Lecture 3, Math 444

Joseph Miles
Created by Jinhyung To

University of Illinois at Urbana-Champaign
Recall: Countably Infinite Sets

Recall: A set $A$ is countably infinite if $\exists$ bijection $f : \mathbb{N} \to A$. 
Lemma: Infinite Subsets of the Natural Numbers

Lemma 1: If \( A \) is an infinite subset of \( \mathbb{N} \), then \( A \) is countably infinite.

Proof: Using Well-Ordering of \( \mathbb{N} \), we let \( a_1 \) be the smallest element of \( A \).

Note \( A - \{a_1\} \).
By Well-Ordering we choose

\(a_2\), the smallest element of \(A - \{a_1\}\).

Note \(a_1 < a_2\) since \(a_1\) was chosen in preference to \(a_2\) at the first step.

Note \(A - \{a_1, a_2\}\) is infinite.
Let $a_3$ be the smallest element of $A - \{a_1, a_2\}$.

Note $a_1 < a_2 < a_3$ since $a_2$ was chosen in preference to $a_3$ at the previous step.

Note $a_3$ is the third smallest element.
Continue, using Well-Ordering Property, to identify for each $n \in \mathbb{N}$ the $n^{th}$ smallest member $a_n$ of $A$.

$a_1 < a_2 < a_3 < ...$
Define $f : \mathbb{N} \rightarrow A$ by $f(n) = a_n$.

(i) $f$ is 1–1: Suppose $i$ and $j$ are different positive integers. WLOG, $i < j$,

then $f(i) = a_i < a_j = f(j)$ \( \checkmark \)

Thus $f(i) \neq f(j)$. So $f$ is 1–1.
(ii) Consider an arbitrary $n_0 \in A$.

Certainly $n_0$ is the $i^{th}$ smallest member of $A$ for some $i < n_0$. So $f(i) = a_i = n_0$.

So, $n_0$ is in the range of $f$. $f$ is surjective.

∴ $f$ is a bijection. ✓
Lemma: Surjection from the Natural Numbers

Lemma 2: Suppose $f : \mathbb{N} \rightarrow B$ is a surjection. Then $B$ is countable.

Proof: Case I: Suppose $B$ is finite. Then $B$ is countable by definition.
Case II: Suppose $B$ is infinite.

Consider $b \in B$. Let $L_b = \{ n \in \mathbb{N} : f(n) = b \}$.

Since $f$ is a surjection, $L_b$ is a nonempty set of positive integers.
Define $g : B \to \mathbb{N}$ by $g(b) = \text{smallest member of } L_b$ using Well-Ordering Property. Note since $g(b) \in L_b$ that $f(g(b)) = b \ \forall b \in B$.

Claim: $g$ is $1-1$. 
Justify the claim.

Suppose $b_1$ and $b_2$ are in $B$ and

$g(b_1) = g(b_2)$. Must show $b_1 = b_2$. 
Thus

\[
\begin{align*}
    b_1 &= f(g(b_1)) = f(g(b_2)) = b_2 \\
    \text{Since } g(b_1) &= g(b_2)
\end{align*}
\]

This shows that \( g \) is an injection.

Consider the direct image \( g(B) = \{g(b) : b \in B\} \subset \mathbb{N} \)

Clearly \( g \) is a surjection of \( B \) onto \( g(B) \)
\[ \mathbb{N} \xrightarrow{f \text{ surjection}} B \]

\[ g \text{ bijection} \]

\[ g^{-1} \text{ bijection} \]

\[ g(B) \]
Claim: $g(B)$ is an infinite subset of $\mathbb{N}$.

Justify: Suppose $g(B)$ is finite. Seek $\otimes$

If $g(B) = \{g(b) : b \in B\}$ is finite, then $\{f(g(b)) : b \in B\}$ is also finite.

But $\{f(g(b)) : b \in B\} \not= \{b : b \in B\} = B$

Desired $\otimes$ Claim $\checkmark$ infinite set
By Lemma 1, \( \exists \) bijection \( h : \mathbb{N} \to g(B) \)

Then \( g^{-1} \circ h : \mathbb{N} \to B \) is a bijection
$\mathbb{N} \xrightarrow{f \text{ surjection}} B \xrightarrow{g \text{ bijection}} g(B) \xrightarrow{g^{-1} \text{ bijection}} B \xrightarrow{h} \mathbb{N}$
By Lemma 1, \( \exists \) bijection \( h : \mathbb{N} \to g(B) \)

Then \( g^{-1} \circ h : \mathbb{N} \to B \) is a bijection

Thus \( B \) is countably infinite by definition.
Prop: Suppose for each $n \in \mathbb{N}$ that $A_n$ is a countable set. Then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Proof: We treat the case (most challenging) where each $A_n$ is countably infinite.
\[ A_1 = \{a_{11}, a_{12}, a_{13}, a_{14}, \ldots\} \]

\[ A_2 = \{a_{21}, a_{22}, a_{23}, a_{24}, \ldots\} \]

\[ A_3 = \{a_{31}, a_{32}, a_{33}, a_{34}, \ldots\} \]

\[ A_4 = \{a_{41}, a_{42}, a_{43}, a_{44}, \ldots\} \]
Define $f : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$ as follows

\begin{align*}
  f(1) &= a_{11} & f(4) &= a_{31} & f(7) &= a_{41} \\
  f(2) &= a_{21} & f(5) &= a_{22} & \vdots \\
  f(3) &= a_{12} & f(6) &= a_{13} & \vdots
\end{align*}
Clearly this leads to a surjection $f : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$

and so $\bigcup_{n=1}^{\infty} A_n$ is countable by Lemma 2.

Check: $a_{ij} = f \left( \frac{(i + j - 2)(i + j - 1)}{2} + j \right)$
Positive Rational Numbers

Def: $\mathbb{Q}^+ = \mathbb{Q} \cap (0, \infty)$, the positive rationals

$A_1 = \left\{ \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \ldots \right\}$

$A_2 = \left\{ \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \ldots \right\}$

$A_3 = \left\{ \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \ldots \right\}$

$A_n = \left\{ \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n}, \ldots \right\}$

Note each $A_n$ is countable.

Also $\mathbb{Q}^+ = \bigcup_{n=1}^{\infty} A_n$
Note if $x \in \mathbb{Q}^+$, then $x = \frac{p}{q}$, where $p$ and $q$ are in $\mathbb{N}$.

Note $x$ appears in $p^{th}$ position of $A_q$.

Thus $\mathbb{Q}^+$ is countably infinite.
Negative Rational Numbers

Let \( \mathbb{Q}^- = \mathbb{Q} \cap (-\infty, 0) \) negative rationals.

\[ f : \mathbb{Q}^+ \to \mathbb{Q}^- \text{ given by } f(x) = -x \text{ is} \]

clearly a bijection.

We conclude that \( \mathbb{Q}^- \) is countably infinite.
The whole Rational Numbers

Finally, \( \mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^- \)

is a union of 3 countable sets and so is countable.
Easier Cases of the Proposition

Going back -

In our proof that $\bigcup_{n=1}^{\infty} A_n$

is countable, if some $A_n$ are finite

or there are only finitely many $A_n$’s,

an obvious modification of our argument

shows $\bigcup_{n=1}^{\infty} A_n$ is countable and those (easier cases)