An Index for H. L. Royden’s *Real Analysis*, Third Edition
Prepared by Jesse Miller

References to definitions and statements of theorems appear in boldface. References to occurrences in problems appear in italics. References to footnotes are followed by an “n”. Terms containing “σ-” are alphabetized as if “sigma” were spelled out in full.

A

Abelian semigroups of linear operators, 224
absolute continuity, 108
  bounded variation and, 108
  Cantor ternary function and, 111
differentiability and, 109
indefinite integrals and, 110

absolute values of measures, 274–275
absolutely continuous measures, 276
absolutely measurable Borel sets, 414
accumulation points, 46
actions, 376–378
adjoints, 392–393
Alaoglu’s theorem, 237
Alexandroff one-point compactifications, 201
algebras, 17–19
  Boolean, 394
  algebras of subsets and, 395
generated by semialgebras, 297
generated by sets, 17
measure, 398
measures on, 291
of functions,
  σ-, 18–19, 253, 289
  Boolean, 395, 398
generated by sets, 19
of measurable sets, 59–60
almost everywhere, 69
annihilators of subspaces of Banach spaces, 228
antisymmetric relations, 24
approximation of measurable functions, 69, 127–128
arcwise connected topological spaces, 183
Ascoli–Arzelá theorem, 169
atlases, see under manifolds
atom-free parts of Borel measures, 408
atomic parts of Borel measures, 408
atomless standard metric measure spaces, 410
atoms
  for Borel measures, 408
  in measure algebras, 398
axiom
  of Archimedes, 35
  of choice, 19
  of completeness, 33
axiomatic treatment, 7

axioms
  of countability, 177
  of order, 32

B

Baire category, 158
Baire category theorem, 159, 200–201
Baire classes of Borel maps, 402
Baire measures, 332, 335
duals of spaces of continuous functions and, 358
extensions to Borel measures, 343, 349–350
invariant, 361
  unicity of, 378–386
  on homogeneous spaces, 361
  on the real line, 299, 300, 301
quasi regularity and, 341
regularity and, 338–340
Riesz representation theorem and, 357
Baire sets, 331–335
  alternative definitions of, 334
  Borel sets and, 334–335
balls in metric spaces, 141
Banach limits, 228, 367
Banach spaces, 124, 218
  annihilators of subspaces of, 228
  of bounded linear operators, 221
  of continuous functions, 126
  uniformly convex, 418
bases
  for locally convex topological vector spaces, 240
  for topologies, 175–176
bijective functions, see one-to-one correspondences
binary expansions, 40
binary operations, 24
Bolzano–Weierstrass property
  in metric spaces, 153
  compactness and, 153, 155
  sequential compactness and, 154, 155
  in topological spaces, 193
  countable compactness and, 194
  first axiom of countability and, 194
  sequential compactness and, 194
Boolean algebras, see under algebras
Borel equivalences, 402, 410
  of uncountable complete separable metric spaces, 406, 407
Borel fields, see algebra: \(\sigma\)-
Borel mappings, 402
Borel measurable functions, 71
Borel measures see under measures
Borel sets, 52–53, 332, 333–334, 402
   absolutely measurable, 414
   Baire sets and, 335
Borel–Lebesgue theorem, 155
bounded convergence theorem, 84
bounded linear functionals, see under linear functionals
bounded linear operators, see under linear operators
bounded sets in topological spaces, 332
bounded variation, 103
   absolute continuity and, 108

