d’Alembert solution of the wave equation.

Edwards and Penney have a typo in the d’Alembert solution (equations (37) and (39) on page 639 in section 9.6). This is an easier way to derive the solution.

Suppose we have the wave equation

$$u_{tt} = a^2 u_{xx}. \quad (1)$$

And we wish to solve the equation (1) given the conditions

$$u(0, t) = u(L, t) = 0 \quad \text{for all } t, \quad (2)$$
$$u(x, 0) = f(x) \quad 0 < x < L, \quad (3)$$
$$u_t(x, 0) = g(x) \quad 0 < x < L. \quad \quad (4)$$

We will transform the equation into a simpler form where it can be solved by simple integration. We change variables to

$$\xi = x - at, \ \eta = x + at$$

and we use the chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$
$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta} = -a^2 \frac{\partial}{\partial \xi} + a^2 \frac{\partial}{\partial \eta}$$

We compute

$$\frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) = \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$
$$\frac{\partial^2 u}{\partial t^2} = \left( -a^2 \frac{\partial}{\partial \xi} + a^2 \frac{\partial}{\partial \eta} \right) \left( -a^2 \frac{\partial u}{\partial \xi} + a^2 \frac{\partial u}{\partial \eta} \right) = a^2 \frac{\partial^2 u}{\partial \xi^2} - 2a^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + a^2 \frac{\partial^2 u}{\partial \eta^2}$$

Then

$$0 = a^2 u_{xx} - u_{tt} = 4a^2 \frac{\partial^2 u}{\partial \xi \partial \eta}$$

And therefore the wave equation (1) transforms into $u_{\xi\eta} = 0$. It is easy to find the general solution to this equation by integration twice. First suppose you integrate with respect to $\eta$ and notice that the constant of integration depends on $\xi$ to get $u_\xi = C(\xi)$. Now integrate with respect to $\xi$ and notice that the constant of integration must depend on $\eta$. Thus, $u = \int C(\xi)d\xi + B(\eta)$. The solution must then be of the following form for some functions $A$ and $B$.

$$u = A(\xi) + B(\eta) = A(x - at) + B(x + at).$$

We will need to solve for the given conditions. First let $F(x)$ denote the odd extension of $f(x)$ and $G(x)$ denote the odd extension of $g(x)$. We let

$$A(x) = \frac{1}{2} F(x) - \frac{1}{2a} \int_0^x G(s) \, ds \quad B(x) = \frac{1}{2} F(x) + \frac{1}{2a} \int_0^x G(s) \, ds.$$

The solution is explicitly

$$u(x, t) = \frac{1}{2} F(x - at) - \frac{1}{2a} \int_0^{x-at} G(s) \, ds + \frac{1}{2} F(x + at) + \frac{1}{2a} \int_0^{x+at} G(s) \, ds$$
Checking our work.

Let us check that this works. So

\[ u(x, 0) = \frac{1}{2} F(x) - \frac{1}{2a} \int_0^x G(s) \, ds + \frac{1}{2} F(x) + \frac{1}{2a} \int_0^x G(s) \, ds = F(x). \]

So far so good. Assume for simplicity \( F \) is differentiable. By the fundamental theorem of calculus we have

\[ u_t(x, t) = \frac{-a}{2} F'(x - at) + \frac{1}{2} G(x - at) + \frac{a}{2} F'(x + at) + \frac{1}{2} G(x + at) \]

So

\[ u_t(x, 0) = \frac{-a}{2} F'(x) + \frac{1}{2} G(x) + \frac{a}{2} F'(x) + \frac{1}{2} G(x) = G(x). \]

Yay! We’re smoking now. OK, now the boundary conditions. Note that \( F \) and \( G \) are odd. Also \( \int_0^x G(s) \, ds \) is an even function of \( x \) because \( G \) is odd (do the substitution \( s = -v \) to see that). So

\[ u(0, t) = \frac{1}{2} F(-at) - \frac{1}{2a} \int_0^{-at} G(s) \, ds + \frac{1}{2} F(at) + \frac{1}{2a} \int_0^{at} G(s) \, ds \]

\[ = \frac{-1}{2} F(at) - \frac{1}{2a} \int_0^{at} G(s) \, ds + \frac{1}{2} F(at) + \frac{1}{2a} \int_0^{at} G(s) \, ds = 0 \]

Now \( F \) and \( G \) are \( 2L \) periodic as well. Furthermore

\[ u(L, t) = \frac{1}{2} F(L - at) - \frac{1}{2a} \int_0^{L-at} G(s) \, ds + \frac{1}{2} F(L + at) + \frac{1}{2a} \int_0^{L+at} G(s) \, ds \]

\[ = \frac{1}{2} F(-L - at) - \frac{1}{2a} \int_0^L G(s) \, ds - \frac{1}{2a} \int_0^{-at} G(s) \, ds + \frac{1}{2} F(L + at) + \frac{1}{2a} \int_0^L G(s) \, ds + \frac{1}{2a} \int_0^{at} G(s) \, ds \]

\[ = \frac{-1}{2} F(L + at) - \frac{1}{2a} \int_0^{at} G(s) \, ds + \frac{1}{2} F(L + at) + \frac{1}{2a} \int_0^{at} G(s) \, ds = 0 \]

Notes

It is best to memorize the procedure rather than the formula itself. You should remember that a solution to the wave equation is a superposition of two waves traveling at opposite directions. That is

\[ u(x, t) = A(x - at) + B(x + at). \]

If you think about it, the formulas for \( A \) and \( B \) are then not hard to guess. Also note that when \( g(x) = 0 \) (and hence \( G(x) = 0 \)) we have

\[ u(x, t) = \frac{F(x - at) + F(x + at)}{2} \]

Here is where the book got it wrong. If you let

\[ H(x) = \int_0^x G(s) \, ds, \]

then assuming that \( F(x) = 0 \) the solution is

\[ \frac{-H(x - at) + H(x + at)}{2a}. \]

So by superposition we get a solution in the general case when neither \( f \) nor \( g \) are identically zero.

\[ u(x, t) = \frac{F(x - at) + F(x + at)}{2} + \frac{-H(x - at) + H(x + at)}{2a}, \]

which is what the book was going for but it missed the minus sign.

**Warning:** Make sure you use the odd extensions \( F \) and \( G \), when you have formulas for \( f \) and \( g \). The thing is, those formulas in general hold only for \( 0 < x < L \), and are note equal to \( F \) and \( G \) for other \( x \).