Algorithm to find all solutions of system of linear equations

**TASK:** Write algorithm to find all solutions of

\[
\begin{align*}
  a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 & \quad E_1 \\
  \vdots & & \\
  a_{m1}x_1 + \cdots + a_{mn}x_n &= b_n & \quad E_m
\end{align*}
\]
ex

\[ \begin{align*}
x_1 + x_2 &= 1 & \text{E1} \\
x_1 + 3x_2 &= 2 & \text{E2}
\end{align*} \]

\[ E_2 - E_1 \Rightarrow 2x_2 = 1 \quad \text{← new E}_2 \]

\[ \begin{align*}
x_1 + x_2 &= 1 & \text{E1} \\
2x_2 &= 1 & \text{E2}
\end{align*} \]

\[ \frac{1}{2}E_2 \Rightarrow x_2 = \frac{1}{2} \]

sub back into E_1

\[ x_1 = \frac{1}{2} \]

system has unique solution \((x_1, x_2) = \left( \frac{1}{2}, \frac{1}{2} \right)\)
Strategy

1) Manipulate equations to get to simpler system

2) Get a simplest form and read solutions from it

① \( E_i \leftrightarrow E_j \)

② \( E_i \leftrightarrow cE_i \quad c \neq 0 \)

③ \( E_i \leftrightarrow E_i + cE_j \) (work horse)
Thm 1 None of these change the set of solutions

Pf
①, ② obvious
③ Suppose \((x_1,\ldots,x_n)\) is a solution of

\[
\begin{align*}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \quad E_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \quad E_2
\end{align*}
\]

\[
\uparrow
\]

\[
\begin{align*}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \quad E_1 \\
(a_{21} + ca_{11})x_1 + \cdots + (a_{2n} + ca_{1n})x_n &= b_2 + cb_1 \quad E_2 + cE_1
\end{align*}
\]
Augmented matrix & Elementary row operations

Observation: in performing ①, ②, ③ the variables are placeholders.

\[
\begin{align*}
    a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
    \vdots \\
    a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

This \( m \times (n + 1) \) is the augmented matrix of the system.

① \( R_i \leftrightarrow R_j \)

② \( R_i \leftrightarrow cR_i \quad c \neq 0 \)

③ \( R_i \leftrightarrow R_i + cR_j \)
\[ x_1 + x_2 = 1 \]
\[ x_1 + 3x_2 = 2 \]

\[ \rightarrow \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \]

\[ R_2 \rightarrow R_2 - R_1 \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \]

\[ R_2 \rightarrow \frac{1}{2} R_2 \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \end{pmatrix} \]

\[ \mathcal{L}S \quad x_1 + x_2 = 1 \]
\[ x_2 = 1/2 \]

Q. To what form should we try to put our linear system (augmented matrix)
Row Echelon Form

**Def** The *leading entry* in a row is the leftmost nonzero entry

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 \\
1 & 2 & 3 & 4
\end{pmatrix}
$$

**Def** A matrix is in Row Echelon Form (REF) if

a) All zero rows are below all nonzero rows
b) the leading entry of each row to be to the right of the leading entry of the row above
Reduced Row Echelon Form

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

in REF

**Def** A matrix is in Reduced Row Echelon Form (RREF) if it is in REF and

c) the leading entries are all 1

d) the leading entries are the only nonzero entry in their columns
\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 2 & 1
\end{pmatrix}
\text{ REF } \checkmark
\]

\[R_2 \rightarrow \frac{1}{2} R_2\]
\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1/2
\end{pmatrix}
\]

\[R_1 \rightarrow R_1 - R_2\]
\[
\begin{pmatrix}
1 & 0 & 1/2 \\
0 & 1 & 1/2
\end{pmatrix}
\text{ RREF } \checkmark
\]
**Thm 2** Every matrix can be put in RREF by a finite sequence of elementary row operations.

**Thm 3** The RREF of a matrix is unique.

**Thm 4** The solution set of a linear system with augmented matrix in RREF is easily described in a standard way.
Applying RREF to linear systems

Let’s understand Thm 4 in examples.

ex Consider the system w/ augmented matrix.

\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{array}{c}
x_1 \\
x_2 \\
0
\end{array}
\begin{array}{c}
- x_3 = 0 \\
x_2 = 0 \\
0 = 1
\end{array}
\]

This system has no solutions. \( \text{vskip -0.2cm} \) (It is inconsistent.)

Moral A leading entry in the rightmost column of RREF \( \Rightarrow \) system is inconsistent.
\[
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\overset{LS}{\rightarrow}
\begin{aligned}
\begin{align*}
x_1 &- x_3 = 0 \\
x_2 &= 0 \\
0 &= 0
\end{align*}
\end{aligned}
\]

There are infinitely many solutions.

Convention: use variable corresponding to columns in RREF with no leading entries to parameterize solutions set.

Set \( x_3 = t \)

\( E_1 \Rightarrow x_1 = t \)

\( E_2 \Rightarrow x_2 = 0 \)

The solution set is

\((x_1, x_2, x_3) = \{(t, 0, t) \mid t \in \mathbb{R}\}\)


\[
\begin{pmatrix}
1 & 2 & 0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 & 4 \\
\end{pmatrix}
\]

\[x_2 = t_1, x_4 = t_2\]

\[x_1 = -2t_1 - t_2 + 2\]

\[x_3 = -3t_2 + 3\]

\[x_5 = 4\]