C
   canonical representations of simple functions, 77
Cauchy criterion for convergence, 37
Cauchy sequences, 37
   convergent sequences and, 39
   in metric spaces, 146
   in normed linear spaces, 123
   uniform, 188
Cauchy–Schwarz inequality, 245–246
changes of variables, 112–113
characteristic functions, 70, 77
choice functions, 19
closed convex hulls, 242
closed graph theorem, 231
closed sets, 40, 43–45
   compact sets and, 44–45
   complements of, 44
   completeness and, 152
   in compact spaces, 156, 191
   in metric spaces, 142
   in topological spaces, 172
   unions of, 52, 172
   see also closures; points of closure
   in metric spaces, 142
   closures (continued)
      in the real numbers, 43
      in topological spaces, 172
      of finite unions, 43
      see also closed sets; points of closure
   cluster points, 37–38
      convergent subsequences and, 38
      in metric spaces, 146
      in topological spaces, 173
      largest and smallest, 38
      limit points and, 38
      of infinite sequences, 147, 153
      of nets, 189
co-meager sets, 158
   see also residual sets
cruder topologies, 174
compact sets, 190
   closed and bounded sets and, 44–45, 156, 191
   continuous images of, 156
   see also compactness; metric spaces: compact; topological spaces: compact
compact-open topologies, 193
compactifications
   Alexandroff one-point, 201
   Stone–Čech, 209
compactness
   completeness and, 156
   countable, 193
      Bolzano–Weierstrass property and, 194
      continuity and, 195
      of unit ball in weak* topology, 237
      sequential, 153
      Bolzano–Weierstrass property and, 154, 155
      compactness and, 155
      total boundedness and, 156
      uniform continuity and, 157
   see also compactness; metric spaces: compact; topological spaces: compact
complements of sets, 13
complete measure spaces, 256–257
complete measures, 291, 297, 310
complete metric spaces, 147
complete normed linear spaces, 124
complete orthonormal systems, 246
completely regular topologies, 179
completeness
   closed sets and, 152
   compactness and, 156
   of normed linear spaces and summability, 124
   of outer measures, 291
completeness axiom, 33
completions of measure spaces, 257
complex measures, 280
composition of functions, 10
congruent subsets, 365
connected topological spaces, see under topological spaces
continuity at a point in a topological space, 173
at a real number, 47
countable compactness and, 195
even, 364
uniform convergence of functions and, 49
continuous convergence of functions, 162
continuous extensions of uniformly continuous functions, 149
continuous functions, 47–52
approximating measurable functions, 69, 127–128
approximation of by polynomials, 213
extensions of, 165
Hölder, 170
inverse images of open sets and, 47
maximum and minimum values of, 47
on compact metric spaces, 154, 157
on metric spaces, 144
on topological spaces, 173
proper, 201
uniformly, 48

closed and bounded sets of the real numbers and, 48
compactness and, 157
continuous extensions of, 149
see also continuity
continuous parts of Borel measures, 408
continuum, 193
continuum hypothesis, 55n
corrections
Cauchy criterion for, 37
continuous, 162
in mean of order \( p \), 123
in measure, 95–96, 262
in metric spaces, 146
in normed linear spaces, 123
convergent subsequences and, 95
of measurable functions, 72–73
of sequences of measures, 269
of sequences of set functions, 268–269
pointwise, 49
almost everywhere, 73
in normed linear spaces, 123
uniform, 49, 188
continuity and, 49
convergent sequences, 37
Cauchy sequences and, 39
in topological spaces, 173
convergent subsequences and sequential compactness, 153
convex functions, 113–116
convex hulls, 242
convex sets, 239–245
extreme points of, 241
separated, 240
weak/strong closure and, 241
convolution operators, 316
coordinate charts, 207
coordinate maps, 207
coordinate neighborhoods, 207
countable additivity, 55, 66, 253–254
of Lebesgue measure, 62
countable compactness, see under compactness
countable ordinals, 26
countable sets, 11, 20–23
countable subadditivity, 55, 288
of outer measure, 57
countable unions of countable sets, 22
countable/co-countable measure, 254
countably additive measures, 55
counting measures, 55, 254
covariant norms of integral kernels, 313
covariant types of integral kernels, 313
covering numbers, 366
covers, see measurable covers; open coverings
cross sections of sets, 304
cubes, 184
cumulative distribution functions, 299, 300, 301

