Flatness in algebraic geometry: a family of conics in $\mathbb{A}^3_{\mathbb{C}}$

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June 1, 2018

Assume that all rings are commutative with 1.

1 A family of conic curves

Here is the real part of a family of conics defined by the equation $x^2 + y^2 - t = 0$ in the affine space $\mathbb{A}^3_{\mathbb{C}} = \text{Spec}(\mathbb{C}[x, y, t])$.

The $\mathbb{C}$-algebra morphism $\varphi : \mathbb{C}[t] \to \frac{\mathbb{C}[x, y, t]}{(x^2 + y^2 - t)}$ defined by $\varphi(t) = t + (x^2 + y^2 - t)$ gives the morphism of affine schemes

$$f : \text{Spec} \left( \frac{\mathbb{C}[x, y, t]}{(x^2 + y^2 - t)} \right) \to \text{Spec}(\mathbb{C}[t])$$

defined by $f(p) = \varphi^{-1}(p)$ for each prime ideal $p \in \text{Spec} \left( \frac{\mathbb{C}[x, y, t]}{(x^2 + y^2 - t)} \right)$.

In this section we will show that $f$ is a flat morphism using local criterion of flatness and also other criterion for flatness.

First we observe this morphism. The only prime ideals in $\mathbb{C}[t]$ are $(t + a)$ or $(0)$ for $a \in \mathbb{C}$, so the image of $\varphi$ is $(t + a)$ or $(0)$.

For example,

- $p = (x + (x^2 + y^2 - t), y + (x^2 + y^2 - t)) \Rightarrow f(p) = \varphi^{-1}(p) = (0)$,
- $p = (x + a + (x^2 + y^2 - t), y + b + (x^2 + y^2 - t)) \Rightarrow f(p) = \varphi^{-1}(p) = (0)$ for any $a, b \in \mathbb{C}$,
- $p = (x + (x^2 + y^2 - t), y + (x^2 + y^2 - t), t + (x^2 + y^2 - t)) \Rightarrow f(p) = \varphi^{-1}(p) = (t)$,
- $p = (t + (x^2 + y^2 - t)) \Rightarrow f(p) = \varphi^{-1}(p) = (t)$,
- $p = (t + a + (x^2 + y^2 - t)) \Rightarrow f(p) = \varphi^{-1}(p) = (t + a)$ for $a \in \mathbb{C}$.
The topological fiber at $q$ is $f^{-1}(q)$, and the fiber at $q$ is

$$\text{Spec} \left( \frac{\mathbb{C}[x,y,t]}{(x^2 + y^2 - t)} \right) \times_{\text{Spec}(\mathbb{C}[t])} \text{Spec}(\mathbb{C}(q)),$$

where $q \in \text{Spec}(\mathbb{C}[t])$, and $\mathbb{C}(q)$ is the residue field $\mathbb{C}[t]_q / \mathfrak{m}$ of the local ring $(\mathbb{C}[t]_q, \mathfrak{m})$, where $\mathfrak{m} = q\mathbb{C}[t]_q$, and $t = (t - a)$ or $(0)$ for $a \in \mathbb{C}$. 

For example, the residue field at $q = (t)$ is $\mathbb{C}(q) = \mathbb{C}[t]_{(t)}/(t)\mathbb{C}[t]_{(t)} \cong \mathbb{C}$ (The kernel of the map $\mathbb{C}[t] \to \mathbb{C}[t]_{(t)} \to \mathbb{C}[t]_{(t)}/(t)\mathbb{C}[t]_{(t)}$ is $(t)$), so

$$\text{Spec} \left( \frac{\mathbb{C}[x,y,t]}{(x^2 + y^2 - t)} \right) \times_{\text{Spec}(\mathbb{C}[t])} \text{Spec}(\mathbb{C}) = \text{Spec} \left( \frac{\mathbb{C}[x,y,t]}{(x^2 + y^2 - t)} \otimes_{\mathbb{C}[t]} \mathbb{C} \right) = \text{Spec} \left( \frac{\mathbb{C}[x,y]}{(x^2 + y^2)} \right).$$

We observe the following. Let $X = \text{Spec} \left( \frac{\mathbb{C}[x,y,t]}{(x^2 + y^2 - t)} \right)$, $Y = \text{Spec} (\mathbb{C}[t])$. We will prove that $f$ is a flat morphism.

