RESEARCH STATEMENT

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Macdonald polynomials are a distinguished family of symmetric functions with roles in an astoundingly broad set of fields. For example, they have appeared in algebraic combinatorics, enumerative geometry, integrable systems, knot theory, representation theory, and probability. In the geometric representation theory of quantum groups, they don the avatar of torus-fixed point classes in the equivariant K-theory of Hilbert schemes of points on the plane. This perspective easily lends itself to generalizations in at least two directions: one can replace the Hilbert schemes with certain other geometries and one can replace equivariant K-theory with other equivariant cohomology theories. My research studies Macdonald theory from this geometric perspective in order to access and understand these generalizations. Moreover, I have in mind some concrete applications of such "new Macdonald theories" to a few of the aforementioned fields.

1. Precursors

I will first review the well-established connections between Macdonald polynomials, Hilbert schemes of points on the plane, and the quantum toroidal algebra of $\mathfrak{gl}_1$. Overall, I wish to show that much of the standard fare of Macdonald theory—norm formulas, Macdonald operators, and Pieri rules—can be studied and derived from equivariant K-theory of the Hilbert schemes in conjunction with an action of the toroidal algebra.

1.1. Three faces of Macdonald polynomials.

Let Λ be the ring of symmetric functions in infinitely many variables and $\Lambda_{q,t} := \Lambda \otimes \mathbb{C}(q,t)$. Recall that the Macdonald polynomials $\{P_\lambda\}$ are a basis of $\Lambda_{q,t}$ with at least three well-known characterizations ([Hai03],[Mac15]). First, they can be characterized as the basis obtained by performing Gram-Schmidt orthogonalization on the Schur functions $\{s_\lambda\}$ with respect to the *Macdonald inner product*. This follows an existing pattern of defining distinguished bases of symmetric functions by deforming the Hall inner product. Second, they can be defined as joint eigenfunctions for a infinite family of commuting difference operators called the *Macdonald operators*. The Macdonald operators come from quantum Hamiltonians of the *trigonometric Ruijsenaars-Schneider* (tRS) system, so this is a characterization via integrable systems.

For the final characterization, recall the relationship between Λ and the representation theory of the symmetric groups $\{\Sigma_n\}$. There is a well-studied isomorphism $\Lambda \cong \bigoplus_n \mathrm{Rep}(\Sigma_n)$ called the *Frobenius characteristic* sending s_λ to the irreducible representation $[V_\lambda]$. We will abuse notation here and freely jump between both sides of the isomorphism. Letting $\mathbb{C}^n$ be the reflection representation of Σ_n, the *modified* Macdonald polynomial H_λ for $\lambda \vdash n$ is determined by the following:

(1) $H_\lambda \otimes \sum_{i=0}^n (-q)^i \left[\bigwedge^i \mathbb{C}^n\right]$ lies in the span of $\{s_\mu\}_{\mu \geq \lambda}$;
(2) $H_\lambda \otimes \sum_{i=0}^n (-t)^{-i} \left[\bigwedge^i \mathbb{C}^n\right]$ lies in the span of $\{s_\mu\}_{\mu \leq \lambda}$;
(3) the coefficient of the trivial representation of Σ_n is 1.

One obtains P_λ by renormalizing the tensor product in (2) so that the coefficient of s_λ is 1.

1.2. Hilbert schemes.

The third characterization can be understood via Haiman's proof of the *Macdonald positivity conjecture* using the geometry of Hilbert schemes [Hai03]. The Hilbert scheme of n points on the plane $\mathrm{Hilb}_n\mathbb{C}^2$ parameterizes ideals I of $\mathbb{C}[x,y]$ such that the quotient ring is n-dimensional. It carries an action of the two-dimensional torus T inherited from the one on $\mathbb{C}^2$, and the fixed points of this action are the monomial ideals, which can be indexed by partitions of n. The Macdonald positivity conjecture states that the expansion of H_λ in terms of $\{s_\mu\}$ has coefficients in $\mathbb{N}[q,t]$. One can try to prove such a statement by constructing an *actual* bigraded representation of Σ_n satisfying the conditions for H_λ. Haiman was able to do this by constructing a $T\times\Sigma_n$-equivariant vector bundle $\mathcal{P}$, the *Procesi bundle*, on $\mathrm{Hilb}_n\mathbb{C}^2$ whose fiber at the fixed point indexed by λ is such a representation.