D

Daniell functionals, 420
Daniell integrals, see under integrals
De Morgan’s laws, 13, 14, 16
decomposable measures, 281
dense sets, 46
density of the rational numbers, 35, 46
denumerable sets, see countable sets
derivatives of functions, 99
derivatives, 97, 100–101
absolute continuity and, 109
bounded variation and, 104
of integrals, 106–108
Radon-Nikodym, 278, 279
see also Radon-Nikodym theorem
diagonal process, 23
diameters of subsets of metric spaces, 140
diffeomorphisms, 388
differences of sets, 13
differentiability, 100
absolute continuity and, 109
bounded variation and, 104
see also derivatives
differentiable structures on manifolds, 208
differential measures, 388, 390
dini’s theorem, 195
generalization, 210–211
direct products, 20, 150
total boundedness and, 158
see also Cartesian products
direct summands, 185
direct unions, 185
directed systems, 188
<table>
<thead>
<tr>
<th>Term</th>
<th>Page(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>discontinuities of functions</td>
<td>104</td>
</tr>
<tr>
<td>discrete topologies</td>
<td>172</td>
</tr>
<tr>
<td>disjoint collections</td>
<td>13</td>
</tr>
<tr>
<td>disjoint sets</td>
<td>13</td>
</tr>
<tr>
<td>distributive laws for sets</td>
<td>12, 14, 16</td>
</tr>
<tr>
<td>domains</td>
<td>9</td>
</tr>
<tr>
<td>duals of normed linear spaces</td>
<td>226</td>
</tr>
<tr>
<td>Egoroff’s theorem</td>
<td>73, 263</td>
</tr>
<tr>
<td>empty set</td>
<td>8</td>
</tr>
<tr>
<td>entourages</td>
<td>187</td>
</tr>
<tr>
<td>envelopes of functions</td>
<td>52</td>
</tr>
<tr>
<td>equicontinuous families of</td>
<td>167–170, 188</td>
</tr>
<tr>
<td>functions</td>
<td>362</td>
</tr>
<tr>
<td>equivalence between functions</td>
<td>9</td>
</tr>
<tr>
<td>and graphs</td>
<td>9</td>
</tr>
<tr>
<td>equivalence classes</td>
<td>24</td>
</tr>
<tr>
<td>equivalence relations</td>
<td>23</td>
</tr>
<tr>
<td>compatible with binary</td>
<td>24</td>
</tr>
<tr>
<td>operations</td>
<td></td>
</tr>
<tr>
<td>essential supremum norm</td>
<td>119</td>
</tr>
<tr>
<td>Euclidean spaces</td>
<td>139</td>
</tr>
<tr>
<td>even continuity</td>
<td>364</td>
</tr>
<tr>
<td>exhaustions of topological</td>
<td>204</td>
</tr>
<tr>
<td>spaces</td>
<td></td>
</tr>
<tr>
<td>extended real numbers</td>
<td>36</td>
</tr>
<tr>
<td>extended real-valued functions</td>
<td>36</td>
</tr>
<tr>
<td>extensions of measures</td>
<td></td>
</tr>
<tr>
<td>extreme points</td>
<td>241</td>
</tr>
<tr>
<td>F</td>
<td></td>
</tr>
<tr>
<td>$F_\alpha$ sets</td>
<td>53</td>
</tr>
<tr>
<td>Fatou’s lemma</td>
<td>86, 89, 92–93, 96, 264</td>
</tr>
<tr>
<td>Daniell integrals</td>
<td>and, 426</td>
</tr>
<tr>
<td>generalization of</td>
<td>269</td>
</tr>
<tr>
<td>field axioms</td>
<td>31–32</td>
</tr>
<tr>
<td>fields</td>
<td>32</td>
</tr>
<tr>
<td>finer topologies</td>
<td>174</td>
</tr>
<tr>
<td>finite additivity</td>
<td>55n, 254</td>
</tr>
<tr>
<td>finite coverings</td>
<td>44</td>
</tr>
<tr>
<td>finite intersection property</td>
<td>45, 153, 190–191</td>
</tr>
<tr>
<td>finite measures</td>
<td>256</td>
</tr>
<tr>
<td>finite ordinals</td>
<td>26</td>
</tr>
<tr>
<td>finite sequences</td>
<td>10, 20–21</td>
</tr>
<tr>
<td>finite sets</td>
<td>11, 21</td>
</tr>
