Some remarks on topological K-theory of dg categories

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Abstract

Using techniques from motivic homotopy theory, we prove a conjecture of Anthony Blanc about semi-topological K-theory of dg categories with finite coefficients. Along the way, we show that the connective semi-topological K-theories defined by Friedlander-Walker and by Blanc agree for quasi-projective complex varieties and we study étale descent of topological K-theory of dg categories.

Key Words. Semi-topological K-theory, motivic homotopy theory, dg categories.

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1 Introduction

Blanc defines [Bla16] semi-topological and topological K-theory functors

$$\mathrm{K}^{\mathrm{st}}, \mathrm{K}^{\mathrm{top}} : \mathrm{Cat}_{\mathbb{C}} \to \mathcal{S}\mathrm{p},$$

where $\mathrm{Cat}_{\mathbb{C}}$ denotes the ∞-category of small idempotent complete pretriangulated dg categories over $\mathbb{C}$[1] ($\mathbb{C}$-linear dg categories in this paper for short) and $\mathcal{S}\mathrm{p}$ is the ∞-category of spectra. When $\mathcal{C}$ is a $\mathbb{C}$-linear dg category, there are natural maps $\mathrm{K}(\mathcal{C}) \to \mathrm{K}^{\mathrm{st}}(\mathcal{C}) \to \mathrm{K}^{\mathrm{top}}(\mathcal{C})$. Moreover, $\mathrm{K}^{\mathrm{st}}(\mathcal{C})$ is a ku-module spectrum, and, by definition,

$$\mathrm{K}^{\mathrm{top}}(\mathcal{C}) \simeq \mathrm{K}^{\mathrm{st}}(\mathcal{C})[\beta^{-1}] \simeq \mathrm{K}^{\mathrm{st}}(\mathcal{C}) \otimes_{\mathrm{ku}} \mathrm{KU},$$

where $\beta \in \pi_2\mathrm{ku}$ is the Bott element.

Let $\mathrm{Sch}_{\mathbb{C}}$ denote the category of separated $\mathbb{C}$-schemes of finite type. If $F : \mathrm{Cat}_{\mathbb{C}} \to \mathcal{S}\mathrm{p}$ is a functor and $\mathcal{C}$ is a $\mathbb{C}$-linear dg category, then we define a presheaf $\underline{F}(\mathcal{C}) : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ by the formula $\underline{F}(\mathcal{C})(X) \simeq F(\mathrm{Perf}_X \otimes_{\mathbb{C}} \mathcal{C})$. In other words, $\underline{F}(\mathcal{C})$ is the composition of the functor $\mathrm{Perf} : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Cat}_{\mathbb{C}}$, the endofunctor

$$\mathrm{Cat}_{\mathbb{C}} \to \mathrm{Cat}_{\mathbb{C}} \qquad \mathcal{D} \mapsto \mathcal{D} \otimes_{\mathbb{C}} \mathcal{C},$$

and the functor $F : \mathrm{Cat}_{\mathbb{C}} \to \mathcal{S}\mathrm{p}$. In many cases, we will use the restriction of $\underline{F}(\mathcal{C})$ to $\mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}}$, $\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}$, or $\mathrm{Aff}_{\mathbb{C}}^{\mathrm{sm,op}} \simeq \mathrm{Sm}_{\mathbb{C}}^{\mathrm{aff,op}}$ the opposites of the categories of affine $\mathbb{C}$-schemes of finite type, smooth separated $\mathbb{C}$-schemes of finite type, and smooth affine $\mathbb{C}$-schemes, respectively.

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[1]By work of L. Cohn [Coh13], $\mathrm{Cat}_{\mathbb{C}}$ is equivalent to the ∞-category of small idempotent complete $\mathbb{C}$-linear stable ∞-categories.