This connection between symmetric functions and Hilbert schemes can be succinctly summarized using *localized* equivariant K-theory. The Hilbert scheme is a *symplectic resolution* of the singularity $(\mathbb{C}^n\oplus\mathbb{C}^n)/\Sigma_n$. A common question asked for such resolutions is if the two spaces have equivalent derived categories of coherent sheaves. The answer is affirmative in this case using $\mathcal{P}$:

$$R\Gamma(\mathcal{P}\otimes^L -): D^b_T(\mathrm{Hilb}_n\mathbb{C}^2)\xrightarrow{\sim} D^b_{T\times\Sigma_n}(\mathbb{C}^n\oplus\mathbb{C}^n)$$

Parts (1) and (2) in the definition of H_λ can then interpreted as conditions on the tensor products of H_λ with the Koszul resolutions for the two halves of $\mathbb{C}^n\oplus\mathbb{C}^n$. In localized equivariant K-theory, this yields

$$K_T(\mathrm{Hilb}_n\mathbb{C}^2)_{loc}\xrightarrow{\sim} K_{T\times\Sigma_n}(\mathbb{C}^n\oplus\mathbb{C}^n)_{loc}\cong \mathrm{Rep}(\Sigma_n)\otimes\mathbb{C}(q,t)$$

By Atiyah-Bott localization [AB84], $K_T(\mathrm{Hilb}_n\mathbb{C}^2)_{loc}$ has a basis $\{|\lambda\rangle\}$ given by classes of structure sheaves of torus-fixed points. The isomorphism above sends $|\lambda\rangle$ to H_λ. Considering the K-theories of all $\mathrm{Hilb}_n\mathbb{C}^2$ at once yields the isomorphism

$$K_T(\mathrm{Hilb}) := \bigoplus_n K_T(\mathrm{Hilb}_n\mathbb{C}^2)_{loc}\cong\Lambda_{q,t}$$

This K-theoretic avatar of symmetric functions naturally carries analogous structures characterizing Macdonald polynomials: fixed point classes form an orthogonal basis for the intersection pairing and also diagonalize operators given by tensoring with equivariant bundles.

1.3. **Quantum toroidal and shuffle algebras.** To properly treat $K_T(\mathrm{Hilb})$ as an incarnation of $\Lambda_{q,t}$, one would need operators given by mulitplication by symmetric functions. These can be built out of the *1-step Nakajima correspondences*:

$$P_{n+1,n}=\{(I_{n+1},I_n)|I_{n+1}\subset I_n\}\subset \mathrm{Hilb}_{n+1}\mathbb{C}^2\times\mathrm{Hilb}_n\mathbb{C}^2$$

On $P_{n+1,n}$, there exists a tautological line bundle $\mathcal{L}$ whose fiber is the line I_n/I_{n+1}. Letting p and q denote the projections onto the first and second factors, one can consider the operators on $K_T(\mathrm{Hilb})$ given by $p_*(\mathcal{L}^i\otimes q^*)$ and $q_*(\mathcal{L}^i\otimes p^*)$ for various i. Combining them with our geometric Macdonald operators from the previous paragraph, these define on $K_T(\mathrm{Hilb})$ the structure of a module for the *quantum toroidal algebra* $U_{q,t}(\ddot{\mathfrak{gl}}_1)$ [FT11]. It is not obvious that these operators can recreate the desired multiplication operators. For example, matrix elements for multiplication by the elementary symmetric function e_n with respect to the basis of Macdonald polynomials are given by the *Pieri rules*. On the other hand, it is not at all straightforward to find a combination of the toroidal operators that will have the same matrix elements.

The *shuffle algebra*, first considered by Feigin and Odesskii, is an essential tool for understanding these multiplication operators in this geometric setting. To define it, let $\mathbb{S}_n:=\mathbb{C}(q,t)(x_1,\ldots,x_n)^{\Sigma_n}$. On $\mathbb{S}:=\bigoplus_n\mathbb{S}_n$, we can define a *shuffle product* in the following way: for $F\in\mathbb{S}_n$ and $G\in\mathbb{S}_m$, $F\star G\in\mathbb{S}_{n+m}$ is given by

$$F\star G=\mathrm{Sym}\left(F(x_1,\ldots,x_n)G(x_{n+1},\ldots,x_{n+m})\prod_{1\le a\le n}^{n<b\le m}\omega(x_a/x_b)\right)$$

where Sym denotes the total symmetrization of the $n+m$ variables and $\omega(z)$ is a prescribed rational function. The shuffle algebra S is then a subspace of $\mathbb{S}$ defined by certain restrictions on the zeroes and poles of the functions. By a theorem of Neguţ [Neg14], S is isomorphic a certain half of $U_{q,t}(\ddot{\mathfrak{gl}}_1)$.

Moreover, matrix elements for the action of $F \in S$ on $K(\mathrm{Hilb})$ with respect to the basis $\{|\lambda\rangle\}$ are given by evaluations of F and are thus completely explicit and combinatorial. For example, Feigin and Tsymbaliuk were able to show in [FT11] that certain explicit functions in S recreated the Pieri rules.