<tr>
<td>finite subcoverings</td>
<td>153</td>
</tr>
<tr>
<td>first axiom of countability</td>
<td>177</td>
</tr>
<tr>
<td>first category</td>
<td>158, 159, 160</td>
</tr>
<tr>
<td>sets of measure zero</td>
<td>and, 159</td>
</tr>
<tr>
<td>Fubini’s theorem</td>
<td>307–308</td>
</tr>
<tr>
<td>functions (continued)</td>
<td></td>
</tr>
<tr>
<td>continuous</td>
<td>144</td>
</tr>
<tr>
<td>differentiable on the left/right</td>
<td>113</td>
</tr>
<tr>
<td>discontinuities of</td>
<td>104</td>
</tr>
<tr>
<td>domains</td>
<td>9</td>
</tr>
<tr>
<td>envelopes of</td>
<td>52</td>
</tr>
<tr>
<td>equicontinuous families of</td>
<td>167–170, 188</td>
</tr>
<tr>
<td>equivalence with graphs</td>
<td>9</td>
</tr>
<tr>
<td>extended real-valued</td>
<td>36</td>
</tr>
<tr>
<td>Hölder continuous</td>
<td>170</td>
</tr>
<tr>
<td>holomorphic</td>
<td>170</td>
</tr>
<tr>
<td>homomorphisms</td>
<td>393</td>
</tr>
<tr>
<td>integrable</td>
<td>266</td>
</tr>
<tr>
<td>isometries</td>
<td>145, 415–418</td>
</tr>
<tr>
<td>limits inferior/superior of</td>
<td>50</td>
</tr>
<tr>
<td>lower envelopes of</td>
<td>52</td>
</tr>
<tr>
<td>lower semicontinuous</td>
<td>51</td>
</tr>
<tr>
<td>monotone increasing sequences</td>
<td>and, 51–52</td>
</tr>
<tr>
<td>measurable</td>
<td>259</td>
</tr>
<tr>
<td>negative parts of</td>
<td>89</td>
</tr>
<tr>
<td>one-to-one</td>
<td>10</td>
</tr>
<tr>
<td>onto</td>
<td>9, 144</td>
</tr>
<tr>
<td>open</td>
<td>see open mappings</td>
</tr>
<tr>
<td>piecewise linear</td>
<td>50</td>
</tr>
<tr>
<td>polygonal</td>
<td>50</td>
</tr>
<tr>
<td>positive parts of</td>
<td>89</td>
</tr>
<tr>
<td>ranges of</td>
<td>9</td>
</tr>
<tr>
<td>simple</td>
<td>70, 77, 260</td>
</tr>
<tr>
<td>canonical representations of</td>
<td>77</td>
</tr>
<tr>
<td>integrals of</td>
<td>263</td>
</tr>
<tr>
<td>singular</td>
<td>111</td>
</tr>
<tr>
<td>somas of</td>
<td>396</td>
</tr>
<tr>
<td>step</td>
<td>51, 76</td>
</tr>
<tr>
<td>approximating measurable</td>
<td>functions, 69, 127–128</td>
</tr>
<tr>
<td>functions</td>
<td></td>
</tr>
<tr>
<td>strictly monotone increasing</td>
<td>49</td>
</tr>
<tr>
<td>successor-preserving</td>
<td>27</td>
</tr>
<tr>
<td>uniformly continuous</td>
<td>148</td>
</tr>
<tr>
<td>images of Cauchy sequences</td>
<td>and, 148</td>
</tr>
<tr>
<td>and</td>
<td></td>
</tr>
<tr>
<td>upper envelopes of</td>
<td>52</td>
</tr>
<tr>
<td>upper semicontinuous</td>
<td>51</td>
</tr>
<tr>
<td>on a topological space</td>
<td>195</td>
</tr>
<tr>
<td>see also homeomorphisms;</td>
<td></td>
</tr>
<tr>
<td>isomorphisms</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td></td>
</tr>
<tr>
<td>$G_\delta$ sets</td>
<td>53</td>
</tr>
<tr>
<td>complete metrics</td>
<td>and, 164, 165–166</td>
</tr>
<tr>
<td>generalized Cantor ternary</td>
<td>set, 64</td>
</tr>
<tr>
<td>set</td>
<td></td>
</tr>
<tr>
<td>generalized principle of</td>
<td>recursive definition, 11, 12</td>
</tr>
<tr>
<td>recursive definition</td>
<td></td>
</tr>
<tr>
<td>graphs</td>
<td></td>
</tr>
<tr>
<td>determining relations</td>
<td>23</td>
</tr>
<tr>
<td>of functions</td>
<td>9</td>
</tr>
</tbody>
</table>
greatest lower bounds, 33
of unbounded sets, 36
group actions, 376
group topologies, 371