In this paper, we prove three theorems about semi-topological and topological K-theory of dg categories. First, we prove that $\mathrm{K}^{\mathrm{sst}}(X) \simeq \mathrm{K}^{\mathrm{cn,st}}(\mathrm{Perf}_X)$ when X is a quasi-projective complex variety, where $\mathrm{K}^{\mathrm{sst}}(X)$ is the semi-topological K-theory spectrum defined by Friedlander and Walker in [FW01] and $\mathrm{K}^{\mathrm{cn,st}}(\mathrm{Perf}_X)$ is the connective version of Blanc's semi-topological K-theory. Second, we prove a conjecture of Blanc, stating that $\mathrm{K}(\mathcal{C})/n \simeq \mathrm{K}^{\mathrm{st}}(\mathcal{C})/n$ for $n \geqslant 1$ and any $\mathbb{C}$-linear dg category $\mathcal{C}$. Third, we prove that $\underline{\mathrm{K}}^{\mathrm{top}}(\mathcal{C})$ is $\mathbb{A}^1$-invariant and a hypersheaf for the étale topology on $\mathrm{Sm}_{\mathbb{C}}$ for any $\mathbb{C}$-linear dg category $\mathcal{C}$. Put together, the last two theorems imply that $\underline{\mathrm{K}}(\mathcal{C})/n$ satisfies étale hyperdescent after inverting the Bott element.

The first theorem has also been obtained by Blanc and Horel and they also made progress toward the second theorem along the same lines as the argument we give.

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2 Comparison of semi-topological K-theories

The original definition of semi-topological K-theory is for complex varieties and goes back to work of Friedlander and Walker [FW02, FW01]. They construct spectra $\mathrm{K}^{\mathrm{semi}}(X)$ and $\mathrm{K}^{\mathrm{sst}}(X)$ when X is quasi-projective and they give a natural map $\mathrm{K}^{\mathrm{sst}}(X) \to \mathrm{K}^{\mathrm{semi}}(X)$. When X is projective and weakly normal, this map is an equivalence by [FW01, Theorem 1.4]. In their survey [FW05], they settle on $\mathrm{K}^{\mathrm{sst}}(X)$ as the 'correct' definition of semi-topological K-theory of quasi-projective complex varieties. It is natural to wonder about the relationship between $\mathrm{K}^{\mathrm{sst}}(X)$ and $\mathrm{K}^{\mathrm{st}}(\mathrm{Perf}_X)$. We prove that they are in fact equivalent. Blanc has communicated to us that he was aware of this fact, although it was open at the time of [Bla16].

We recall the definition of semi-topological K-theory of dg categories from [Bla16]. Let $\mathrm{Sch}_{\mathbb{C}} \to \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_{\mathbb{C}})$ be the spectral Yoneda functor, where $\mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_{\mathbb{C}})$ is the stable presentable ∞-category of presheaves of spectra on $\mathrm{Sch}_{\mathbb{C}}$. Let $\mathrm{Sch}_{\mathbb{C}} \to \mathcal{S}\mathrm{p}$ be the composition of

$$\mathrm{Sch}_{\mathbb{C}} \to \mathcal{S} \qquad X \mapsto \mathrm{Sing}\, X(\mathbb{C})$$

with the suspension spectrum functor $\Sigma^\infty_+ : \mathcal{S} \to \mathcal{S}\mathrm{p}$, where $\mathcal{S}$ denotes the ∞-category of topological spaces. Define the topological realization $\mathrm{Re} : \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_{\mathbb{C}}) \to \mathcal{S}\mathrm{p}$ as the left Kan extension

$$\begin{tikzcd} \mathrm{Sch}_{\mathbb{C}} \arrow[r, "X \mapsto \Sigma^\infty \mathrm{Sing}\, X(\mathbb{C})_+"] \arrow[d] & \mathcal{S}\mathrm{p} \\ \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_{\mathbb{C}}). \arrow[ur, dotted, "\mathrm{Re}"'] & \end{tikzcd} \tag{1}$$

Given $\mathcal{C} \in \mathrm{Cat}_{\mathbb{C}}$, there is the presheaf $\underline{\mathrm{K}}(\mathcal{C})$ as defined in Section 1, where $\mathrm{K} : \mathrm{Cat}_{\mathbb{C}} \to \mathcal{S}\mathrm{p}$ denotes nonconnective K-theory as defined for example in [BGT13].

