Deformation Theory : families of vector bundles over the dual numbers

Jin Hyung To

May 11, 2018

1 Introduction

The purpose of this article is to survey the deformation of algebraic objects such as vector bundles, coherent sheaves, schemes, etc to study the moduli spaces of these objects.

2 The moduli space of vector bundles over a nonsingular algebraic curve

The moduli space $\mathcal{M} := \mathcal{M}(r, d)$ of semistable vector bundles of rank $r$ and degree $d$ over a nonsingular algebraic curve over an algebraically closed field $k$ is a projective variety of dimension $1 - r^2(1 - g)$, where $g$ is the genus of the curve. The open subset $\mathcal{M}^s$ of stable vector bundles is a fine moduli space and every stable vector bundle represents a smooth point. We have a bijective map

$$T_x(\mathcal{M}) \cong \text{Hom}_x(D, \mathcal{M}) \to \{\text{equivalence classes of deformations of } E \text{ over } D\},$$

where $x$ is the point corresponding to a stable vector bundle $E$, and $D = \text{Spec}(k[t]/(t^2))$, and $T_x(\mathcal{M})$ is the tangent space at $x$ to $\mathcal{M}$.

3 The Zariski tangent space to the moduli space of vector bundles

Let $E$ be a vector bundle of rank $r$ over an algebraic curve $X$ over $\mathbb{C}$, and $\{U_i\}$ an open covering which defines a local trivialization. Define $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. Let $g_{ij} : U_{ij} \to \text{GL}(r, \mathbb{C})$ be the transition map of $E$. $\text{GL}(r, \mathcal{O}_{D \times X}) = \text{GL}(r, \mathcal{O}_X) + \varepsilon \mathcal{M}_{r \times r}(\mathcal{O}_X)$, where $D = \text{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon))$. Let $\tilde{g}_{ij} = g_{ij} + \varepsilon b_{ij} = g_{ij}(1 + \varepsilon a_{ij}) \in \text{GL}(r, \mathcal{O}_X(U_{ij})) + \varepsilon \mathcal{M}_{r \times r}(\mathcal{O}_X(U_{ij}))$ with $\varepsilon^2 = 0$. Then the cocycle condition

$$\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} = \text{Id} \text{ on } U_{ijk}$$

implies that

$$g_{ij}(1 + \varepsilon a_{ij})g_{jk}(1 + \varepsilon a_{jk})g_{ki}(1 + \varepsilon a_{ki}) = \text{Id}$$

$$\text{Id} + g_{ij}g_{jk}g_{ki} \varepsilon a_{ki} + g_{ij}g_{jk} \varepsilon g_{ki} + g_{ij} \varepsilon a_{ij}g_{jk}g_{ki} = \text{Id}$$

$$\varepsilon(\text{Id} a_{ki} + g_{ik}a_{jk}g_{ki} + g_{ij}a_{ij}g_{ji}) = 0$$

$$a_{ki} + g_{ik}a_{jk}g_{ki} + g_{ij}a_{ij}g_{ji} = 0$$

(1)
If \( r = 1 \), then they are commutative, so we get a cocycle condition for \( \mathcal{O}_X \).

\[
a_{ki} + a_{jk} + a_{ij} = 0.
\]

Let \( L = E \). So, the deformation associates a transition map \( \{a_{ij}\} \) is an element of \( H^1(\mathcal{O}_X) \), and \( \operatorname{Ext}^1(L, L) = H^1(\mathcal{E}nd(L, L)) = H^1(L^{-1} \otimes L) = H^1(\mathcal{O}_X) \) (since \( L \) is locally free.)

Now assume that \( r > 1 \). We will review the deformation theory of coherent sheaves on a scheme. Let \( k[\varepsilon] = k[t]/(t^2) \) be the (local) ring of dual number. Here \( \varepsilon = t + (t^2) \) so that \( \varepsilon^2 = 0 \). Let \( D = \text{Spec}(k[\varepsilon]) \). Then \( D \) is a scheme consisting of a single point \( \xi \) corresponding to the unique maximal ideal \( (\varepsilon) \), and \( \mathcal{O}_D(D) = \mathcal{O}_{\xi,D} = k[\varepsilon] \). The scheme \( D \) has a unique closed subscheme \( \text{Spec}(k[\varepsilon]/(\varepsilon)) = \text{Spec}(k) \). We will consider the projection \( X \times D \to D \) to define the deformation of a coherent sheaf on \( X \).