H
Haar measures, 373, 379
Hahn decomposition theorem, 273
Hahn decompositions, 274
Hahn–Banach theorem, 223
for complex vector spaces, 227
Hausdorff dimension, 330
of Cantor ternary set, 330
Hausdorff maximal principle, 25
Hausdorff measures, 329
Hausdorff spaces, 178
Heine–Borel theorem, 44
general metric spaces and, 152
Hilbert cubes, 184
Hilbert spaces, 245
nonseparable, 249
orthogonal elements in, 246
orthogonal systems in, 246
complete, 246
orthonormal systems in, 246, 247–248
separable, 247–248
Hölder continuous functions, 170
Hölder inequality, 121, 282
for sequences, 122
homeomorphisms, 50
between metric spaces, 144
between topological spaces, 174
transitive groups of, 361
uniform, 148
homogeneous spaces, 361
homomorphisms, 393

I
images of sets under functions, 9
indefinite integrals, see under integrals
indexed subsets, 15
indiscrete topologies, see trivial topologies
induction, 7
infinite sequences, 10, 20–21
infinite sets, 11
infinity (∞) as an element of the extended real numbers, 36
inherited topologies, 173
injective functions, see functions: one-to-one
inner measures, see under measures
inner products, 245
inner regular Borel measures, 369
inner regular measures, 337
integers as a subset of ordered fields, 35
integrability of absolute value, 93
integrable functions, 266
for Daniell integrals, 425
measurable, 90
non-negative measurable, 88
Riemann, 76, 84–85
integrable sets, 429
integral kernels, 313
integral operators, 313
absolute operator types of, 315
integrals
as bounded linear functionals, 131
Daniell, 419, 420, 421
extensions of, 422–427
uniqueness of extensions of, 427–429
indefinite, 104
absolute continuity and, 110
derivatives of, 106, 107
Lebesgue
as a generalization of the Riemann integral, 81
basic properties of, 82, 85, 90
of bounded measurable functions, 81
of non-negative measurable functions, 85
of simple functions, 77–79
of non-negative measurable functions, 264
of simple functions, 263
Riemann, 75–77, 93
as Daniell integrals, 420
integration, 263–268
as a linear functional, 282–283
basic properties of, 266
order of, 307–308
interior points
in the real numbers, 46
in topological spaces, 172
interiors of sets
in metric spaces, 143
in topological spaces, 172
intermediate value theorem, 48
generalization of, 182
internal points, 239
intersections, 12, 44
of collections, 14
of indexed sets, 15
of open sets, 52
intervals of real numbers, 40
invariant measures, 361
invariant topologies, 370
inverse images, 9–10
isolated points, 402
isolated sets, 46
isometries, 145, 415–418
isomorphic measures, 407
isomorphisms, 220
of measure algebras, 398
isomorphisms
  of measure algebras (continued)
    point mappings and, 398, 412
  of topological vector spaces, 234
isotropy subgroups, 378

J
Jensen inequality, 115
Jordan decompositions of signed measures, 274

K
kernels
  integral, 313
    measurable, 324
  of operators, 222
Krein–Milman theorem, 242
Kuratowski fourteen-subset problem, 175

L
$L^\infty$ spaces, 119, 282
$L^p$ spaces, 118, 282
  lattices, 210
    homomorphisms of, 393
    polar sets of, 421
    vector, 419, 420
  least upper bounds, 33
    of unbounded sets, 36
Lebesgue convergence theorem, 91, 92–93, 96, 267
  Daniell integrals and, 426
  generalization of, 92, 269, 270
Lebesgue decompositions of measures, 278
  Lebesgue integrals, see under integrals
Lebesgue measure, 61, 254, 256, 383
  as a normalized Hausdorff measure, 330
  countable additivity of, 62
  separable atomless measure algebras and, 399
Lebesgue–Stieltjes integral, 302
  left actions, 376
  left invariant measures, 373
  left translations, 371
Lie groups, 388
  limits
    inferior (lim inf), 38, 39
      of functions, 50
    of nets, 189
    of sequences, 37
      convergent subsequences and, 39
      points of closure and, 46
    superior (lim sup), 38, 39
      of functions, 50
Lindelöf property
  in the real numbers, 42
  in topological spaces, 177, 193, 203, 207
  Lindelöf theorem, 142
linear functionals, 130, 222
  bounded, 130, 355
    as integrals, 131–135, 282–287, 352
    extensions of, 222–223, 224
    positive, 352
linear manifolds, see under manifolds
linear operators, 220, 313
  bounded, 220
  Banach spaces of, 221
  continuity of, 220
linear orderings, 25
linear spaces, 118, 217
  normed, 118
    complete, 124
    duals of, 226
    isometrically isomorphic, 226
    reflexive, 227
    subspaces of, 218
linear transformations, see linear operators
Lipschitz condition, 112
Littlewood’s three principles, 72, 84, 294
  first principle, 407
  second principle, 127, 282
local bases for topological vector spaces, 233
local properties, 186
locally compact topological spaces, 199
locally connected topological spaces, 183
locally convex topological spaces, 240
  strongest locally convex topologies on, 244
locally Euclidean topological spaces, 207
locally finite collections, 204
locally measurable sets, 257
lower bounds, 33
lower envelopes of functions, 52
lower Riemann integrals, 76
lower semicontinuous functions, see under functions
Lusin’s theorem, 74

