Review quiz

Quiz An equivalence relation $\sim$ on a set $X$ is a relation satisfying 3 conditions.

What are they? Define them.
Review the Definition of an Equivalence Relation

A relation $\sim$ on a set $X$ is an equivalence relation if it is:

- **reflexive**: $\forall a \in X, a \sim a$
- **symmetric**: $\forall a, b \in X$, if $a \sim b$, then $b \sim a$
- **transitive**: $\forall a, b, c \in X$, if $a \sim b$, $b \sim c$, then $a \sim c$

**Ex:** $<$ on $\mathbb{Z}$ (Neither reflexive or symmetric)
Example: Let $X = \{(a, b) \in \mathbb{Z}^2 | b \neq 0\}$

Define $(a, b) \sim (c, d)$ if $ad = bc$

reflexive?: $(a, b) \in X$, $ab = ba \Rightarrow (a, b) \sim (a, b)$

symmetric?: $(a, b), (c, d) \in X$. Suppose $(a, b) \sim (c, d)$

\[ ad = bc \iff bc = ad \iff (c, d) = (a, b) \]

transitive?: $(a, b), (c, d), (u, v) \in X$. Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (u, v)$
Have: \( ad = bc, \quad cv = ud \) \hspace{1cm} \text{Need:} \quad av = ub

\[ \Downarrow \quad \Downarrow \]

\( adv = bcv, \quad bcv = bud \)

\( adv = bud \)

\( av = bu \quad \checkmark \)

\( (a, b) \sim (u, v) \)

So, \( \sim \) is an equivalence relation.
Definition of a Partition

Example $f : X \to Y$ is a function, define

$\sim_f$ on $X$ by $a \sim_f b$ if $f(a) = f(b)$

$X$ a set, a partition of $X$ is a collection $\Omega$

of subsets of $X$ such that

(1) $\forall A, B \in \Omega$ either $A = B$ or $A \cap B = \emptyset$

(2) $\bigcup_{A \in \Omega} A = X$
Equivalence relations are the same as partitions

If $X$ is a set w/ an equivalence relation $\sim$, then

for $x \in X$, the equivalence class of $x$ is the set $[x] = \{y \in X | y \sim x\}$

Note: $x \in [x]$

$x \sim y$ iff $[x] = [y]$
Theorem 1.2.7  Let $X$ be any nonempty set, $\sim$ an equivalence relation on $X$. Then the set of equivalence classes $X/\sim = \{[x]|x \in X\}$ is a partition. Conversely, given a partition $\Omega$ of $X$, there exists a unique equivalence relation $\sim$ such that $X/\sim = \Omega$. 
Equivalence relations are the same as partitions

proof of 1st half Let \( x, y \in X, [x] \cap [y] \neq \emptyset \)

Need to show \([x] = [y]\)

Let \( z \in [x] \cap [y] \Rightarrow z \sim x \) and \( z \sim y \)

Let \( w \in [x] \Rightarrow w \sim x \) \(\Rightarrow\) \( x \sim w \)

\(\Rightarrow\) \( z \sim w \) \(\Rightarrow\) \( w \sim z \) \(\Rightarrow\) \( w \sim y \) \(\Rightarrow\) \( w \in [y] \)

\(\therefore [x] \subseteq [y]\)
Symmetric argument shows that \([y] \subset [x]\), so \([x] = [y]\)

The second property of a partition

Need \(\bigcup_{[x] \in X/\sim} [x] = X\)

Let \(x \in X, x \in [x] \subset \bigcup_{[y] \in X/\sim} [y] \checkmark\)
Examples of Equivalence Classes

Equivalence relations are the same as partitions

Example: $U = \{\text{students at U of I}\}$

$a, b \in U, a \sim b$ if $\text{age}(a) = \text{age}(b)$

Equivalence classes? $[a] = \{b | \text{age}(a) = \text{age}(b)\}$

Example: $f : X \rightarrow Y, \sim_f$ on $X,$ $x \sim_f y$ if $f(x) = f(y)$
Equivalence classes are fibers. $f(x) = f(y) \Rightarrow y \in f^{-1}(z)$

