Some problems to solve with solutions ...

1. Let \( C \) be the curve \( x = \cos t, \ y = \sin t, \ z = t, \) for \( 0 \leq t \leq 2\pi \) oriented in the direction of increasing \( t \). Evaluate:

\[
\int_C z \, dx + x \, dy + y \, dz
\]

**Solution:** First draw a picture of the curve to see what we are integrating over. We differentiate to get \( dx = -\sin t \, dt, \ dy = \cos t \, dt, \ dz = dt \).

\[
\int_C z \, dx + x \, dy + y \, dz = \int_0^{2\pi} \left( t(-\sin t) + (\cos t)(\cos t) + (\sin t) \right) \, dt
\]

\[
= \int_0^{2\pi} -t \sin t \, dt + \int_0^{2\pi} \cos^2 t \, dt + \int_0^{2\pi} \sin t \, dt
\]

\[
= \left[ -\sin t + t \cos t \right]_{t=0}^{2\pi} + \left[ \frac{1}{2} t + \frac{1}{4} \sin 2t \right]_{t=0}^{2\pi} + 0
\]

\[
= 2\pi + \pi + 0 = 3\pi
\]

2. Let \( S \) be the unit sphere, \( x^2 + y^2 + z^2 = 1 \). Let \( \mathbf{n} \) be the outward unit normal. Evaluate the following surface integral using Gauss’s theorem (the divergence theorem).

\[
\iint_S x \, dy \, dz + y \, dz \, dx + z \, dx \, dy
\]

**Solution:** First draw a picture. Let \( S \) be the sphere and let \( R \) be the interior of the sphere. Note that the vector field \( x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \) is continuous and has continuous partial derivatives everywhere and so we can apply Gauss’s theorem.

\[
\iiint_R (1 + 1 + 1) \, dx \, dy \, dz
\]

\[
= 3 \iiint_R \, dx \, dy \, dz
\]

\[
= 3(\text{“volume of } R\text{”})
\]

\[
= 3\left( \frac{4}{3}\pi \right) = 4\pi
\]
3. Let \( F(x, y, z) = 2x^2 - y^2 - z^2 \). Let \( R \) be the unit cube, \( 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1 \). Let the surface \( S \) be the boundary of \( R \). Let \( \mathbf{n} \) be the outward unit normal. Evaluate the following surface integral.

\[
\iint_S \nabla F \cdot \mathbf{n} \, d\sigma
\]

**Solution:** First draw a picture. Note that the vector field \( \nabla F \) is continuous and has continuous partial derivatives everywhere and so we can apply Gauss’s theorem. Also note that \( \text{div} \nabla F = \nabla^2 F \) (the Laplacian of \( F \)).

\[
\iint_S \nabla F \cdot \mathbf{n} \, d\sigma = \iiint_R \text{div} \nabla F \, dx \, dy \, dz
\]
\[
= \iiint_R \nabla^2 F \, dx \, dy \, dz
\]
\[
= \iiint_R (4 - 2 - 2) \, dx \, dy \, dz
\]
\[
= \iiint_R 0 \, dx \, dy \, dz = 0
\]

Notice that for any harmonic function \( F \) (recall \( \nabla^2 F = 0 \)) we have

\[
\iint_S \nabla F \cdot \mathbf{n} \, d\sigma = 0
\]
4. Let $S$ be the upper hemisphere of the unit sphere. Let $n$ be the upper unit normal. Let $v = zxi + zyj + yk$. Evaluate

$$\oiint_S v \cdot n \, d\sigma$$

**Solution:** Let $R_{xy}$ be the region $x^2 + y^2 \leq 1$ in the $xy$-plane. Notice that $S$ is the graph $z = \sqrt{1 - x^2 - y^2}$. Draw a picture.

First compute

$$\frac{\partial z}{\partial x} = \frac{-2x}{2\sqrt{1 - x^2 - y^2}} = \frac{-x}{\sqrt{1 - x^2 - y^2}} = \frac{-x}{z}$$

$$\frac{\partial z}{\partial y} = \frac{-2y}{2\sqrt{1 - x^2 - y^2}} = \frac{-y}{\sqrt{1 - x^2 - y^2}} = \frac{-y}{z}$$

$$\oiint_S v \cdot n \, d\sigma = \oiint_S zx \, dy \, dz + zy \, dz \, dx + y \, dx \, dy$$

$$= \oiint_{R_{xy}} \left[ -zx \left( \frac{-x}{z} \right) - zy \left( \frac{-y}{z} \right) + y \right] \, dx \, dy$$

$$= \oiint_{R_{xy}} \left[ x^2 + y^2 + y \right] \, dx \, dy$$

$$= \int_0^{2\pi} \int_0^1 \left( r \cos \theta \right)^2 + \left( r \sin \theta \right)^2 + r \sin \theta \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left( r^3 + r^2 \sin \theta \right) \, dr \, d\theta$$

$$= 2\pi \int_0^1 r^3 \, dr + \int_0^1 r^2 \left[ \int_0^{2\pi} \sin \theta \, d\theta \right] \, dr$$

$$= 2\pi \frac{1}{4} + 0 = \frac{\pi}{2}$$
5. Let $S$ be the whole unit sphere. Let $\mathbf{n}$ be the outward unit normal. Let $\mathbf{v} = zx\mathbf{i} + zy\mathbf{j} + y\mathbf{k}$ (as in the last question). Evaluate

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, d\sigma$$

Use this and result of problem 4 to find $\iint_L \mathbf{v} \cdot \mathbf{n} \, d\sigma$ where $L$ is the lower hemisphere of $S$ and $\mathbf{n}$ the lower unit normal (hence the same normal as in the surface integral over all of $S$).

**Solution:** Again we apply Gauss's theorem. Let $R$ be the interior of $S$.

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, d\sigma = \iiint_R \text{div} \, \mathbf{v} \, dx \, dy \, dz$$

$$= \iiint_R z + z \, dx \, dy \, dz$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 2z \, dz \, dx \, dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 0 \, dz \, dx \, dy \quad \text{(an integral of an odd function over a symmetric interval equals 0)}$$

$$= 0$$

We let $U$ denote the upper hemisphere. Notice that

$$0 = \iint_S \mathbf{v} \cdot \mathbf{n} \, d\sigma = \iint_L \mathbf{v} \cdot \mathbf{n} \, d\sigma + \iint_U \mathbf{v} \cdot \mathbf{n} \, d\sigma$$

Hence

$$\iint_L \mathbf{v} \cdot \mathbf{n} \, d\sigma = -\iint_U \mathbf{v} \cdot \mathbf{n} \, d\sigma$$

We have computed in problem 4 that $\iint_U \mathbf{v} \cdot \mathbf{n} \, d\sigma = \frac{\pi}{2}$. Hence

$$\iint_L \mathbf{v} \cdot \mathbf{n} \, d\sigma = -\frac{\pi}{2}.$$