**Definition 1.** A module \( M \) over a Noetherian ring \( A \) is flat if the functor \( N \mapsto N \otimes_A M \) is exact on the category of \( A \)-modules. Consider a morphism of schemes \( f : X \to Y \). The morphism \( f \) is flat if for every point \( x \in X \), \( \mathcal{O}_{x,X} \) is flat as an \( \mathcal{O}_{f(x),Y} \)-module. A sheaf of \( \mathcal{O}_X \)-module \( \mathcal{F} \) over \( Y \) is flat if for every point \( x \in X \), the stalk \( \mathcal{F}_x \) is flat as an \( \mathcal{O}_{f(x),Y} \)-module.

**Lemma 1.** A module \( M \) over a Noetherian ring \( A \) is flat if and only if for every prime ideal \( p \subseteq A \), \( \operatorname{Tor}^A_1(M, A/p) = 0 \).

**Proof.** \( M \) is flat if and only if \( \operatorname{Tor}^A_1(M, N) = 0 \) for all \( A \)-modules \( N \). Since \( N \) is a direct limit of its finitely generated submodules and tensor product commutes with direct limit, this is equivalent to that \( \operatorname{Tor}^A_1(M, N) = 0 \) for all finitely generated \( A \)-modules \( N \). Indeed, \( \operatorname{Tor}^A_1(M, N) = \operatorname{Tor}^A_1(M, \lim N_\alpha) = \lim \operatorname{Tor}^A_1(M, N_\alpha) \), where \( \{N_\alpha\} \) is the direct system of all finitely generated submodules of \( N \). In more detail, \( \operatorname{Tor}^A_1(M, \lim N_\alpha) = H_1(M \otimes \lim \to N_\alpha) = H_1((\lim \to M \otimes N_\alpha) = \lim H_1((M \otimes N_\alpha) = \lim \operatorname{Tor}^A_1(M, N_\alpha) \), where \( (N_\alpha) \) is the projective resolution

\[
\ldots \to P_2 \to P_1 \to P_0 \to N_\alpha
\]

of \( N_\alpha \). Now every finitely generated module over \( A \) has a filtration whose quotient is isomorphic to \( A/p_i \) for some prime ideals \( p_i \subseteq A \). Say \( N = N_\alpha \). In detail, there is a filtration \( 0 = N_0 \subset N_1 \subset \ldots \subset N_{m-1} \subset N_m = N \) such that \( N_i/N_{i-1} \cong A/p_i \) for some prime ideal \( p_i \subseteq A \). (Here \( p_i \) is the associated prime of \( N_i/N_{i-1} \), i.e., \( p_i = \text{Ann}(x_i) \) for some \( x_i \in N_i/N_{i-1} \). Indeed \( p_i \) is a maximal ideal among the ideals \( \text{Ann}(x) \) for all \( x \neq 0 \) in \( N_i/N_{i-1} \).) We will show that \( \operatorname{Tor}^A_1(M, N) = 0 \) if \( \operatorname{Tor}^A_1(M, A/p_i) = 0 \) for all \( i = 1, 2, \ldots, m \). Assume that \( \operatorname{Tor}^A_1(M, A/p_i) = 0 \) for all \( i = 1, 2, \ldots, m \). For each \( i \),

\[
0 \to N_{i-1} \to N_i \to N_i/N_{i-1} \cong A/p_i \to 0
\]

is exact, and there is an associated long exact sequence

\[
\ldots \to \operatorname{Tor}^A_1(M, N_{i-1}) \to \operatorname{Tor}^A_1(M, N_i) \to \operatorname{Tor}^A_1(M, A/p_i) \to \ldots
\]

For \( i = 1 \), \( \operatorname{Tor}^A_1(M, N_1) = \operatorname{Tor}^A_1(M, A/p_1) = 0 \),

for \( i = 2 \), \( \operatorname{Tor}^A_1(M, N_2) = 0 \) and \( \operatorname{Tor}^A_1(M, N_2/N_1) = \operatorname{Tor}^A_1(M, A/p_2) = 0 \), so \( \operatorname{Tor}^A_1(M, N_2) = 0 \).