1. Every fiber of the map $f$ has the same dimension 1.

The fiber at the point corresponding to a prime ideal $q \subset \mathbb{C}[t]$ is

$$\text{Spec} \left( \frac{\mathbb{C}[x,y,t]}{(x^2 + y^2 - t)} \right) \times_{\text{Spec}(\mathbb{C}[t])} \text{Spec}(\mathbb{C}(q)) = \text{Spec} \left( \frac{\mathbb{C}[x,y,t]}{(x^2 + y^2 - t)} \otimes_{\mathbb{C}[t]} \mathbb{C}(q) \right).$$

We compute the tensor product of rings.

$$\frac{\mathbb{C}[x,y,t]}{(x^2 + y^2 - t)} \otimes_{\mathbb{C}[t]} \mathbb{C}(q) = \begin{cases} \frac{\mathbb{C}[x,y]}{(x^2 + y^2 - t)} & \text{if } q = (t - a) \\ \frac{\mathbb{C}(t)[x,y]}{(x^2 + y^2 - t)} & \text{if } q = (0). \end{cases}$$

Since $t^n = t^n - a^n + a^n = (t - a)(t^{n-1} + t^{n-2}a + \ldots + a^{n-1}) + a^n$, tensoring $\mathbb{C}((t - a))$ is plugging in $t = a$. At the generic point $q = (0)$ the residue field is $\mathbb{C}(q) = \mathbb{C}(t)$, so $t$ is a unit. Hence the dimension of each of the fibers is 1.

The fiber at $q = (t)$ is the union of two affine lines, and has singularity at $(0,0)$. The fiber at $q = (t - a)$ with $a \neq 0$ is a nonsingular conic curve.

Since the dimension of fiber is 1, $\dim X_a = \dim X - \dim Y$ at a point $a \in Y$. Here $\dim X = 2$ and $\dim Y = 1$.

The fiber $X_a = \begin{cases} V(x^2 + y^2) & \text{if } a = 0 \\ V(x^2 + y^2 - a) & \text{if } a \neq 0. \end{cases}$

The fiber $X_0$ is the union of two affine lines, and has singularity at $(0,0)$. $X_a$ is nonsingular conic curve for every $a \neq 0$.

Exercises Problems:

2. Prove that $f(D(h))$ is an open set for every $h \in \frac{\mathbb{C}[x,y,t]}{(x^2 + y^2 - t)}$, i.e., $f$ is an open morphism.

3. Is $f$ smooth of relative dimension 1? In other words, prove or disprove that
   (i) $f$ is flat;
   (ii) if $X' \subseteq X$ and $Y' \subseteq Y$ are irreducible components such that $f(X') \subseteq Y'$, then $\dim X' = \dim Y'$. 

dim \( Y' + 1 \);

(iii) for each point \( x \in X \) (closed or not),

\[
dim_\mathbb{K}(x)(\Omega_{X/Y} \otimes k(x)) = 1.
\]

(4) Is \( f \) faithfully flat? In other words, prove or disprove that for every prime ideal \( p \subset \mathbb{C}[x,y,t] \), \( \mathcal{O}_{p,X} \) is a flat \( \mathcal{O}_{f(p),Y} \)-module, and for every prime ideal \( p \subset \mathbb{C}[x,y,t] \) \( \mathcal{O}_{p,X} \otimes \mathcal{O}_{f(0),Y} \not\equiv 0 \) for every nonzero \( \mathcal{O}_{f(p),Y} \)-module \( M \).

The \( \mathbb{C} \)-algebra morphism \( \varphi \) makes \( \mathbb{C}[x,y,t] \) a \( \mathbb{C}[t] \)-algebra, and \( \varphi \) induces a local morphism \( f^# : \mathbb{C}[t]_{f(p)} \to \left( \mathbb{C}[x,y,t] \right)_p \) for each \( p \in \text{Spec} \left( \mathbb{C}[x,y,t] \right) \). \( X \) is flat over \( Y \) via the morphism \( f \) (or \( f \) is a flat morphism) if \( \mathcal{O}_X \) is flat over \( Y \), where \( X = \text{Spec} \left( \mathbb{C}[x,y,t] \right) \) and \( Y = \text{Spec} \left( \mathbb{C}[t] \right) \), i.e., \( \mathcal{O}_{p,X} \) is a flat \( \mathcal{O}_{f(p),Y} \)-module for all \( p \in X \).

Clearly, \( \mathcal{O}_{p,X} \) is a finitely generated \( \mathcal{O}_{p,X} \)-module for all \( p \in X \). So, by the local criterion for flatness, \( \mathcal{O}_{p,X} \) is a flat \( \mathcal{O}_{f(p),Y} \)-module if and only if \( \text{Tor}_1^{\mathbb{C}[t]} (\mathcal{O}_{p,X}, \mathbb{C}(f(p))) = 0 \) for every \( p \in X \).

