Lecture 01, Math 423

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Goals / Applications

Math 423    Differential Geometry
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Two goals
(i) extend calculus to “curved” spaces
(ii) study “geometric properties” of objects using calculus

Pre-req
- Multivariable calculus (241)
- familiarity w/ proofs (347, 400-level)
- linear algebra (recommend)

Applications
- Computer graphics (Discrete Differential Geometry (DDG))
- Physics   general relativity, gauge theory
- Engineering    Navier-Stokes
Extending calculus

Extending calculus

Single variable calculus deals with functions $f : \mathbb{R} \to \mathbb{R}$ or
$f : (a, b) \to \mathbb{R}$

Multivariable calculus $f : \mathbb{R}^n \to \mathbb{R}^m$ (241, $m, n \leq 3$) or
$f : U \to \mathbb{R}, U \subseteq \mathbb{R}^n_{\text{open}}$

Unrealistic for some problems

**Ex** $f(p) =$ temperature in °F at point $p$ on earth

![Diagram showing a function $f$ mapping a point $p$ in a circle to a value $f(p)$ on the real number line.](attachment:image.png)
Input \[ S^2(r) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\} \]

sphere of radius \( r \)

\[ f : S^2(r) \rightarrow \mathbb{R} \]

What is gradient?

How is \( f \) changing in a given “direction”?

Ex 

Wind vector field on earth

What is divergence?

What is curl?
Geometric properties using calculus

Geometric properties using calculus: curvature

Ex Curves in plane

Curvature of planar curve measures how much curve differs from being a straight line.

Given a parameterized closed curve with curvature $k(t)$

$k(t)$ admits at least one minimum & one maximum.

$k'(t) = 0$ at these “vertices”
**4-vertex theorem / local v.s. global / geodesic**

**Thm 4-vertex theorem**
Every closed plane curve has at least 4 vertices.

**Typical feature of DG**  **local versus global**

*Ex* \( f : [0, T] \rightarrow S^2(r) \)

\( f(t) = \) position of aircraft \( t \) time units after take-off

Shortest paths lie along great circles.  **geodesic**
Goals of course

For sufficiently close by points $\exists$ unique shortest path (local)
**Not true** globally (e.g. NP & SP).

(i) functions $f: M \rightarrow N$, $M, N$ are $1$-dim curves & $2$-dim surfaces
Helps w/ further of $n$-dimensional (manifolds)

Möbious, Klein
(ii) length, curvature

Gaussian curvature of surface $M$

How $M$ deviates from being a plane.
Extrinsic / Intrinsic view point

Extrinsic view point \([M \subset \text{ambient space } \mathbb{R}^2, \mathbb{R}^3]\)

Intrinsic view point [without ambient space]

Thm (Theorema Egregium of Gauss)

“Gaussian curvature is intrinsic property”
Gauss-Bonnet formula

Thm (Gauss-Bonnet formula)
M is “nice” surface with no holes

\[ \iint_M KdA = 4\pi \]

M with \( g \) holes

\[ \iint_M KdA = 4\pi(1 - g) \]
2-2-2 viewpoints of the course

2 extrinsic v.s. intrinsic

2 local v.s. global

2-2-2