Constant mean curvature surfaces with three ends

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We announce the classification of complete almost embedded surfaces of constant mean curvature, with three ends and genus zero. They are classified by triples of points on the sphere whose distances are the asymptotic necksizes of the three ends.

Surfaces that minimize area under a volume constraint have constant mean curvature (CMC); this condition can be expressed as a nonlinear partial differential equation. We are interested in complete CMC surfaces properly embedded in $\mathbb{R}^3$; we rescale them to have mean curvature one. For technical reasons, we consider a slight generalization of embeddedness [introduced by Alexandrov (1)]: An immersed surface is *almost embedded* if it bounds a properly immersed three-manifold.

Alexandrov (1, 2) showed that the round sphere is the only *compact* almost embedded CMC surface. The next case to consider is that of *finite-topology* surfaces, homeomorphic to a compact surface with a finite number of points removed. A neighborhood of any of these punctures is called an *end* of the surface. The *unduloids*, CMC surfaces of revolution described by Delaunay (3), are genus-zero examples with two ends. Each is a solution of an ordinary differential equation; the entire family is parametrized by the unduloid *necksize*, which ranges from zero (at the singular chain of spheres) to π (at the cylinder).

Over the past decade there has been increasing understanding of finite-topology almost embedded CMC surfaces. Each end of such a surface is asymptotic to an unduloid (4). Meeks showed (5) there are no examples with a single end. The unduloids themselves are the only examples with two ends (4). Kapouleas (6) has constructed examples (near the limit of zero necksize) with any genus and any number of ends greater than two.

In this note we announce the classification of all almost embedded CMC surfaces with three ends and genus zero; we call these *triunduloids* (see Fig. 1). In light of the trousers decomposition for surfaces, triunduloids can be seen as the building blocks for more complicated almost embedded CMC surfaces (7). Our main result determines explicitly the *moduli space* of triunduloids with labeled ends, up to Euclidean motions. Because triunduloids are transcendental objects, and are not described by any ordinary differential equation, it is remarkable to have such a complete and explicit determination for their moduli space.

THEOREM. *Triunduloids are classified by triples of distinct labeled points in the two-sphere (up to rotations); the spherical distances of points in the triple are the necksizes of the unduloids asymptotic to the three ends. The moduli space of triunduloids is therefore homeomorphic to an open three-ball.*

The proof of the theorem has three parts. First we define the classifying map from triunduloids to spherical triples, and observe that it is proper; then we prove it is injective; and finally we show it is surjective.

To define the classifying map, we use the fact that any triunduloid has a reflection symmetry that decomposes the surface into mirror-image halves (8). Each half is simply connected, so Lawson's construction (9) gives a conjugate cousin minimal surface in the three-sphere. Using observations of Karcher (10), we find that its boundary projects under the Hopf map to the desired spherical triple. The composition of these steps defines our classifying map. It follows from curvature estimates (11) that the map is proper.

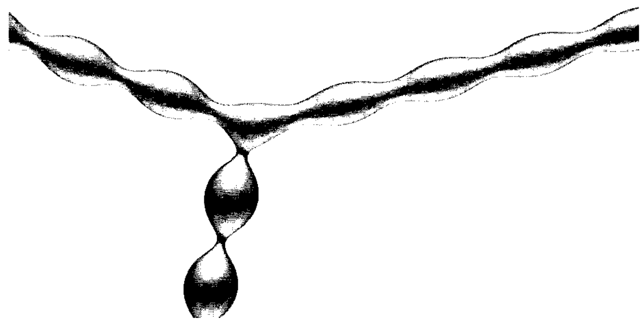

Fig. 1. A triunduloid is an embedded surface of constant mean curvature with three ends, each asymptotic to a Delaunay unduloid.

The injectivity of our classifying map is really a uniqueness result. We use the Hopf circle bundle to construct a trivial circle bundle over the disk representing the Lawson conjugate. Its total space is locally isometric to the three-sphere, and so the circle action along the fibers is by isometries. The classifying triple determines the bundle up to isometries. Moreover, any conjugate surface with the same triple defines a minimal section of the bundle. Thus, we are in a situation familiar from minimal graphs, and we can apply a suitable maximum principle to deduce uniqueness.

Finally, we need an existence result showing that our classifying map is surjective. We depend on the fact (12) that the moduli space of CMC surfaces of genus g with k ends is locally a real analytic variety of (formal) dimension $3k - 6$. In particular, near a *nondegenerate* triunduloid, our moduli space has dimension three. We get such a nondegenerate triunduloid by using a nondegenerate minimal trinoid (13) in a recent construction by Mazzeo and Pacard.

To prove surjectivity of our classifying map, we use the fact (14–16) that a three-dimensional analytic variety can be triangulated, with each two-simplex meeting an even number of three-simplices. We then show that a proper, injective map from such a three-complex to a connected three-manifold (here, the three-ball) must be surjective as well. We use the standard lemma (17) that a proper, injective map from any space to a compactly generated space is a homeomorphism onto its image. Once surjectivity is known, this lemma is used once more to show that our classifying map is in fact a homeomorphism.

Note that our geometric picture of the triunduloid moduli space naturally explains necksize bounds for triunduloids. For instance, the symmetric triunduloids constructed previously (18) have three congruent ends, with necksize at most $2\pi/3$. This bound in the symmetric case can now be seen as the maximum

MATHEMATICS

Abbreviation: CMC, constant mean curvature.

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side length for a spherical equilateral triangle. More generally, we have the following.

COROLLARY. *The triple $0 < x, y, z \leq \pi$ can be the necksizes of a triunduloid if and only if it satisfies the spherical triangle inequalities*:

$$x + y + z \leq 2\pi, \quad x \leq y + z, \quad y \leq z + x, \quad z \leq x + y$$

In particular, at most one end of a triunduloid can be asymptotic to a cylinder.

Similar methods apply to genus-zero surfaces with $k > 3$ ends, when those ends still have asymptotic axes in a common plane. The moduli space of such *coplanar k-unduloids* can be understood as a covering of the space of spherical k-gons. More general surfaces, without coplanar ends, will be more difficult to classify.

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