We know that \( \mathcal{O}_{p,X} = \left( \mathbb{C}[x,y,t] \right)_p \) and \( \mathcal{O}_{f(p),Y} = \mathbb{C}[t]_{f(p)} \).

\[
\text{Tor}_1^{\mathbb{C}[t]} (\mathcal{O}_{p,X}, \mathbb{C}(f(p))) = \begin{cases} \text{Tor}_1^{\mathbb{C}[t]} (\mathcal{O}_{p,X}, \mathbb{C}(t)) & \text{if } f(p) = (0), \\ \text{Tor}_1^{\mathbb{C}[t]} (\mathcal{O}_{p,X}, \mathbb{C}) & \text{if } f(p) \not= (0) \text{ (i.e., } f(p) = (t + a)). \end{cases}
\]

Case 1: \( f(p) = (0) \)

\[
\text{Tor}_1^{\mathbb{C}[t]} (\mathcal{O}_{p,X}, \mathbb{C}(f(p))) = \text{Tor}_1^{\mathbb{C}[t]} \left( \mathbb{C}[x,y,t] \left( \frac{C}{x^2 + y^2 - t} \right) \right)_{(0)} \to \mathbb{C}[t]_{(0)} \to \mathbb{C}[t]_{(0)} = 0
\]

since \( \text{Tor}_1^{\mathbb{C}[t]} \left( \mathbb{C}[x,y,t] \left( \frac{C}{x^2 + y^2 - t} \right), \mathbb{C}[t] \right) = \text{Tor}_1^{\mathbb{C}[t]} \left( \mathbb{C}[t], \mathbb{C}[x,y,t] \left( \frac{C}{x^2 + y^2 - t} \right) \right) = 0 \).

Case 2: \( f(p) \not= (0) \) (i.e., \( f(p) = (t + a) \))

We resolve \( \mathbb{C} = \mathbb{C}[t]_{(t+a)} / (t+a)\mathbb{C}[t]_{(t+a)} \) as a \( \mathbb{C}[t] \)-module.

\[
0 \to \mathbb{C}[t] \to \mathbb{C}[t] \to \mathbb{C} \to 0
\]

Tensoring with \( \left( \mathbb{C}[x,y,t] \left( \frac{C}{x^2 + y^2 - t} \right) \right)_p \) gives the following complex of \( \mathbb{C}[t] \)-modules:

\[
0 \to \left( \mathbb{C}[x,y,t] \left( \frac{C}{x^2 + y^2 - t} \right) \right)_p \otimes \mathbb{C}[t] \mathbb{C}[t] \to \left( \mathbb{C}[x,y,t] \left( \frac{C}{x^2 + y^2 - t} \right) \right)_p \otimes \mathbb{C}[t] \mathbb{C}[t] \to 0
\]

Since \( 1 \otimes (t) \) is injective, \( \text{Tor}_1^{\mathbb{C}[t]} (\mathcal{O}_{p,X}, \mathbb{C}) = 0 \). Thus, \( X \) is flat over \( Y \).

We also use the following criterion (Theorem 2) for flatness to show that \( X \) is flat over \( Y \).
Lemma 1. If $M$ is a nonzero module over a ring $A$, then there exists a maximal ideal $I \subset A$ such that $M_I := M \otimes_A A_I \neq 0$. More generally, the homomorphism $m \mapsto (m \otimes 1)_I$ from $M$ to $\prod_I M_I$.

Proof. Let $\varphi : M \to M_I := M \otimes_A A_I$ $(\varphi(m) = m \otimes 1)$. Then $m \in \ker(\varphi)$ if and only if $m \otimes 1 = 0$ if and only if $r(m \otimes 1) = rm \otimes 1 = 0$ for some $r \notin I$ if and only if $rm = 0$ for some $r \notin I$, i.e., $r \in \text{Ann}(m)$ for some $r \notin I$. Note that $\text{Ann}(m) = A$ if and only if $m = 0$. By assumption, $M \neq 0$, so $\text{Ann}(m_0) \neq A$ for some $m_0 \neq 0$. There is a maximal ideal $I_0 \subset A$ containing $\text{Ann}(m_0)$. So, $m_0 \notin \ker(M \to M_{I_0})$, i.e., $m_0 \otimes 1 \neq 0$, so $M_{I_0} \neq 0$.