Continue this process and get

\[
\operatorname{Tor}^A_1(M, N_{m-1}) = 0 \text{ and } \operatorname{Tor}^A_1(M, N_m/N_{m-1}) = \operatorname{Tor}^A_1(M, A/p_m) = 0.
\]

Thus, \( \operatorname{Tor}^A_1(M, N) = \operatorname{Tor}^A_1(M, N_m) = 0 \).
Since $A/p$ is finitely generated $A$-module for every prime ideal $p$ of $A$, we have the following equivalent statement. $\text{Tor}_1^A(M, N) = 0$ for every finitely generated $A$-module $N$ if and only if $\text{Tor}_1^A(M, A/p) = 0$ for every prime ideal $p$ of $A$.

\[ \]

**Proposition 1.** (A special case of local criterion of flatness) Let $A' \to A$ be a surjective homomorphism of Noetherian rings whose kernel $J$ has square zero. Then $A'$-module $M'$ is flat over $A$ if and only if

(1) $M = M' \otimes_{A'} A$ is flat over $A$,

and

(2) the map $M \otimes_A J \to M'$ is injective ($m' \otimes a \otimes j \mapsto ajm'$ for all $m' \in M', a \in A, j \in J$).

Here, $J$ is an $A$-module through the identification $B/J \cong A$. Since $J^2 = 0$, the module action $(b + J) \cdot j = bj + J$ is well-defined for every $b \in B, j \in J$.

**Proof.** Let $M'$ be an $A'$-module. Suppose that $M'$ is flat $A'$-module. By base extension, $M = M' \otimes_{A'} A$ is a flat $A$-module. Since $M'$ is a flat $A'$-module, tensoring $M'$ with the exact sequence of $A'$-modules

$$0 \to J \to A' \to A \to 0$$

gives an exact sequence

$$0 \to M' \otimes_{A'} J \to M' \otimes_{A'} A' \to M' \otimes_{A'} A \to 0$$

So, the map $M' \otimes_A J \to M'$ is injective. Conversely, assume (1) and (2). We show that $M'$ is a flat $A'$-module. By lemma 1, we need to show that $\text{Tor}_1^{A'}(M', A'/p') = 0$ for every prime ideal $p' \subset A'$.

Since $J$ is nilpotent, $J \subset p'$. Let $p = p'/J \subset A$ be a prime ideal. Then we have the following commutative diagram of exact sequences of $A'$-modules.

$$0 \to J \to p' \to p \to 0$$

Since the functor $M' \otimes_{A'}$ is right exact, we obtain the following exact sequences

\[ \]
Note that $M' \otimes_{A'} N = (M' \otimes_{A'} A) \otimes_A N$ for every $A$-module, and every $A'$-module becomes an $A$-module via the map $A' \to A$. So, $M' \otimes_{A'} N = M \otimes_A N$. In particular, $\text{Tor}_1^{A'}(M', A/p) = \text{Tor}_1^A(M, A/p) = 0$ by the assumption (2).

We rewrite the diagram.

By the assumption (1), (*) and (**) are exact. By the snake lemma we obtain an exact sequence

$$0 \to 0 \to \text{Tor}_1^{A'}(M', A'/p') \to \text{Tor}_1^A(M, A/p) \to \cdots$$

So, $\text{Tor}_1^{A'}(M', A'/p') = 0$. By the lemma 1, $M'$ is a flat $A'$-module.

**Definition 2.** Let $\mathcal{F}$ be a coherent sheaf over a scheme $X$ over a field $k$. A deformation of $\mathcal{F}$ over $D$ (a first order deformation of $\mathcal{F}$) is a coherent sheaf $\mathcal{F}'$ over $X' = X \times D$ which is flat over $D$, together with a homomorphism of $\mathcal{O}_X$-modules $\mathcal{F}' \to \mathcal{F}$ such that $\mathcal{F}'|_X = \mathcal{F}$, i.e., $\mathcal{F}' \otimes_{k[]} k \cong \mathcal{F}$. ($\mathcal{F}'$ is flat over $D$ through the projection $X' \to D$.) Indeed, $\mathcal{F}'|_X$ is the restriction to the closed subscheme $X \times \text{Spec}(k) = X \subseteq X'$. Note that $X' \times_D \text{Spec}(k) = X$. Two deformations $\mathcal{F}'_1$ and $\mathcal{F}'_2$ of $\mathcal{F}$ are equivalent if there is an isomorphism $\mathcal{O}_{X'}$-modules $\varphi : \mathcal{F}_1' \to \mathcal{F}_2'$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{F}_1' & \xrightarrow{\varphi} & \mathcal{F}_2' \\
\downarrow & & \downarrow \\
\mathcal{F} & & \mathcal{F}
\end{array}
$$

is commutative.
is commutative.

