incidence axiom says that there is always a unique line between any two points. The sphere and the cylinder show that we should not expect uniqueness in general. On the plane minus a disk, there is not always a line connecting two given points. But it turns out that existence does follow on complete surfaces.

Theorem. *If a surface is geodesically complete, then there exists a geodesic between any two points.*

The protractor axiom asks about the existence of a unique geodesic with a given starting point and direction. This is true on any smooth surface, because of a standard theorem about differential equations.

Theorem. *There exists a unique solution to every ordinary differential equation with given initial conditions (positition and direction).*

The half-plane axiom and the mirror axiom are not true in general. They both can fail, for instance on the cylinder.