Theorem 1. (Proposition 9.2, [Hartshorne])
Let $\varphi : A \to B$ be a homomorphism of rings, and let $M$ be a $B$-module. Let $f : X \to Y$ be the corresponding morphism of affine schemes, where $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$, and let $\mathcal{F} = \widetilde{M}$. Then $\mathcal{F}$ is flat over $Y$ if and only if $M$ is a flat $A$-module.

Proof. We must show that $M_q := M \otimes_B B_q$ is a flat $A_{\varphi^{-1}(q)}$-module for every prime ideal $q \subset B$ if and only if $M$ is a flat $A$-module. Suppose $M$ is a flat $A$-module. Then $M_q := M \otimes_A A_q$ is a flat $A_q$-module for every prime ideal $p \subset A$. So, for every prime ideal $q \subset B, M_{\varphi^{-1}(q)} := M \otimes_A A_{\varphi^{-1}(q)}$ is a flat $A_{\varphi^{-1}(q)}$-module. The ring homomorphism $\varphi : A \to B$ induces the ring homomorphism $A_{\varphi^{-1}(q)} \to B_q$. By the base extension theorem, $M_q := M \otimes_B B_q = M_{\varphi^{-1}(q)} \otimes_{A_{\varphi^{-1}(q)}} B_q$ is a flat $A_{\varphi^{-1}(q)}$-module. Conversely, suppose that for some exact sequence $0 \to L \to N$ of $A$-modules, $L \otimes_A M \to N \otimes_A M$ has a nonzero kernel $K$, i.e.,

$$0 \to K \to L \otimes_A M \to N \otimes_A M.$$

This is an exact sequence of $B$-modules, so by Lemma 1, there exists a prime ideal $q \subset B$ such that $K_q \neq 0$. So,

$$0 \to K \otimes_B B_q \to L \otimes_A (M \otimes_B B_q) \to N \otimes_A (M \otimes_B B_q).$$

$$L \otimes_A M_q = (L \otimes_A A_{\varphi^{-1}(q)}) \otimes_{A_{\varphi^{-1}(q)}} M_q \text{ and } N \otimes_A M_q = (N \otimes_A A_{\varphi^{-1}(q)}) \otimes_{A_{\varphi^{-1}(q)}} M_q.$$
Example 1. For a commutative ring $A$ with 1 and a prime number $p$, let $\mu_{p^n}(A) := \{a \in A \mid a^{p^n} - 1 = 0\}$ be the group scheme with ring $A[x]/(x^{p^n} - 1)$ over Spec$(A)$, i.e., $\mu_{p^n}(A) = \text{Spec}(\mathcal{O}(\mu_n(A))) = \text{Spec}(A[x]/(x^{p^n} - 1))$. Since $\mathcal{O}(\mu_n(A))$ is a free module of rank $p^n$ over $A$, by Corollary 1, the scheme $\mu_n(A)$ is flat over Spec$(A)$. In fact, $\mu_{p^n}$ is a functor from commutative rings to abelian groups for every integer $n \geq 0$.

Example 2. The abelian group $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ is a $\mathbb{Z}$-module. In fact, every $\mathbb{Z}$-module is the same as an abelian group. Since $\text{Tor}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}_2) \neq 0$, $\mathbb{Z}_2$ is not a flat $\mathbb{Z}$-module. This can be interpreted as a ring (a field). Then Spec$(\mathbb{Z}_2)$ is not flat over Spec$(\mathbb{Z})$.

Example 3. The group scheme $\mu_2(\mathbb{Z}) = \text{Spec}(\mathbb{Z}[x]/(x^2 - 1))$ is flat over Spec$(\mathbb{Z})$.

Theorem 2. Let $M$ be a module over a Noetherian ring $A$. Then $M$ is a flat $A$-module if and only if $\text{Tor}^1_{A}(M, A/p) = 0$ for every prime ideal $p \subset A$.

Proof. See [To] for full proof or [Hartshorne2].