**Proposition 1.2.12** If $\sim$ is an equivalence relation on $X$

define $\pi : X \to X/\sim$ by $\pi(x) = [x]$, then $\sim_{\pi} = \sim$

**Example:** $X = \{(a, b) \in \mathbb{Z}^2 | b \neq 0\}$

$X/\sim = \{(a, b) | (a, b) \in X\} = \mathbb{Q}$
Equivalence relations are the same as partitions

Let’s write equivalence of \((a, b)\) as \(\frac{a}{b} = [(a, b)]\)

\[
\begin{align*}
\frac{9 \cdot 4}{9 \cdot 6} &= \frac{6 \cdot 6}{6 \cdot 9} \\
\frac{a}{b} &= \frac{c}{d} & \iff & \frac{ad}{bd} = \frac{bc}{bd} \\
\iff & ad = bc
\end{align*}
\]
Permutations

Permutations: $X$ nonempty set.

$$Sym(X) = \{ f : X \to X \mid f \text{ is a bijection} \} \subset X^X$$

**Proposition** For any nonempty set $X$, $\circ$ is an operation on $Sym(X)$, and

1. $\circ$ is associative

2. $id_X \in Sym(X)$, $id_X \circ \sigma = \sigma \circ id_X = \sigma$

3. $\forall \sigma \in Sym(X), \sigma^{-1} \in Sym(X)$
\[ \sigma \in \text{Sym}(X) \] permutation group of \( X \)

symmetric group of \( X \)

\( \sigma \) is a permutation

\( X = \) green marbles

\[ \sigma : X \rightarrow X \]

\[ \sigma(1) = 2 \]
\[ \sigma(2) = 5 \]
\[ \sigma(3) = 4 \]
\[ \sigma(4) = 3 \]
\[ \sigma(5) = 1 \]

"permutation" of marbles is a rearrangement

label the location of marbles by names in \( X \).
When \( X = \{1, \ldots, n\}, \; n \in \mathbb{Z}, \) write

\[ S_n = \text{Sym}(X) \text{ symmetric group on } n \text{ elements} \]

\[
\begin{align*}
\sigma(1) &= 2 \\
\sigma(2) &= 5 \\
\sigma(3) &= 4 \\
\sigma(4) &= 3 \\
\sigma(5) &= 1
\end{align*}
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 5 & 4 & 3 & 1
\end{pmatrix}
\]
\[ \sigma \in Sym(X), \sigma^2 = \sigma \circ \sigma, \sigma^3 = \sigma \circ \sigma \circ \sigma, \sigma^4 = \sigma \circ \sigma \circ \sigma \circ \sigma \]

\[ \sigma^0 = \text{id}_X \]

\[ \sigma^{-1} = \text{inverse}, \ r > 0, \sigma^{-r} = (\sigma^{-1})^r \]

\[ r, s \in \mathbb{Z}, \sigma^{r+s} = \sigma^r \circ \sigma^s = \sigma^s \circ \sigma^r \]
Cycle Decompositions of Permutations

\begin{align*}
&1 \xrightarrow{\sigma} 2 \xrightarrow{\sigma} 5 \xrightarrow{\sigma} 1, \quad \tau_1 \\
&3 \xrightarrow{\sigma} 4 \xrightarrow{\sigma} 3, \quad \tau_2 \\
\tau_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix}, \quad \text{3-cycle} \\
\tau_2 &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix}, \quad \text{2-cycle}
\end{align*}
\[ \sigma = \tau_1 \circ \tau_2 \quad \text{\(\tau_1\) \& \(\tau_2\) are disjoint cycles} \]

\[ = \tau_2 \circ \tau_1 \quad \text{disjoint cycles commute} \]

Every permutation is a composition of disjoint cycles, uniquely
\( \tau_1 \) in cycle notation \( \tau_1 = (1 2 5) = (2 5 1) = (5 1 2) \)
\( \tau_2 = (3 4) = (4 3) \)

\( \sigma = (1 2 5) \circ (3 4) \)
\[ = (1 2 5)(3 4) \]

Disjoint cycle notation for \( \sigma \)

Example \( \rho = (1 2 5 3)(6 7) \)

\( \rho^{-1} = (3 5 2 1)(6 7) \)