M
manifolds, 206–207, 206–209
  atlases for, 207
    differentiable, 207–208
  linear, 218
    weak/strong topologies and, 236
mappings, see functions
meager sets, 158
  see also first category
measurability
  equality of inner and outer measures and, 320
  of Borel sets, 60–61
measurable covers, 324
measurable functions, 66–72, 67, 259
approximation by continuous and step functions, 69, 127–128
convergence of, 72–73
Daniell integrals and, 429
equivalence classes of, 119
point mappings and, 392
measurable kernels, 324
measurable point mappings, see point mappings
measurable rectangles, 303
measurable sets, 58, 289
Daniell integrals and, 429
locally measurable, 257
measurable spaces, 253
measurable spaces with null sets, 396
measure algebras, 398
measure spaces, 254
complete, 256–257
completions of, 257
isomorphic, 407
standard metric, 408–409
atomless, 410
measure zero and first category, 159
measure-theoretic regularity versus topological regularity, 346
measures, 253
absolute values of, 274–275
absolutely continuous, 276
Baire, see Baire measures
atomic parts of, 408
atoms for, 408
continuous parts of, 408
from Baire measures, 343
non-regular, 344
regularity and, 341
 carriers of, 351, 408
complete, 291, 297, 310
convergence of a sequence of, 269
decomposable, 281
defined by integration, 276
see also integrals
differential, 388, 390
Carathéodory extension theorem, 295
finite, 256
finitely additive, 254a
Haar, 373, 379
Hausdorff, 329
normalized, 330
inner, 317–325
equality with outer measures, 319
superadditivity of, 321
inner regular, 337
invariant, 361
left/right, 373
isomorphic, 407
measures (continued)
Lebesgue decompositions of, 278
mutually singular, 274, 276
on an algebra, 291
outer, 56–61, 288, 295
approximation by Borel sets, 63
approximation by open and closed sets, 63
approximation by open sets, 58
equality with inner open and closed sets, 319
of countable sets, 58
of intervals, 56
outer regular, 337
product, 303
quasi regular, 340
regular, 337
represented as integrals, 276–277
see also Radon–Nikodym theorem
restrictions of, 258
saturated, 257, 297
semifinite, 256
$\sigma$-finite, 256
 signed, 271
Jordan decompositions of, 274
supports of, 408
total variations of, 274–275
see also Lebesgue measure
metric spaces, 139
compact, 153
Bolzano–Weierstrass property and, 153, 155
continuous functions on, 157
complete, 147
expansions to complete metric spaces, 147
first axiom of countability and, 177
second category everywhere, 159
subspaces of, 151
topologically complete, 413
totally bounded, 154
zero-dimensional, 406
metrics, 139
equivalent, 145
extended, 141
proper, 204
uniformly equivalent, 148
metrizable topological spaces, 172
minimal elements, 25
versus smallest elements, 26
Minkowski inequality, 120, 282
for sequences, 122
monotone convergence theorem, 87, 89, 92–93, 96, 265
monotone functions and bounded variation, 103
monotone increasing functions, 49
monotone sequences, 11
monotonicity, 55, 254, 288
Moore manifold, 208
mutually singular measures, 274, 276
N

$n$-tuples, see finite sequences
natural mappings of sets onto quotient spaces, 24
natural numbers, 7
as a subset of ordered fields, 35
as a subset of the real numbers, 34
negative parts of functions, 89
negative sets, 271
see also Hahn decompositions
neighborhoods of points
in metric spaces, 143
in topological spaces, 176
nets, 189
non-finite ordinals, 26–27
nonmeager sets, 158
see also second category
nonmeasurable sets, 64
contained in sets of positive measure, 66
normal spaces, 178
normalized Hausdorff measures, 330
normed linear spaces, see under linear spaces
norms, 118
equivalent, 231
of bounded linear functionals, 130
of bounded linear operators, 220
on linear spaces, 217
subadditivity of, 119–120
superadditivity of, 120
nowhere dense sets, 158, 159, 160
null functions, 428
null sets
of Boolean $\sigma$-algebras, 395–396
of signed measures, 271