Alternatively, we use Theorem 2 to show that the morphism $f : \text{Spec} \left( \frac{\mathbb{C}[x, y, t]}{(x + y^2 - t)} \right) \to \text{Spec}(\mathbb{C}[t])$ of schemes is flat, where $f$ is induced by the morphism $\varphi : \mathbb{C}[t] \to \frac{\mathbb{C}[x, y, t]}{(x + y^2 - t)}$ of rings defined by $\varphi(t) = t + (x^2 + y^2 - t)$. By Corollary 1, $f$ is a flat morphism (or Spec$(\frac{\mathbb{C}[x, y, t]}{(x + y^2 - t)})$ is flat over Spec$(\mathbb{C}[t])$ if and only if $\text{Tor}^1_{\mathbb{C}[t]}(\frac{\mathbb{C}[x, y, t]}{(x + y^2 - t)}, \mathbb{C}[t]/p) = 0$ for every prime ideal $p \subset \mathbb{C}[t]$.

\[
\text{Tor}^1_{\mathbb{C}[t]}(\frac{\mathbb{C}[x, y, t]}{(x^2 + y^2 - t)}, \mathbb{C}[t]/p) = \begin{cases} 
\text{Tor}^1_{\mathbb{C}[t]}(\frac{\mathbb{C}[x, y, t]}{(x^2 + y^2 - t)}, \mathbb{C}[t]) & \text{if } p = (0), \\
\text{Tor}^1_{\mathbb{C}[t]}(\frac{\mathbb{C}[x, y, t]}{(x^2 + y^2 - t)}, \mathbb{C}) & \text{if } p \neq (0) \text{ (i.e., } p = (t + a))
\end{cases}
\]

Case 1: $p = (0)$

\[
\text{Tor}^1_{\mathbb{C}[t]}(\frac{\mathbb{C}[x, y, t]}{(x^2 + y^2 - t)}, \mathbb{C}[t]) = 0 \text{ because the functor } \text{Tor}^1_{\mathbb{C}[t]}(\cdot, \mathbb{C}[t]) \text{ is exact.}
\]

since $\mathbb{C}[t]$ is free over $\mathbb{C}[t]$.

Case 2: $p \neq (0) \text{ (} p = (t + a))$

We resolve $\mathbb{C} = \mathbb{C}[t]_{(t+a)}/(t+a)\mathbb{C}[t]_{(t+a)}$ as a $\mathbb{C}[t]$-module.

\[
0 \to \mathbb{C}[t] \xrightarrow{\cdot t} \mathbb{C}[t] \to \mathbb{C} \to 0.
\]

Tensoring with $\frac{\mathbb{C}[x, y, t]}{(x + y^2 - t)}$ gives the following complex of $\mathbb{C}[t]$-modules:

\[
0 \to \frac{\mathbb{C}[x, y, t]}{(x^2 + y^2 - t)} \otimes_{\mathbb{C}[t]} \mathbb{C}[t] \xrightarrow{1 \otimes (t)} \frac{\mathbb{C}[x, y, t]}{(x + y^2 - t)} \otimes_{\mathbb{C}[t]} \mathbb{C}[t] \to \frac{\mathbb{C}[x, y, t]}{(x^2 + y^2 - t)} \otimes_{\mathbb{C}[t]} \mathbb{C} \to 0
\]

Since $1 \otimes (t)$ is injective, $\text{Tor}^1_{\mathbb{C}[t]}(\frac{\mathbb{C}[x, y, t]}{(x^2 + y^2 - t)}, \mathbb{C}) = 0$. Thus, Spec$(\frac{\mathbb{C}[x, y, t]}{(x^2 + y^2 - t)})$ is flat over Spec$(\mathbb{C}[t])$. 

\[5\]
2 History of flatness

In [Hartshorne2], “The notion of flatness is due to [Serre], who showed that there is a one-to-one correspondence between coherent algebraic sheaves on a projective variety over \( \mathbb{C} \) and the coherent analytic sheaves on the associated complex analytic space. He observed that the algebraic and analytic local rings have the same completion, and that this makes them a “flat couple.” The observation that localization and completion both enjoy this property, and that flat modules are those that are acyclic for the Tor functors, explained and simplified a number of situations by combining them into one concept. Then in the hands of Grothendieck, flatness became a central tool for managing families of structures of all kinds in algebraic geometry. The local criterion of flatness is developed in [Grothendieck2] IV, §5. Our statement is [loc. cit., 5.5]. A note before [loc. cit. 5.2] says “La proposition suivante a été dégagée au moment du Séminaire par Serre; elle permet des simplifications substantielles dans le présent numéro.” (“The next proposition was cleared at the time through Seminar by Serre; it allows the substantial simplification into the present version.”)

The infinitesimal study of the Hilbert scheme is in Grothendieck’s Bourbaki seminar [Grothendieck1] exposé 221.”

In [Eisenbud], Chapter 6 “The notion of flatness was first isolated by Serre [1955 - 1956] and was then systematically developed and mined by Grothendieck. It is now a central theme in algebraic geometry and commutative algebra.”

References


[To] To, Jinhyung: *Deformation Theory*