Definition 2.1 (Blanc [Bla16])**.** The semi-topological K-theory of $\mathcal{C}$ is the spectrum $\mathrm{K}^{\mathrm{st}}(\mathcal{C}) = \mathrm{Re}(\underline{\mathrm{K}}(\mathcal{C}))$. More generally, let $f : X \to \mathrm{Spec}\,\mathbb{C}$ be a separated $\mathbb{C}$-scheme of finite type and let Sch_X be the category of separated X-schemes of finite presentation. There is an adjunction $f^* : \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_{\mathbb{C}}) \rightleftarrows \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_X) : f_*$ defined in the usual way. We let $\underline{\mathrm{K}}^{\mathrm{st}}(\mathcal{C}) : \mathrm{Sch}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ be the presheaf with value at $f : X \to \mathbb{C}$ given by $\underline{\mathrm{K}}^{\mathrm{st}}(\mathcal{C})(X) = \mathrm{Re}(f_* f^* \underline{\mathrm{K}}(\mathcal{C}))$. In particular, $\underline{\mathrm{K}}^{\mathrm{st}}(\mathcal{C})(X) \simeq \mathrm{K}^{\mathrm{st}}(\mathrm{Perf}_X \otimes_{\mathbb{C}} \mathcal{C})$ and $\underline{\mathrm{K}}^{\mathrm{st}}(\mathcal{C})(\mathrm{Spec}\,\mathbb{C}) \simeq \mathrm{K}^{\mathrm{st}}(\mathcal{C})$.

Definition 2.2 (Blanc [Bla16])**.** If we apply the same construction with connective K-theory K^{cn} we obtain a connective version of semi-topological K-theory, namely $\mathrm{K}^{\mathrm{cn,st}}(\mathcal{C}) = \mathrm{Re}(\underline{\mathrm{K}}^{\mathrm{cn}}(\mathcal{C}))$, where $\underline{\mathrm{K}}^{\mathrm{cn}}(\mathcal{C})$ is the presheaf of connective spectra $\underline{\mathrm{K}}^{\mathrm{cn}}(\mathcal{C})(X) \simeq \mathrm{K}^{\mathrm{cn}}(\mathrm{Perf}_X \otimes_{\mathbb{C}} \mathcal{C})$. This is the theory denoted by $\underline{\widetilde{\mathrm{K}}}(\mathcal{C})$ in Blanc's paper.

The following theorem has also been obtained by Blanc and Geoffroy Horel (private communication).

Theorem 2.3. *If X is a quasi-projective complex variety, then there is a natural equivalence $\mathrm{K}^{\mathrm{sst}}(X) \simeq \mathrm{K}^{\mathrm{cn,st}}(\mathrm{Perf}_X)$.*

Proof. To begin, we give the definition of $\mathrm{K}^{\mathrm{sst}}(X)$ after [FW01, Definition 1.1]. Let $\widetilde{\mathcal{T}\mathrm{op}}$ be a small category of topological spaces and continuous maps containing at least the essential image of $r : \mathrm{Sch}_{\mathbb{C}} \xrightarrow{U \mapsto U(\mathbb{C})} \mathcal{T}\mathrm{op}$ and the topological simplices Δ^n_{top}. Friedlander and Walker consider the left Kan extension

$$\begin{tikzcd} \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \arrow[r, "\underline{\mathrm{K}}^{\mathrm{cn}}(X)"] \arrow[d, "U \mapsto U(\mathbb{C})"'] & \mathcal{S}\mathrm{p} \\ \widetilde{\mathcal{T}\mathrm{op}}^{\mathrm{op}}. \arrow[ur, "r^*\underline{\mathrm{K}}^{\mathrm{cn}}(X)"'] & \end{tikzcd}$$