2. Research Objectives

In trying to generalize Macdonald theory by replacing Macdonald polynomials with fixed point classes in equivariant cohomology theories applied to other spaces, one runs into an immediate ontological problem: in what sense can these classes be considered functions? This greatly focuses the scope of generalization onto settings that could plausibly yield special functions. A common thread is that such special functions should be related to an integrable system of some sort. I will consider here *wreath Macdonald polynomials* as well as elliptic analogues of Macdonald polynomials.

2.1. **Wreath Macdonald polynomials.** Proposed by Haiman in [Hai03], the wreath Macdonald polynomials generalize the modified Macdonald polynomials from Σ_n to the wreath products $\Sigma_n^\ell := \mathbb{Z}/\ell\mathbb{Z} \wr \Sigma_n$. Their definition requires some preliminary knowledge on wreath products, ℓ-cores, and ℓ-quotients [Mac15]. The irreducible representations of Σ_n^ℓ are now indexed by ℓ-tuples of partitions whose parts sum up to n. There is also a *wreath Frobenius characteristic* $\Lambda^{\otimes \ell} \cong \bigoplus_n \mathrm{Rep}(\Sigma_n^\ell)$. For an ℓ-tuple of partitions $\vec{\lambda} = (\lambda^0, \ldots, \lambda^{\ell-1})$, we define the *multi-Schur function* to be

$$s_{\vec{\lambda}} := s_{\lambda^0} \otimes \cdots \otimes s_{\lambda^{\ell-1}} \in \Lambda^{\otimes \ell}$$

The wreath Frobenius characteristic then sends $s_{\vec{\lambda}}$ to the irreducible representation $[V_{\vec{\lambda}}]$. Finally, for an ordinary partition λ, one can remove all continguous strips of length of length ℓ to obtain the ℓ-core partition $\mathrm{core}(\lambda)$. There is a way to record the positions of the removed strips into an ℓ-tuple of partitions $\mathrm{quot}(\lambda)$ called the ℓ-quotient. This decomposition yields a bijection

$$\{\text{partitions}\} \leftrightarrow \{\ell\text{-cores}\} \times \{\ell\text{-tuples of partitions}\}$$

The wreath product Σ_n^ℓ has a natural n-dimensional reflection representation $\mathfrak{h}$ which is no longer isomorphic to its dual. For λ such that the parts of $\mathrm{quot}(\lambda)$ sum to n, the wreath Macdonald polynomial $H_\lambda^\ell \in \mathrm{Rep}(\Sigma_n^\ell) \otimes \mathbb{C}(q,t)$ is characterized by

(1) $H_\lambda^\ell \otimes \sum_i (-q)^i \left[\bigwedge^i \mathfrak{h}^*\right]$ lies in the span of $[V_{\mathrm{quot}(\mu)}]$ for $\mu \geq \lambda$ such that $\mathrm{core}(\mu) = \mathrm{core}(\lambda)$;
(2) $H_\lambda^\ell \otimes \sum_i (-t)^{-i} \left[\bigwedge^i \mathfrak{h}^*\right]$ lies in the span of $[V_{\mathrm{quot}(\mu)}]$ for $\mu \leq \lambda$ such that $\mathrm{core}(\mu) = \mathrm{core}(\lambda)$;
(3) the coefficient of the trivial representation is 1.

Similar to before, there exists a geometric story, this time realized by Bezrukavnikov and Finkelberg [BF14]. The singularity $(\mathfrak{h} \oplus \mathfrak{h}^*)/\Sigma_n^\ell$ is now resolved by certain *cyclic Nakajima quiver varieties* $\mathfrak{M}(\mathbf{v})$. On these varieties, the authors of *loc. cit.* were able to produce analogues of Procesi bundles whose fibers at torus-fixed points have character H_λ^ℓ.

Wreath analogues of standard aspects of Macdonald theory did not receive treatment until my recent work [Wen19]. By work of Varagnolo and Vasserot [VV99], the equivariant K-theories of the quiver varieties involved fit together to form a module for the quantum toroidal algebra $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_\ell)$. While $\Lambda_{q,t}^{\otimes \ell}$ does not carry an action of the toroidal algebra, note that it also lacks a way of recording ℓ-cores. However, ℓ-cores are in bijection with the root lattice Q of $\mathfrak{sl}_\ell$, and the tensor product $\Lambda_{q,t}^{\otimes \ell} \otimes \mathbb{C}[Q]$ has an action of the toroidal algebra called the *vertex representation* [Sai98]. The geometric and vertex representations are not isomorphic, but Tsymbaliuk [Tsy19] was able to show that they are after twisting by the highly nontrivial *Miki automorphism* [Mik99]. Utilizing shuffle algebra techniques developed by Neguţ [Neg13] as well as some results of Feigin-Tsymbaliuk [FT16], I was able to prove the following:

Theorem ([Wen19]). *When viewed as elements of the vertex representation, the wreath Macdonald polynomials diagonalize the horizontal Heisenberg subalgebra of* $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_\ell)$.

