Suppose $f : A \to B$ is a bijection.

Suppose $H \subset B$. Then $f^{-1}(H)$ has two interpretations:

(1) $f^{-1}(H)$ is the inverse image of $H$ under the function $f$.

(2) $f^{-1}(H)$ is the direct image of $H$ under the function $f^{-1}$. 
Exercise: Show these two sets are the same.
Well Ordering Property of Natural Numbers

Mathematical Induction

Well-Ordering property of \( \mathbb{N} \).

Every nonempty subset of \( \mathbb{N} \) has

a smallest element.
Principle of Mathematical Induction

Suppose $A \subseteq \mathbb{N}$ with the two following properties:

(i) $1 \in A$.

(ii) If $n \in A$, then $n + 1 \in A$.

Then $A = \mathbb{N}$.
Proof: Consider $\mathbb{N} - A = \{ n \in \mathbb{N} : n \notin A \}$

Wish to show that $\mathbb{N} - A = \emptyset$. Suppose $\mathbb{N} - A \neq \emptyset$.

We seek a $\otimes$. 

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By Well-Ordering property, we consider $n_0$, the smallest element of $\mathbb{N} - A$. Since $n_0 \notin A$ and $1 \in A$, we conclude $n_0 \neq 1$. Thus

$n_0 \neq 1 \in \mathbb{N}$. By the minimality of $n_0$, we have $n_0 - 1 \in A$. By assumption (ii),

$(n_0 - 1) + 1 = n_0 \in A$, $\otimes$.

Then $\mathbb{N} - A = \emptyset$, or $\mathbb{N} = A$. 
Principle of Mathematical Induction II

Suppose for every \( n \in \mathbb{N} \) that \( P(n) \) is a statement that is either true or false.

Suppose

(i) \( P(1) \) is true.

(ii) If \( P(n) \) is true, then \( P(n + 1) \) is true.

Then \( P(n) \) is true for all \( n \).
Justify: Let $A = \{n : P(n) \text{ is true}\}$.

Then (i) $1 \in A$

(ii) If $n \in A$, then $n + 1 \in A$.

By our first version of Mathematical Induction, $A = \mathbb{N}$, i.e., $P(n)$ is true for all $n \in \mathbb{N}$. 
Problem: Show that the sum of the first $n$ odd integers is $n^2$.

or \[ \sum_{k=1}^{n} (2k - 1) = n^2 \quad \leftarrow \quad \text{This statement is } P(n) \]

Proof: (base step) \[ P(1) : \sum_{k=1}^{1} 2k - 1 = 1 \]

\[ \text{or } 2(1) - 1 = 1^2 \quad \checkmark \]
(ii) induction step

Suppose \( P(n) \) is true, i.e.,

\[
\sum_{k=1}^{n} (2k - 1) = n^2
\]

Add \( 2(n + 1) - 1 \) to both sides:

\[
\sum_{k=1}^{n+1} (2k - 1) = n^2 + 2(n + 1) - 1
\]

\[
= n^2 + 2n + 1 = (n + 1)^2
\]
This says $P(n + 1)$ is true. ✓

By Mathematical Induction, $P(n)$ is true for all $n$. 
Problem II: Show $n^3 + 5n$ is divisible 6 for all $n \in \mathbb{N}$.

So $P(n)$ is: $n^3 + 5n$ is divisible 6.

base step: $1^3 + 5(1)$ is divisible 6 $\checkmark$
induction step: Suppose $P(n)$ is true.

Note $(n + 1)^3 + 5(n + 1) = n^3 + 3n^2 + 3n + 1 + 5n + 5$

$$= n^3 + 5n + 3n^2 + 3n + 2$$
\[ n^3 + 5n + 3n(n+1) + 6 \]

- Divisible by 6 since \( P(n) \) is true
- Divisible by 6 since \( n(n+1) \) is even

\( P(n+1) \) is true. ✓

By Mathematical Induction,

\( P(n) \) is true for all \( n \in \mathbb{N} \).
Finite Sets

Countability

Def: $\emptyset$ has 0 elements.

Def: Suppose $n \in \mathbb{N}$. A has $n$ elements if there exists a bijection from

$\{1, 2, 3, \ldots, n\}$ onto $A$. 
Infinite Sets

Def: A set is finite if it has \( n \) elements for some \( n \in \mathbb{N} \) or is \( \emptyset \).

Def: A set is infinite if it is not finite.
Countably Infinite Sets

Def: A set $A$ is countably infinite if there exists a bijection $f : \mathbb{N} \rightarrow A$.

Def: A set $A$ is countable if it is either finite or countably infinite.
Suppose $A$ is countably infinite.

So a bijection $f : \mathbb{N} \rightarrow A$.

For each $n \in \mathbb{N}$, let $a_n = f(n)$.

So, $A = \{a_1, a_2, a_3, \ldots\}$ since $f$ is a surjection.

So, the members of $A$ can be arranged in such a list.
Bijection from a Countably Infinite Set

Remark: Suppose $A$ is countably infinite.
Suppose $g : A \rightarrow B$ is a bijection. Then $B$ is countably infinite.
Justify: Since $A$ is countably infinite,

there exists a bijection $f : \mathbb{N} \rightarrow A$
Recall that a composition of bijection is a bijection. So, $g \circ f : \mathbb{N} \rightarrow B$

is a bijection and So $B$ is countably infinite.
Lemma 1: If $A$ is an infinite subset of $\mathbb{N}$, then $A$ is countably infinite.

Proof: By Well-Ordering property, $A$ has smallest element $a_1$.

Consider $A - \{a_1\}$. This is an infinite subset of $\mathbb{N}$.

By Well-Ordering, $A - \{a_1\}$ has a smallest member $a_2$. 
Note $a_1 < a_2$ since $a_1$ was chosen in preference to $a_2$ at the previous. So $a_2$ is the second smallest member of $A$. to be continued.