Review

Suppose \( \Omega \) is a simply connected domain. Suppose \( f \) is analytic on \( \Omega \). Then

1. \[ \int f(z)\,dz = 0 \text{ for every closed contour } C \text{ in } \Omega. \]
2. \( C_1 \) path independent

3. \( f \) has an anti-derivative on \( \Omega \)

Suppose \( u \) is harmonic on \( \Omega \). Then

4. \( u \) has a harmonic conjugate on \( \Omega \).

Let \( \Omega = \mathbb{C} - \{0\} \) - not simply connected.

1. \[ \oint_{|z|=1} \frac{dz}{z} = 2\pi i \neq 0 \]

Also \( \ln |z| \) is harmonic on \( \mathbb{C} - \{0\} \) with no harmonic conjugate.
Suppose $f'(z_0) \neq 0$. Then

1. angles preserved at $z_0$.
2. $f$ is $1 - 1$ on some open $U$ containing $z_0$.

Ex: $f(z) = z^2$, $z_0 = 0$, $f'(0) = 0$

$z^2$ is a $2 - 1$ mapping of $\Delta(0, \delta)$ to $\Delta(0, \delta^2)$

Suppose $f$ is analytic on $\Delta(z_0, \delta)$ s.t. $f'(z_0) = 0$ and $f''(z_0) \neq 0$.

$f(z) = f(z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + a_3(z - z_0)^3 + \ldots$
Such an $f$ doubles angles at $z_0$ and is a $2−1$ mapping on same open $U$ containing $z_0$.

Example: $w = u + iv = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

$f'(z) = \cos z \quad f'(\frac{\pi}{2}) = 0 \quad f''(z) = -\sin z \quad f''(\frac{\pi}{2}) = -1$

Note angles are doubled at $\frac{\pi}{2}$  \(\checkmark\)

is locally 2−1.
A problem from Exam A

1.(i) Where is \( \sin \bar{z} \) differentiable?

\[
\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y
\]

\[
\sin(x - iy) = \sin x \cosh y - i \cos x \sinh y = u + iv
\]

\[
u = \sin x \cosh y \quad u_x = \cos x \cosh y \quad v_x = \sin x \sinh y
\]

\[
v = -\cos x \sinh y \quad u_y = \sin x \sinh y \quad v_y = -\cos x \cosh y
\]

\[
0 \quad \Rightarrow \quad u_x = v_y \iff \cos x \cosh y = 0 \iff \cos x = 0 \quad x = \frac{\pi}{2} + n\pi, \quad n = \text{integer}
\]

\[
u_y = -v_x \iff \sin x \sinh y = 0 \iff \sinh y = 0 \text{ or } y = 0.
\]

\[
z = \frac{\pi}{2} + n\pi, \quad n = \text{integer}
\]
Review

$$f(z) = \frac{1}{(z^2+1)^3}$$

(1st) $\oint_{|z|=2} \frac{dz}{(1+z^2)^3} = 2\pi i \left\{ \text{Res}(f, i) + \text{Res}(f, -i) \right\} = 0$

Suppose $R >> 2$. by deformation of contours

$$\oint_{|z|=2} \frac{dz}{(z^2+1)^3} = \oint_{|z|=R} \frac{dz}{(1+z^2)^3}$$

(2nd) $\left| \oint_{|z|=R} \frac{dz}{(1+z^2)^3} \right| \leq \frac{2\pi R}{(R^2-1)^3} \to 0$ as $R \to \infty$

(3rd) $\oint_{|z|=2} \frac{dz}{(z^2+1)^3} = 2\pi i \text{Res} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right), 0 \right)$

$$f \left( \frac{1}{z} \right) = \frac{1}{\left( \frac{1}{z} \right)^2+1}^3 = \frac{z^6}{(z^2+1)^3}$$

$$\frac{1}{z^2} f \left( \frac{1}{z} \right) = \frac{z^4}{(z^2+1)^3} \text{ analytic on } \Delta(0, 1)$$

$$\text{Res} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right), 0 \right) = 0 \quad \text{integral} = 0.$$