This shows that the wreath Macdonald polynomials diagonalize a large commutative algebra of operators, hopefully leading to a notion wreath Macdonald operators.

I will now outline some future plans for the subject.

2.1.1. *Development of standard theory.* I would like to develop wreath analogues of what can be found in Chapter VI of [Mac15]: Pieri rules, Macdonald inner product, skew versions, evaluation formulas, and the Cauchy kernel. These are basic elements of Macdonald theory essential to many of its applications, a great example being the *Macdonald processes* in integrable probability [BC14]. In the course of proving my theorem above, I was able to find shuffle elements corresponding to multiplication by e_n in each tensorand of $\Lambda^{\otimes\ell}$. In ongoing work, I am using these shuffle elements to derive Pieri rules for wreath Macdonald polynomials. I can define a wreath analogue of the Macdonald pairing, and I would like to compute the norms of the wreath Macdonald polynomials with respect to it. The combination of the pairing with Pieri rules should lead to a straightforward study of skew wreath Macdonald polynomials.

Analogues of evaluation formulas and the Cauchy kernel may be more subtle as they require one to view the polynomials as actual functions. While this is possible thanks to the Frobenius characteristic, an interesting and possibly different functional interpretation comes from *toroidal Schur-Weyl duality* [VV96]. The module of the toroidal algebra given by K-theory of cyclic quiver varieties is actually the well-known q-Fock space, a deformed version of the fermionic Fock space [Nag09]. It has a construction via toroidal Schur-Weyl duality where the basis $\{|\lambda\rangle\}$ comes from partially symmetrized *nonsymmetric* Macdonald polynomials. Such partial symmetrizations have appeared, for example, in Borodin and Wheeler's study of colored stochastic vertex models [BW18]. I would like to see if the standard theory in the wreath case bears any meaning for these partial symmetrizations.

2.1.2. *Deformed boson-fermion correspondence and stable bases.* Whereas the K-theoretic representation is a deformed fermionic Fock space, the vertex represenation can be viewed as a bosonic Fock space. Thus, Tsymbaliuk's twisted isomorphism between the two is a kind of deformed boson-fermion correspondence. One complaint against such a statement is that the classical boson-fermion correspondence explicitly relates the basis of pure wedges on the fermionic side with the basis of multi-Schur functions on the bosonic side. A consequence of my theorem above is that Tsymbaliuk's isomorphism sends the fixed point basis in K-theory to a multiple of the wreath Macdonald polynomials. I would like to show that deformed pure wedges are mapped to suitably modified multi-Schur functions.

In K-theory, there is another distinguished bases given by the *K-theoretic stable envelopes* of Maulik and Okounkov [MO19]. They depend on a parameter called the slope, and in the Hilbert scheme case, it is known that slope 0 stable envelopes correspond to modified Schur functions. In the same spirit of matching bases, I would like to find a similar correspondence for these cyclic quiver varieties. A possible approach would entail understanding vertex operators in the geometric representation.

2.1.3. *Gordon's conjecture on characteristic cycles.* The cyclic quiver varieties $\{\mathfrak{M}(\mathbf{v})\}$ in this story are quantized by *cyclotomic rational Cherednik algebras* [Gor06]. To certain modules M of these algebras, one can assign a cycle on the variety called the *characteristic cycle* $CC(M)$. The *Verma modules* $\{M_\mu\}$ are a particular class of such modules. Their characteristic cycles are necessarily combinations of attracting sets $\{m_\lambda\}$ for a certain torus action. Define the *wreath Kostka-Macdonald* *coefficients* $K^{\ell}_{\lambda\mu}(q,t)$ by

$$H^{\ell}_{\lambda} = \sum_{\substack{\mu \\ \mathrm{core}(\mu)=\mathrm{core}(\lambda)}} K^{\ell}_{\lambda\mu}(q,t) s_{\mathrm{quot}(\mu)}$$

Mirroring previous results in the Hilbert scheme case ([GS06], [McG12]), Gordon [Gor08] conjectured

$$CC(M_{\mu}) = \sum_{\substack{\lambda \\ \mathrm{core}(\mu)=\mathrm{core}(\lambda)}} K^{\ell}_{\lambda\mu}(0,1) m_{\lambda}$$

The K-theoretic stable envelopes degenerate to their versions in cohomology, which in turn are related to $CC(M_{\mu})$. After 2.1.2, making this relation precise will resolve this conjecture.