O

one-to-one correspondences, 10
finiteness/countability and, 21
onto functions, see under functions
open actions, 376
open coverings, 44, 153
Lebesgue numbers of, 155
refinements of, 192
open mappings, 229
open sets, 41
complements of, 44
countable subcollections of, 42
disjoint unions of intervals and, 42
in metric spaces, 141
in topological spaces, 171
intersections and unions of, 41, 142
operator types of integral kernels, 313
order axioms, 32
order of integration, 307–308
ordered fields, 32
ordered pairs, 8
orderings, 24–27
ordinals, 26
ordinate sets, 260, 261
orthogonal elements in Hilbert spaces, 246
orthogonal systems, 246
orthonormal systems, 246, 247–248
complete, 246
Osgood’s theorem, 170
outer measures, see under measures
outer regular measure, 337

P

paracompact topological spaces, 205, 207
manifolds and, 207
partial orderings, 24–26, 27
piecewise linear functions, 50
point mappings, 392
and isomorphisms of measure algebras, 398, 412
points of closure, 43
limits of sequences and, 46
in metric spaces, 142
in topological spaces, 172
nets and, 189
see also closed sets; closures
pointwise convergence, see under convergence
polar sets of lattices, 421
polygonal functions, 50
positive linear functionals, 352
positive parts of functions, 89
positive sets, 271
power sets, 12
principle of recursive definition, 11
products
of compact topological spaces, 196–199
of sequentially compact topological spaces, 198
of topological spaces, 184
see also Cartesian products; direct products; measures: product
projections, 184
proper actions, 377
proper continuous functions, 201
proper metrics, 204
pseudocompact topological spaces, 199n
pseudometric spaces, 141
of Cauchy sequences, 147
pseudometrics, 140
pseudonorms, 219

Q

quasi regular measures, 340
quotients, 24
### R

- **Radon–Nikodym derivatives**, 278, 279
- **Radon–Nikodym theorem**, 276–277
  - for signed measures, 279
- **ranges**, 9
- **rational numbers**, 7
  - as a subset of ordered fields, 35
  - as a subset of the real numbers, 34
  - countability of, 22
  - density of, 35, 46
- **real numbers**, 31
  - Borel subsets of, 52–53
  - completeness axiom for, 33
  - continuous functions on, 47–52
  - extended, 36
  - field axioms for, 31–32
  - order axioms for, 32
  - sequences of, 37–38
  - topology on, 40–46
- **rectangles**, 303
  - refinements of open coverings, 192
- **reflexive partial orderings**, 25
- **reflexive relations**, 23–24
- **regular measures**, 337
  - topological versus measure-theoretic regularity, 346
- **regular topological spaces**, 178
- **relations**, 23–24
- **relative closures**, 151
- **relative complements**, 13
- **residual sets**, 158, 159, 160
- **restrictions**
  - of functions, 10
  - of measures, 258
- **Riemann integrable functions**, 76, 84–85
  - see also integrable functions
- **Riemann integrals**, see under integrals
- **Riemann–Lebesgue theorem**, 94
- **Riemann’s convergence criterion**, 303
- **Riesz representation theorem**
  - for general measure spaces, 284
  - for the real numbers, 132
  - for topological spaces, 357
- **Riesz–Fisher theorem**, 125, 282
- **Riesz–Markov theorem**, 352
- **right actions**, 377
- **right invariant measures**, 373
- **right translations**, 371

### S

- **saturated measures**, 257, 297
- **Schwarz inequality**, 245–246
- **second axiom of countability**, 177
- **second category**, 158
  - (continued)
    - locally compact topological spaces and, 201
  - everywhere, 159, 164
- **second countable topological spaces**, 177, 193, 207
- **segments of well-ordered sets**, 27
- **semialgebras**, 297
  - of measurable rectangles, 303
- **semicontinuous functions**, 51
- **semitfinite measures**, 51
- **separable of subspaces**, 152
- **separable measure algebras**, 398
- **separable metric spaces**, 142
- **separated convex sets**, 240
- **separating points**, 210, 241
- **separating sets by functions**, 327
- **separation axioms**, 178
- **sequences**, 10
  - of real numbers, 37
  - of subsets, 14
  - summable, 39
- **separability of subspaces**, 152
- **sequential compactness**, see under compactness
- **sequentially compact topological spaces**, see under topological spaces
  - series
    - absolutely summable, 124
    - summable, 124
- **set functions**, 54, 253
  - setwise convergence of, 268–269
- **set membership**, 7
- **set theory**, 6
- **set-theoretic operations**, 12–20
- **σ-algebras**, see under algebras
- **σ-bounded sets**, 332
- **σ-compact topological spaces**, 203, 207
- **σ-finite measures**, 256
- **σ-homomorphisms**, 393
- **σ-ideals**, 159, 396
- **σ-rings**, 258
- **signed measures**, see under measures
  - simple functions, see under functions
  - simple orderings, see linear orderings
- **singleton sets**, 8
- **singular functions**, 111
- **somas of functions**, 396
- **stability of measurable functions under operations**, 67–69
- **standard metric measure spaces**, 408–409
- **star-finite collections**, 205
- **step functions**, 51, 76
- **Stone–Čech compactifications**, 209
- **Stone’s theorem**, 431
- **Stone–Weierstrass theorem**, 212–213
  - alternative proof of, 359
- **strict partial orderings**, 25
- **strictly monotone increasing functions**, 49
- **strong topologies on vector spaces**, 236
- **stronger topologies**, 174
- **subadditivity**, 288