**Remark 1.** \(F'|_X = i^{-1}F' \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_X = F' \otimes_{\mathcal{O}_X} \mathcal{O}_X = F' \otimes_{\mathcal{O}_X \otimes k[\varepsilon]} \mathcal{O}_X \cong F' \otimes_{k[\varepsilon]} k\), where \(X \hookrightarrow X' := X \times D\) is the closed embedding and \(F'\) is a coherent sheaf on \(X'\) which is flat over \(D\). The restriction \(F' \otimes_{k[\varepsilon]} k \cong F\) can be compared to the deformation \(X\) (over \(D\)) of a scheme \(X\). The scheme \(X\) is flat over \(D\) and its fiber at the closed subscheme \(\text{Spec}(k) \subset D\) is \(X \times_D \text{Spec}(k) \cong X\). The subscheme of \(D\) corresponds to the unique maximal ideal \((\varepsilon) \subset k[\varepsilon]\), i.e., \(\text{Spec}(k[\varepsilon]/(\varepsilon)) \cong \text{Spec}(k)\). Or, equivalently the closed subscheme is the closed embedding \(\text{Spec}(k) \hookrightarrow \text{Spec}(k[\varepsilon])\) given by the evaluation map \(a + b\varepsilon \mapsto a\) from \(k[\varepsilon]\) to \(k\).

**Lemma 2.** The two \(k[\varepsilon]\)-modules \((\varepsilon)\) and \(k = k[\varepsilon]/(\varepsilon)\) are isomorphic.

**Proof.** The homomorphism \(a + b\varepsilon \mapsto (a + b\varepsilon)\varepsilon\) from \(k[\varepsilon]\) to \(k[\varepsilon]\) has kernel \((\varepsilon)\) and image \((\varepsilon)\). So, \(k = k[\varepsilon]/(\varepsilon) \cong (\varepsilon)\).

**Theorem 1.** There is a canonical bijection between the equivalence classes of deformations of \(E\) and the vector space \(H^1(X, \mathcal{End}(E))\).

**Proof.** Let \(E'\) be a deformation of \(E\) over \(D\). Consider the surjective morphism \(a + b\varepsilon \mapsto a\) from \(k[\varepsilon]\) to \(k\). The kernel of this map is \((\varepsilon)\) which has square zero. By Proposition 1, \(E'\) is flat over \(D\) if and only if

1. \(E' \otimes_{k[\varepsilon]} k\) is flat over \(k\), and
2. \(E' \otimes_{k[\varepsilon]} k \otimes_k (\varepsilon) \to E'\) is injective.

Since \(k\) is a field, (1) is obvious.

For (2), \(E' \otimes_{k[\varepsilon]} k \otimes_k (\varepsilon) = E' \otimes_{k[\varepsilon]} (\varepsilon) \cong E' \otimes_{k[\varepsilon]} k\).

So, \(E'\) is flat over \(D\) if and only if

\[
0 \to E' \otimes_{k[\varepsilon]} k \xrightarrow{1 \otimes \varepsilon} E' \otimes_{k[\varepsilon]} k[\varepsilon] \to E' \otimes_{k[\varepsilon]} k \to 0
\]

is an exact sequence of \(\mathcal{O}_{X'}\)-modules, where \(X' = X \times D\). Note \(E' \otimes_{k[\varepsilon]} k[\varepsilon] = E' \otimes_{k[\varepsilon]} 1 + E' \otimes_{k[\varepsilon]} \varepsilon\).

Since \(E \cong E' \otimes_{k[\varepsilon]} k\) \((E'\) is a deformation of \(E)\) and \(E'\) is also an \(\mathcal{O}_X\)-module,

\[
0 \to E \xrightarrow{\varepsilon} E' \to E \to 0
\]

is also an exact sequence of \(\mathcal{O}_X\)-modules.

Say \(\alpha : 0 \to E \xrightarrow{\varepsilon} E' \to E \to 0\). Then we pull-back (Yoneda’s correspondence) the exact sequence

\[
0 \to E \to I_0 \to I_0/E \to 0,
\]

where \(0 \to E \xrightarrow{\varepsilon} I_0 \xrightarrow{\psi_0} I_1 \xrightarrow{\psi_1} \ldots\) is an injective resolution of \(E\).