2.1.4. *Connections with spin tRS systems and Coulomb branches.* Following my diagonalization result, one can ask if there is an integrable system for which the commuting operators can be viewed as quantum Hamiltonians and the wreath Macdonald polynomials as their eigenfunctions. The "Jack degeneration" of this story was studied by Uglov [Ugl98], wherein he was able to make concrete connections with the *spin trigonometric Calogero-Moser system.* Thus, I expect the wreath Macdonald polynomials to be connected to the spin trigonometric Ruijsenaars-Schneider (spin tRS) systems, and I believe this can be made precise via quantum toroidal algebras.

A totalizing perspective on the ordinary Macdonald polynomials and tRS systems comes from the *spherical double affine Hecke algebra of* GL_n (GL_n sDAHA) [Che05], an algebraic structure containing both the Macdonald operators and symmetric functions. By the work of Jordan [Jor14], the GL_n sDAHA quantizes the phase space of the n-particle tRS system, matching the operators/polynomials with Hamiltonians/eigenfunctions. Quantum toroidal algebras appear via an isomorphism of Schiffmann and Vasserot [SV13]: one can make sense of a "GL_{∞} sDAHA" and they prove it is isomorphic to $U_{q,t}(\ddot{\mathfrak{gl}}_1)$.

I would like to reverse engineer this story starting with $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_{\ell})$. Phase spaces for spin tRS systems have been recently studied by Chalykh and Fairon [CF18], and my plan is to find suitable truncations of the quantum toroidal algebras [FT17] that quantize them. *K-theoretic Coulomb branches* for the cyclic quiver (cf. *loc. cit.* and [BFN18]) are also expected to be quantized by truncated quantum toroidal algebras, and it would be interesting to compare truncations.

2.2. **Elliptic Macdonald theory.** Another way to generalize Macdonald theory comes from studying *elliptic deformations* of the tRS system. In the phase space for the tRS system, each position and momentum coordinate lies on the multiplicative group $\mathbb{C}^*$. The *elliptic Ruijsenaars-Schneider system* (eRS) arises from placing the position coordinates on an elliptic curve $E \cong \mathbb{C}^*/p^{\mathbb{Z}}$. Alternatively, one obtains the *dual eRS system* by changing the momenta instead. Performing both alterations yields the mysterious *double elliptic* (DELL) integrable system [KS19].

A general program would be to understand eigenfunctions for the quantum Hamiltonians of these systems much like we do the Macdonald polynomials. To make this less vague, here are some directions of inquiry where I believe geometric representation theory can be helpful:

(1) *Formal eigenfunctions:* In [KS19], the authors assert that eigenfunctions for the dual eRS system for n particles (n variables) can be presented as a single formal eigenfunction

$$\mathcal{I}^{dual}_{eRS}(\mathbf{x},\mathbf{z}) = \sum_{\alpha\in\mathbb{N}} \mathbf{x}^{\alpha}\mathcal{F}_{\alpha}(\mathbf{z})$$

Here, $\mathbf{x} = (x_1,\ldots,x_n)$ are the n position variables, $\mathbf{z} = (z_1,\ldots,z_n)$ are auxilliary variables, each $\mathcal{F}_{\alpha}(\mathbf{z})$ is a product of theta functions in the z_i, and we use multinomial notation for $\mathbf{x}^{\alpha}$. For a partition λ with at most n parts, the eigenfunction corresponding to λ is obtained from $\mathcal{I}^{dual}_{eRS}$ by performing the specialization $z_i \mapsto q^{\lambda_i}t^{n-i}$. A formula for the formal eigenfunction

$\mathcal{I}_{eRS}(\mathbf{x}, \mathbf{z})$ of the n-particle eRS system has been conjectured by Shiraishi [Shi19]. These two formal eigenfunctions should be related by *bispectral duality*:

$$\mathcal{I}_{eRS}(\{q^{\mu_i}t^{n-i}\}, \{q^{\lambda_i}t^{n-i}\}) = \mathcal{I}_{eRS}^{dual}(\{q^{\lambda_i}t^{n-i}\}, \{q^{\mu_i}t^{n-i}\})$$

(2) *Hydrodynamical limit:* Similar to how we worked with symmetric functions in infinitely many variables in the usual Macdonald theory, there should be a framework for studying these eigenfunctions when the number of variables goes to infinity.
(3) *Orthogonality:* These eigenfunctions should form an orthogonal basis for some natural elliptic deformation of the Macdonald inner product. This has already been worked out for the eRS system [Rui09].
(4) *Elliptic DAHA:* The quantum Hamiltonians and eigenfunctions for each system should form some algebraic structure deforming the spherical DAHA. Moreover, for eRS and dual eRS, these structures should be the same except with the Hamiltonian/eigenfunction roles interchanged. Analogues of such elliptic DAHA have been proposed and studied by Ginzburg-Kapranov-Vasserot [GKV97] and Rains [Rai17].