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9
subbases for topologies, 176
subordinate collections of functions, 200
subsequences, 11
subsets, 7
subspaces of linear spaces, 218
successor-preserving functions, 27
summable sequences, 39
sums of arbitrary sets of real numbers, 40
supports
  of atoms, 408
  of functions, 200, 331
  of measures, 351, 408
surjective functions, see functions: onto
symmetric differences, 13
symmetric relations, 23

T

ternary expansions, 40
thick sets, 402, 410–411
  very, 402
Tietze’s extension theorem, 179
Tonelli’s theorem, 309
topological closures, 172
topological equicontinuity, 362
topological groups, 372
topological neighborhoods, 176
topological properties, 144
topological regularity versus measure-theoretic regularity, 346
topological spaces
  bounded sets in, 332
  compact, 190
    continuous images of, 191
    products of, 196–199
    regularity of measures on, 338
  connected, 182
    arcwise, 183
countably compact, 193
exhaustions of, 204
Hausdorff, 178
homeomorphic, 174
Lindelöf, 193
local properties of, 186
locally compact, 199
locally connected, 183
metrizable, 172
normal, 178
paracompact, 205
pseudocompact, 199n
regular, 178
  topological versus measure-theoretic regularity, 346
  second countable, 177, 193, 207
  separations of, 182
  sequentially compact, 194
topological spaces
  sequentially compact (continued)
    products of, 198
  σ-compact, 203, 207
Tychonoff, 178
topological vector spaces, see under vector spaces
topologically complete metric spaces, 413

topologies, 171
  bases for, 175–176
  coarser, 174
  completely regular, 179
  discrete, 172
  finer, 174
group, 371
  inherited, 173
  invariant, 370
on the real numbers, 40–53
pointwise convergence, 184
product, 184
  stronger, 174
subbases for, 176
  trivial, 172
  weak, 179
  weaker, 174
Zariski, 181
total boundedness, 154
total orderings, see linear orderings
total variation of measures, 274–275
transitive groups of homeomorphisms, 361
transitive relations, 23
translates of subsets, 365
translation invariance, 54, 55
translations, 371
triangle inequality, 140
trivial topologies, 172
Tychonoff spaces, 178
Tychonoff’s theorem, 196–199, 197

U

uncountability
  of the real numbers, 40
  of the unit interval, 58
uncountable ordinals, 26
uncountable sets, 26
uncountable sums, 40
uniform boundedness principle, 160, 232
uniform Cauchy sequences, 188
uniform convergence, see under convergence
uniform properties, 148
uniform spaces, 187
uniformly continuous functions, see under continuous functions
uniformly convex Banach spaces, 418
unimodular groups, 383
unions, 12
  of closed sets, 52, 172
  of collections, 14
  of indexed sets, 15
  of open sets, 41, 142
unit sets, 8
univalent functions, see functions: one-to-one
unordered pairs, 8
upper bounds, 33
upper envelopes of functions, 52
upper Riemann integrals, 75
upper semicontinuous functions, see under functions
Urysohn metrization theorem, 179
Urysohn’s Lemma, 179

V
vector lattices, 419, 420
vector spaces, 217
  strong topologies on, 236
  topological, 233
    local bases for, 233
    locally convex, 240
  see also linear spaces
very thick sets, 402
Vitali coverings, 97

W
weak topologies, 179
  on vector spaces, 236
weak* topologies, 237
  sets of finite Baire measures and, 358
weaker topologies, 174
Weierstrass theorem, 214
  see also Stone-Weierstrass theorem
well-ordering principle, 7, 26
well orderings, 26–27

Z
Zariski topologies, 181
zero-dimensional metric spaces, 406