Math 241, Calculus III
Midterm I
19 February 2015

1. (15 points) Consider the points \( A = (1, 2, 3) \), \( B = (0, 2, 1) \), and \( C = (4, 0, 3) \) in \( \mathbb{R}^3 \).

(a) Find a normal vector \( \mathbf{n} \) to the plane \( P \) containing the points \( A, B, C \).

\[
\overrightarrow{AB} = \langle -1, 0, -2 \rangle \\
\overrightarrow{AC} = \langle 3, -2, 0 \rangle \\
\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-1 & 0 & -2 \\
3 & -2 & 0
\end{vmatrix} = \langle 4 \mathbf{i} - (0 + 6) \mathbf{j} + (2 - 3) \mathbf{k} \rangle \\
= \langle -4, -6, 2 \rangle
\]

(b) Give an equation for the plane \( P \).

\[
\langle -4, -6, 2 \rangle \cdot \langle x-1, y-2, z-3 \rangle = 0 \\
-4(x-1) + 6(y-2) + 2(z-3) = 0
\]

(c) Find the area of the parallelogram spanned by the vectors \( \mathbf{AB} \) and \( \mathbf{AC} \).

\[
\text{Area} = |\overrightarrow{AB} \times \overrightarrow{AC}| = |\langle -4, -6, 2 \rangle| \\
= \sqrt{(-4)^2 + (-6)^2 + 2^2} \\
= \sqrt{16 + 36 + 4} \\
= \sqrt{56}
\]

(d) Consider the line \( L \) given by the parametrization \( \mathbf{r}(t) = (3 + t, -2 - t, 2) \). Is \( L \) parallel to the plane \( P \)? Why or why not?

For \( L \) to be parallel to \( P \), it must be perpendicular to the direction vector of \( L \).

\[
\langle -4, -6, 2 \rangle \cdot \langle 1, 1, 0 \rangle = -4(1) + 6(1) + 2(0) = 2 \neq 0
\]

Since the dot product is not 0, this is not the case.
2. (6 points) Let \( L \) be the line parametrized by \( \mathbf{r}(t) = \mathbf{a} + t\mathbf{b} \) where \( \mathbf{a} = (-3, 0, -3) \) and \( \mathbf{b} = (3, 2, 4) \). Find the point \( Q \) of intersection of the line \( L \) with the plane \( 3x - y + 2z = 15 \).

\[
\begin{align*}
  x &= -3 + 3t \\
  y &= 2t \\
  z &= -3 + 4t
\end{align*}
\]

\[
3((-3 + 3t) - (2t) + 2(-3 + 4t)) = 15
\]

\[
-9 + 9t - 2t - 6 + 8t = 15
\]

\[
-15 + 15t = 15
\]

\[
15t = 30
\]

\[
t = 2
\]

\[
Q = (-3, 4, 5)
\]

3. (5 points) Find the midpoint \( M \) of the straight-line segment between the points \( P = (5, 2, 6) \) and \( Q = (7, -1, 3) \).

\[
\begin{align*}
  M &= \frac{P + Q}{2} \\
  &= \left( \frac{5 + 7}{2}, \frac{2 - 1}{2}, \frac{6 + 3}{2} \right) \\
  &= \left( 6, \frac{1}{2}, \frac{9}{2} \right)
\end{align*}
\]
4. (18 points) For each function

\[ (a) \ x^2 - y^2 \quad (b) \ y \cos(x) \quad (c) \ e^{-x^2-y^2} \]

label its graph and its level set diagram from the options below. Here each level set diagram consists of level sets \( \{ f(x, y) = c_i \} \) drawn for evenly spaced \( c_i \).
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5. (15 points) EXACTLY one of the following limits exists. Circle the one that exists AND either evaluate it or prove the other limit does not exist.

\[ \lim_{(x,y) \to (0,0)} \frac{yx^2 + y^2x}{x^3 + y^3} \]

Along \( x = 0 \), \( F(0,y) = \frac{0}{y^3} = 0 \)
Along \( y = x \), \( F(x,x) = \frac{x^3 + x^3}{x^3 + x^3} = 1 \)

Thus \( F \) approaches different values depending on how we approach \( (0,0) \)
Thus the first limit does not exist and the second one does.

\[ \lim_{(x,y) \to (0,0)} \frac{xy^4}{(x^2 + y^2)^2} \]

6. (8 points) Is \( u = e^{-4t} \sin(2x) \) a solution of the heat equation, that is, does \( u_t = u_{xx} \)?

\[ u_t = -4e^{-4t} \sin(2x) \]
\[ u_x = 2e^{-4t} \cos(2x) \]
\[ u_{xx} = -4(2) e^{-4t} \sin(2x) \]
\[ = -4 e^{-4t} \sin(2x) \]
\[ = u_t \]

Yes \( u = e^{-4t} \sin(2x) \) is a solution of the heat equation.
7. (15 points) This question has two parts. Level curves are shown for a function \( f \).

(a) Determine whether the following are positive, negative, or zero at the point \( P \).

\[
\begin{align*}
&f_x & f_y & f_{xx} & f_{yy} & f_{xy} & D_v(f) \\
&+ & + & + & + & + & \\
&- & - & - & - & - & \\
\end{align*}
\]

(b) Draw a vector in the direction of \( \nabla f(p) \) on the picture.

8. (10 points) Find an equation of the tangent plane to \( z = xy + \ln(x^2 + y^2) \) at \((x, y) = (0, 1)\).

\[
\begin{align*}
F(x, y) &= xy + \ln(x^2 + y^2) \\
F_x &= y + \frac{2x}{x^2 + y^2} \\
F_y &= x + \frac{2y}{x^2 + y^2} \\
F_x(0, 1) &= 1 + \frac{0}{1} = 1 \\
F_y(0, 1) &= 0 + \frac{2}{1} = 2 \\
F(0, 1) &= 0 + \ln(1) = 0 \\
Z &= F(0, 1) + F_x(0, 1)(x-0) + F_y(0, 1)(y-1) \\
&= 0 + 1(x-0) + 2(y-1) \\
&= x + 2y - 2
\end{align*}
\]
9. (8 points) Suppose $f$ is a differentiable function of $x$ and $y$, and $g(u,v) = f(u^2 + \cos v, e^v + \sin u)$. Use the table of values to calculate $g_u(0,0)$ and $g_v(0,0)$.

<table>
<thead>
<tr>
<th></th>
<th>$f$</th>
<th>$g_x$</th>
<th>$f_x$</th>
<th>$f_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>5</td>
<td>2</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>(1,1)</td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>1</td>
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</tbody>
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