On the geometric end, one can obtain elliptic deformations by replacing equivariant K-theory with *equivariant elliptic cohomology* [Gan14]. For a space X with a T-action (T is still a torus), note that $\operatorname{Spec} K_T(X)$ can be viewed as a scheme over $\operatorname{Spec} K_T(pt) \cong (\mathbb{C}^*)^{\dim T}$. Equivariant elliptic cohomology $Ell_T(X)$ now replaces $\mathbb{C}^*$ with an elliptic curve $E = \mathbb{C}^*/p^{\mathbb{Z}}$, and so $Ell_T(X)$ is now a scheme over the non-affine $Ell_T(pt) \cong E^{\dim T}$. One can still make sense of localization, characteristic classes, pushforward, and pullback and thus play the same game as before.

I will now outline some concrete geometric approaches.

2.2.1. *Elliptic shuffle algebras revisited.* A natural starting point is to replicate the work of Feigin-Tsymbaliuk in K-theory [FT11] and consider operators on $\coprod_n Ell_T(\mathrm{Hilb}_n\mathbb{C}^2)$ given by 1-step Nakajima correspondences. This yields an action of the elliptic shuffle algebra considered in [FHH$^+$09] on rational sections of $\bigoplus_n \mathcal{O}(Ell_T(\mathrm{Hilb}_n\mathbb{C}^2))$. Throwing in the operators given by multiplication by *elliptic Chern classes* of vector bundles on the various $\mathrm{Hilb}_n\mathbb{C}^2$, it would be interesting to compare the resulting algebra of operators with the elliptic DAHAs mentioned in (4) above.

To make contact with eigenfunctions, I would like to define a matching action of the elliptic shuffle algebra on dual eRS eigenfunctions by applying difference operators in the z variables of $\mathcal{I}_{eRS}^{dual}$. By "matching", I mean that it is defined so that matrix elements with respect to the eigenfunction basis and those with respect to the fixed point basis are the same when the eigenfunctions have sufficiently many variables. To make this matching one of substance, I would then like to see if under this identification, the elliptic shuffle elements of [FHH$^+$09] that deform multiplication by e_n still act on eigenfunctions via multiplication by e_n. After this, an ambitious route would be to use the intersection pairing in $Ell_T(\mathrm{Hilb}_n\mathbb{C}^2)$ to define a suitable adjoint to multiplication by e_n and derive evaluation formulas, giving one side of the bispectral duality equation. Altogether, these would shed light on aspects of (1)-(3) above.

2.2.2. *(Affine) Laumon spaces.* Realizing a proposal of Braverman [Bra06], Neguţ [Neg09] showed that the formal eigenfunction for the n-particle trigonometric Calogero-Moser system can be obtained from a generating function of equivariant integrals:

$$Z(m) = \sum_{\alpha \in \mathbb{N}^n} \mathbf{x}^\alpha \int_{\mathcal{M}_\alpha} c(T\mathcal{M}_\alpha, m)$$

Here, the $\mathcal{M}_\alpha$ are $SL(n)$ *Laumon spaces*—moduli spaces parametrizing n-step flags of torsion-free subsheaves in $\mathcal{O}_{\mathbb{P}^1}^n$—and the $c(T\mathcal{M}_\alpha, m)$ are equivariant Chern polynomials of their tangent bundles. A K-theoretic version of this formula was considered by Braverman-Finkelberg-Shiraishi [BFS14], and they were able to show that the resulting generating function was the formal eigenfunction for the n-particle tRS system. For these systems, the positions stay living on $\mathbb{C}^*$ while the momenta are upgraded from $\mathbb{C}$ to $\mathbb{C}^*$. It is then natural to conjecture that a generating function of appropriate elliptic genera of Laumon spaces recovers the formal eigenfunction of the n-particle dual eRS system.

To elliptically deform the position variables, one should consider *affine* Laumon spaces. These are now certain moduli spaces of parabolic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. In later work [Neg11], Neguţ was able to show as well that an analogous generating function of equivariant integrals for affine Laumon spaces yields the formal eigenfunction for the *non-stationary deformation* of the elliptic Calogero-Moser system. Shiraishi's conjectural formula for the formal eigenfunction of the eRS system comes from upgrading this to K-theory, although one obtains directly from the geometry a similar non-stationary deformation which no longer has an integrable systems intepretation. Proving Shiraishi's conjecture would be the next rung of the ladder. Following that, it would be interesting to apply the correct elliptic cohomological analogue found from ordinary Laumon spaces to this affine setting to obtain a conjectural formula for the formal eigenfunction of the DELL system.

