MATH 241, Calculus III

Exam III

23 April 2015

Name: ____________________________________________________________

- You have 50 minutes to complete this exam.
- Show ALL your work.
- There are NO calculators allowed on this exam.
- This exam has 8 questions worth a total of 100 points.
- Check to see that you have all of the pages. Including the cover sheet, each exam has 7 pages.

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1
1. (10 points) Calculate the iterated integral by changing the order of integration.

\[ \int_0^1 \int_x^1 x \sqrt{y^2 - x^2} \, dy \, dx \]

\[ \int_0^1 \int_0^y x \sqrt{u} \, dx \, dy \]

\( u = y^2 - x^2 \)
\( du = -2x \, dx \)
\( -\frac{du}{2} = x \, dx \)

\[ \int_0^y u \, du = \frac{2}{3} u^{3/2} \]

\[ -\frac{1}{2} \int_0^1 \frac{2}{3} (y^2 - u^2)^{3/2} \, dy \]

\[ = -\frac{1}{2} \int_0^1 -\frac{2}{3} y^3 \, dy \]

\[ = \frac{1}{3} \left( -\frac{2}{3} y^3 \right)_0 \]

\[ = \frac{1}{3} \left[ \frac{1}{4} y^4 \right]_0 \]

\[ = \frac{1}{12} \left[ \frac{1}{4} - 0 \right] \]

\[ = \frac{1}{12} \]
2. Consider the following triple integral:

\[ \int_0^1 \int_0^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z^2 \, dz \, dx \, dy \]

and let \( E \) denote the region of integration.

(a) (5 points) Circle the picture of \( E \).

(b) (10 points) Transform the integral to spherical coordinates (do NOT evaluate).

\[ \int_0^\pi \int_0^\frac{\pi}{4} \int_0^{\rho \cos \phi} \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta \]

Cone:
\[ z = \rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi} = \rho \sin \phi \quad (0 \leq \phi \leq \pi) \]

\[ \Rightarrow \phi = \frac{\pi}{4} \]
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3. Let $R$ be the region bounded by $x + y = 1$, $x + y = -1$, $x - y = 1$, and $x - y = -1$. Let $S = [-1, 1] \times [-1, 1]$.

\[ T \]

\[ y = x + 1 \]
\[ y = 1 - x \]
\[ y = -1 - x \]
\[ y = x - 1 \]

(a) (10 points) Give a transformation, $T(u, v)$, from the square $S$ to the diamond $R$.

\[
\begin{align*}
u &= x + y \\
v &= y - x \\
u + v &= 2y \\
u - v &= 2x
\end{align*}
\]

\[ T(u, v) = \left( \frac{u - v}{2}, \frac{u + v}{2} \right) \]

\[-1 \leq u \leq 1 \quad -1 \leq v \leq 1\]

(b) (10 points) Using the transformation from part (a), evaluate $\int_R e^{x+y} \, dA$.

\[ \int_{-1}^{1} \int_{-1}^{1} e^{\frac{u+v}{2}} \left( \frac{1}{2} \right) \, du \, dv \]

\[ J = \left| \frac{1}{2} \right| \left( \frac{1}{2} \right) = \frac{1}{4} - \left( \frac{1}{4} \right) \]

\[ = \frac{1}{2} \]

\[ = \frac{1}{2} \int_{-1}^{1} \left( \frac{1}{2} \right) \, dv \]

\[ = \frac{1}{2} \int_{-1}^{1} e^{\frac{u}{2}} \, dv \]

\[ = e^{-\frac{1}{2}} \]

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4. (10 points) Let $C_1$ and $C_2$ be two curves in the $xy$-plane shown below, each with clockwise orientation. Let $\mathbf{F}(x,y) = (7y, 2x + e^y)$.

![Diagram of two curves $C_1$ and $C_2$ in the $xy$-plane]

**True or False:** $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

**JUSTIFY** your answer.

$$\int_{C_1}^{C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_D (P_x - Q_y) \, dA = \iint_D 2 \, dA = 2\iint_D dA$$

$$\int_{C_1}^{C_2} \mathbf{F} \cdot d\mathbf{r} = -\iint_D dA 
eq 0$$

5. (10 points) Let $C$ be a simple closed planar curve, oriented counterclockwise, and let $D$ be the region bounded by $C$. Which ONE of the following does NOT give the area of $D$? (No partial credit without justification.)

A. $\int_C x \, dy = \iint_D 1 - 0 \, dA$

B. $\iint_D 1 \, dA$

C. $\int_C x \, dy - \frac{y}{2} \, dx = \iint_D \left( \frac{1}{2} - \frac{1}{2} \right) \, dA = \iint_D dA$

D. $\int_C y \, dx = \iint_D 0 - 1 \, dA \neq \iint_D 1 \, dA$
6. Let $S$ be the portion of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

(a) (10 points) Find a parametrization for $S$.

\[ \vec{r}(r, \theta) = (r^2 \cos \theta, r \sin \theta, 0) \quad \text{for} \quad 0 \leq r \leq 3, \quad 0 \leq \theta \leq 2\pi \]

\[ \vec{r}(u, v) = (u^2 + v^2, u, v) \]

(b) (10 points) Set up an integral which gives the surface area of $S$ (do NOT evaluate).

\[ \int_{-3}^{3} \int_{-\sqrt{9-u^2}}^{\sqrt{9-u^2}} \sqrt{1 + \left( \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial x}{\partial v} \right)^2} \, dv \, du \]

\[ \int_{-3}^{3} \int_{-\sqrt{9-u^2}}^{\sqrt{9-u^2}} \sqrt{1 + \left( \frac{\partial x}{\partial u} \right)^2 + \left( \frac{\partial x}{\partial v} \right)^2} \, dv \, du \]

\[ \int_{0}^{2\pi} \int_{0}^{3} \sqrt{r^2 + 4r^4} \, r \, dr \, d\theta \]
7. (5 points) Circle the graph of the parametric surface given by:

\[ x = u, \quad y = \cos u \sin 2v, \quad z = \sin u \sin 2v \]

A.  
B.  
C.  
D.  

8. (10 points) Circle ALL that are TRUE. (No partial credit without justification.)

A. The integral

\[ \int_0^{2\pi} \int_0^2 \int_0^2 r \, dz \, dr \, d\theta \]

represents the volume enclosed by the cone \( z = \sqrt{x^2 + y^2} \) and the plane \( z = 2 \).

B. There is a vector field \( \mathbf{F} \) such that \( \text{curl} (\mathbf{F}) = (3, x, z) \).

C. If \( P \) and \( Q \) have continuous partial derivatives on an open region containing \( D \), then

\[ \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl} \, dA \]

D. If \( P \) and \( Q \) have continuous partial derivatives on an open region containing \( D \), then

\[ \oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \text{div} (\mathbf{F}) \, dA \]

E. A-D are all False.