The pull-back is as follows:

Since an element of \(\text{Ext}_X^1(E, E)\) is represented by a cycle \(v \in \text{Hom}_{\mathcal{O}_X}(E, I_1)\) such that \(\psi_1 v = 0\). So, \(\text{Im}(v) \subseteq \text{Ker}(\psi_1), \ v : E \to \text{Ker}(\psi_1) = \text{Im}(\psi_0) \cong I_0/E\).

\[
\alpha : 0 \to E \to E' \to E \to 0
\]

\[
\begin{array}{cccccc}
0 & \to & E & \to & E' & \to & E & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & E & \to & I_0 & \to & I_0/E & \to & 0
\end{array}
\]
We get a corresponding element \([v] \in \text{Ext}_X^1(E, E)\).

We need to show that the mapping from \(\{\text{deformations of } E\}/\sim_1 \to \{\text{extensions of } E \text{ by } E\}/\sim_2\) is well-defined, where \(\sim_1\) and \(\sim_2\) are the equivalences of deformations and extensions, respectively.

By Yoneda, there is a bijection between \(\{\text{extensions of } E \text{ by } E\}/\sim_2\) and \(\text{Ext}_X^1(E, E)\).

Exercise: Two deformations \(E'_1\) and \(E'_2\) of \(E\) are equivalent if and only if the corresponding extensions \(\alpha_1\) and \(\alpha_2\) are equivalent, where

\[
\alpha_1 : 0 \to E \to E'_1 \to E \to 0 \quad \text{and} \quad \alpha_2 : 0 \to E \to E'_2 \to E \to 0.
\]

Now, we have a well-defined injective map

\[
\{\text{deformations of } E\}/\sim_1 \to \{\text{extensions of } E \text{ by } E\}/\sim_2 \cong \text{Ext}_X^1(E, E).
\]

The next step is to show that this map is surjective. For a given exact sequence of \(O_X\)-modules,

\[
0 \to E \xrightarrow{i} E' \xrightarrow{\pi} E \to 0,
\]

we should construct an \(O_X\)-module \(E'\) with a homomorphism of \(O_X\)-modules \(E' \to E\) such that \(E'|_X = E' \otimes_k \kappa \cong E\). i.e., we should define a multiplication by \(\varepsilon\) on \(E'\) such that \(E'|_X = E' \otimes_k \kappa \cong E\). For \(e' \in E'\) define \(\varepsilon \cdot e' = i \pi(e')\). Then \(E' \otimes_k \kappa \cong E\) by the mapping \(e' \otimes 1 \mapsto \pi(e')\). This is surjective. To show that the map is injective, assume that \(\pi(e') = 0\). Then \(e' \in \text{Ker}(\pi) = \text{Im}(i)\), so \(i(e) = e'\) for some \(e \in E\). Since \(\pi\) is surjective, \(\pi(e'_1) = e\) for some \(e'_1 \in E'\), so \(e' = i \pi(e'_1) = \varepsilon \cdot e'_1\).

Now, \(e' \otimes 1 = (\varepsilon \cdot e'_1) \otimes 1 = e'_1 \otimes (\varepsilon \cdot 1) = e'_1 \otimes 0 = 0\). The multiplication by \(\varepsilon\) on \(k\) is defined by the closed embedding \(\text{Spec}(k) \hookrightarrow \text{Spec}(k[\varepsilon])\) given by the evaluation map \(a + b\varepsilon \mapsto a\) from \(k[\varepsilon]\) to \(k\).

Finally, since \(E\) is locally free, \(\text{Ext}_X^1(E, E) \cong H^1(X, \mathcal{E}nd(E))\).

\[\square\]

**Remark 2.** \(H^1(X, \mathcal{E}nd(E)) \cong H^1(X, \mathcal{E}^* \otimes \mathcal{E}) \cong H^0(X, \mathcal{E}^* \otimes \mathcal{E} \otimes K) \cong \text{Hom}(\mathcal{E}, \mathcal{E} \otimes K)\).

**Remark 3.** More generally for any coherent sheaf \(\mathcal{F}\) on a scheme over an algebraically closed field \(k\) there is a canonical bijection between the deformations of \(\mathcal{F}\) and the vector space \(\text{Ext}_X^1(\mathcal{F}, \mathcal{F})\).

**Remark 4.** The vector space \(\text{Ext}_X^1(E, E)\) is canonically isomorphic to the tangent space to the moduli space of vector bundles at the closed point corresponding to a stable vector bundle \(E\).