2.2.3. *Dynamical parameters and elliptic stable envelopes.* The equivariant elliptic cohomology of symplectic resolutions has received much recent attention thanks to the beautiful *3d mirror symmetry conjecture* of Aganagic and Okounkov for their [AO16]. A slight difference is that the authors consider what they call *extended equivariant elliptic cohomology.* Here, the base $Ell_T(pt) = E^{\dim T}$ is extended to $E^{\dim T} \times (\mathrm{Pic}(X) \otimes_{\mathbb{Z}} E^\vee)$. On $E \times E^\vee$, one has the *Poincaré line bundle* $\mathbb{L}$. Roughly speaking, in extended equivariant elliptic cohomology, one replaces $\mathcal{O}(Ell_T(X))$ with a suitable pullback of $\mathbb{L}$ onto $Ell_T(X)$. The extra $E^\vee$ factors contribute analogues of the *dynamical parameters* from the theory of elliptic quantum groups. Aganagic and Okounkov define in this extension their *elliptic stable envelopes.*

The particular case of $\mathrm{Hilb}_n\mathbb{C}^2$ has been studied by A. Smirnov [Smi18]. Here, $\mathrm{Pic}(\mathrm{Hilb}_n\mathbb{C}^2)$ has rank 1, so there is a single dynamical parameter. Dynamical extensions of elliptic shuffle algebras have been defined and studied by Yang and Zhao [YZ17]. After working out 2.2.1 and the first paragraph of 2.2.2, I would be interested in seeing how much of those results could be lifted to this dynamical setting. This would give a functional interpretation of the elliptic stable envelopes, which should be some remarkable special functions. Moreover, I would be curious to see if the dynamical parameter is bispectral dual to the non-stationary deformation parameter.

References

[AB84] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[AO16] Mina Aganagic and Andrei Okounkov. Elliptic stable envelopes. *arXiv preprint arXiv:1604.00423*, 2016.

[BC14] Alexei Borodin and Ivan Corwin. Macdonald processes. *Probab. Theory Related Fields*, 158(1-2):225–400, 2014.

[BF14] Roman Bezrukavnikov and Michael Finkelberg. Wreath Macdonald polynomials and the categorical McKay correspondence. *Camb. J. Math.*, 2(2):163–190, 2014. With an appendix by Vadim Vologodsky.

[BFN18] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $N = 4$ gauge theories, II. *Adv. Theor. Math. Phys.*, 22(5):1071–1147, 2018.

[BFS14] Alexander Braverman, Michael Finkelberg, and Jun'ichi Shiraishi. Macdonald polynomials, Laumon spaces and perverse coherent sheaves. In *Perspectives in representation theory*, volume 610 of *Contemp. Math.*, pages 23–41. Amer. Math. Soc., Providence, RI, 2014.

[Bra06] Alexander Braverman. Spaces of quasi-maps into the flag varieties and their applications. In *International Congress of Mathematicians. Vol. II*, pages 1145–1170. Eur. Math. Soc., Zürich, 2006.

[BW18] Alexei Borodin and Michael Wheeler. Coloured stochastic vertex models and their spectral theory. *arXiv preprint arXiv:1808.01866*, 2018.

[CF18] Oleg Chalykh and Maxime Fairon. On the Hamiltonian formulation of the trigonometric spin Ruijsenaars-Schneider system. *arXiv preprint arXiv:1811.08727*, 2018.

[Che05] Ivan Cherednik. *Double affine Hecke algebras*, volume 319 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[FHH+09] B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, and S. Yanagida. A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials. *J. Math. Phys.*, 50(9):095215, 42, 2009.

[FT11] B. L. Feigin and A. I. Tsymbaliuk. Equivariant K-theory of Hilbert schemes via shuffle algebra. *Kyoto J. Math.*, 51(4):831–854, 2011.

[FT16] Boris Feigin and Alexander Tsymbaliuk. Bethe subalgebras of $U_q(\widehat{\mathfrak{gl}}_n)$ via shuffle algebras. *Selecta Math. (N.S.)*, 22(2):979–1011, 2016.

[FT17] Michael Finkelberg and Alexander Tsymbaliuk. Multiplicative slices, relativistic Toda and shifted quantum affine algebras. *arXiv preprint arXiv:1708.01795*, 2017.

[Gan14] Nora Ganter. The elliptic Weyl character formula. *Compos. Math.*, 150(7):1196–1234, 2014.

[GKV97] Victor Ginzburg, Mikhail Kapranov, and Eric Vasserot. Residue construction of Hecke algebras. *Adv. Math.*, 128(1):1–19, 1997.

[Gor06] Iain Gordon. A remark on rational Cherednik algebras and differential operators on the cyclic quiver. *Glasg. Math. J.*, 48(1):145–160, 2006.

[Gor08] I. G. Gordon. Quiver varieties, category $\mathcal{O}$ for rational Cherednik algebras, and Hecke algebras. *Int. Math. Res. Pap. IMRP*, (3):Art. ID rpn006, 69, 2008.

[GS06] I. Gordon and J. T. Stafford. Rational Cherednik algebras and Hilbert schemes. II. Representations and sheaves. *Duke Math. J.*, 132(1):73–135, 2006.

[Hai03] Mark Haiman. Combinatorics, symmetric functions, and Hilbert schemes. In *Current developments in mathematics, 2002*, pages 39–111. Int. Press, Somerville, MA, 2003.

[Jor14] David Jordan. Quantized multiplicative quiver varieties. *Adv. Math.*, 250:420–466, 2014.

[KS19] Peter Koroteev and Shamil Shakirov. The quantum DELL system. *arXiv preprint arXiv:1906.10354*, 2019.

[Mac15] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, second edition, 2015. With contribution by A. V. Zelevinsky and a foreword by Richard Stanley, Reprint of the 2008 paperback edition [MR1354144].

[McG12] Kevin McGerty. Microlocal KZ-functors and rational Cherednik algebras. *Duke Math. J.*, 161(9):1657–1709, 2012.

[Mik99] Kei Miki. Toroidal braid group action and an automorphism of toroidal algebra $U_q(\mathrm{sl}_{n+1,\mathrm{tor}})$ $(n \geq 2)$. *Lett. Math. Phys.*, 47(4):365–378, 1999.

[MO19] Davesh Maulik and Andrei Okounkov. Quantum groups and quantum cohomology. *Astérisque*, (408):ix+209, 2019.

[Nag09] Kentaro Nagao. K-theory of quiver varieties, q-Fock space and nonsymmetric Macdonald polynomials. *Osaka J. Math.*, 46(3):877–907, 2009.

[Neg09] Andrei Neguţ. Laumon spaces and the Calogero-Sutherland integrable system. *Invent. Math.*, 178(2):299–331, 2009.

[Neg11] Andrei Neguţ. Affine Laumon spaces and integrable systems. *arXiv preprint arXiv:1112.1756*, 2011.

[Neg13] Andrei Neguţ. Quantum toroidal and shuffle algebras. *arXiv preprint arXiv:1302.6202*, 2013.

[Neg14] Andrei Neguţ. The shuffle algebra revisited. *Int. Math. Res. Not. IMRN*, (22):6242–6275, 2014.

[Rai17] Eric M Rains. Elliptic double affine Hecke algebras. *arXiv preprint arXiv:1709.02989*, 2017.

[Rui09] S. N. M. Ruijsenaars. Hilbert-Schmidt operators vs. integrable systems of elliptic Calogero-Moser type. II. The A_{N-1} case: first steps. *Comm. Math. Phys.*, 286(2):659–680, 2009.

[Sai98] Yoshihisa Saito. Quantum toroidal algebras and their vertex representations. *Publ. Res. Inst. Math. Sci.*, 34(2):155–177, 1998.

[Shi19] Jun'ichi Shiraishi. Affine screening operators, affine Laumon spaces, and conjectures concerning non-stationary Ruijsenaars functions. *arXiv preprint arXiv:1903.07495*, 2019.

[Smi18] Andrey Smirnov. Elliptic stable envelope for hilbert scheme of points in the plane. *arXiv preprint arXiv:1804.08779*, 2018.

[SV13] Olivier Schiffmann and Eric Vasserot. The elliptic Hall algebra and the K-theory of the Hilbert scheme of $\mathbb{A}^2$. *Duke Math. J.*, 162(2):279–366, 2013.

[Tsy19] Alexander Tsymbaliuk. Several realizations of Fock modules for toroidal $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. *Algebr. Represent. Theory*, 22(1):177–209, 2019.

[Ugl98] Denis Uglov. Yangian Gelfand-Zetlin bases, $\mathfrak{gl}_N$-Jack polynomials and computation of dynamical correlation functions in the spin Calogero-Sutherland model. *Comm. Math. Phys.*, 191(3):663–696, 1998.

[VV96] M. Varagnolo and E. Vasserot. Schur duality in the toroidal setting. *Comm. Math. Phys.*, 182(2):469–483, 1996.

[VV99] M. Varagnolo and E. Vasserot. On the K-theory of the cyclic quiver variety. *Internat. Math. Res. Notices*, (18):1005–1028, 1999.

[Wen19] Joshua Jeishing Wen. Wreath Macdonald polynomials as eigenstates. *arXiv preprint arXiv:1904.05015*, 2019.

[YZ17] Yaping Yang and Gufang Zhao. Quiver varieties and elliptic quantum groups. *arXiv preprint arXiv:1708.01418*, 2017.