By definition, if Y is a topological space, then

$$r^*\underline{\mathrm{K}}^{\mathrm{cn}}(X)(Y) \simeq \operatorname*{colim}_{Y \to U(\mathbb{C})} \underline{\mathrm{K}}^{\mathrm{cn}}(X)(U) \simeq \operatorname*{colim}_{Y \to U(\mathbb{C})} \underline{\mathrm{K}}^{\mathrm{cn}}(X \times_{\mathbb{C}} U).$$

Evaluating $r^*\underline{\mathrm{K}}^{\mathrm{cn}}(X)$ at the cosimplicial space $\Delta^\bullet_{\mathrm{top}}$, we obtain a simplicial spectrum $r^*\underline{\mathrm{K}}^{\mathrm{cn}}(X)(\Delta^\bullet_{\mathrm{top}})$. The semi-topological K-theory of Y is defined to be

$$\mathrm{K}^{\mathrm{sst}}(X) = |\underline{\mathrm{K}}^{\mathrm{cn}}(X)(\Delta^\bullet_{\mathrm{top}})|.$$

Note that this process is precisely the composition of the functors

$$\widetilde{\mathrm{Re}}_{\mathcal{S}\mathrm{p}} : \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_{\mathbb{C}}) \xrightarrow{r^*} \mathcal{P}_{\mathcal{S}\mathrm{p}}(\widetilde{\mathcal{T}\mathrm{op}}) \xrightarrow{s_*} \mathcal{P}_{\mathcal{S}\mathrm{p}}(\Delta) \to \mathcal{S}\mathrm{p}$$

applied to $\underline{\mathrm{K}}^{\mathrm{cn}}(X)$, where s denotes the inclusion of Δ into $\widetilde{\mathcal{T}\mathrm{op}}$ classifying the cosimplicial space $\Delta^\bullet_{\mathrm{top}}$ and the final arrow is geometric realization of a simplicial spectrum. This composition is the stabilization of

$$\widetilde{\mathrm{Re}} : \mathcal{P}(\mathrm{Sch}_{\mathbb{C}}) \xrightarrow{r^*} \mathcal{P}(\widetilde{\mathcal{T}\mathrm{op}}) \xrightarrow{s_*} \mathcal{P}(\Delta) \xrightarrow{|-|} \mathcal{S}, .$$

where $\mathcal{P}(\Delta)$ is the ∞-category of presheaves of spaces on the simplex category Δ, or in other words the ∞-category of simplicial spaces, and $|-|$ denotes geometric realization. To prove the theorem it suffices to prove that $\widetilde{\mathrm{Re}} : \mathcal{P}(\mathrm{Sch}_{\mathbb{C}}) \to \mathcal{S}$ is equivalent to the functor $\mathcal{P}(\mathrm{Sch}_{\mathbb{C}}) \to \mathcal{S}$ obtained via the unstable version of the left Kan extension in (1):

$$\begin{tikzcd} \mathrm{Sch}_{\mathbb{C}} \arrow[r, "U \mapsto \mathrm{Sing}\, U(\mathbb{C})"] \arrow[d] & \mathcal{S} \\ \mathcal{P}(\mathrm{Sch}_{\mathbb{C}}). \arrow[ur, "\mathrm{Re}"] & \end{tikzcd}$$

To prove that $\widetilde{\mathrm{Re}} \simeq \mathrm{Re}$, note first that both functors are left adjoints because $\mathcal{P}(\mathrm{Sch}_{\mathbb{C}})$ is presentable. The only thing to check is that s_* preserves colimits, but this follows because colimits in presheaf categories are computed pointwise (see for example [Lur09, Corollary 5.1.2.3]). Thus, it suffices to prove that the restrictions of $\widetilde{\mathrm{Re}}$ and Re to $\mathrm{Sch}_{\mathbb{C}}$ are equivalent. On the one hand, we know that $\mathrm{Re}(U) \simeq U(\mathbb{C})$ in $\mathcal{S}$ for $U \in \mathrm{Sch}_{\mathbb{C}}$. On the other hand, by definition, $r^*U(\Delta^n_{\mathrm{top}}) \simeq \mathrm{colim}_{\Delta^n_{\mathrm{top}} \to V(\mathbb{C})} \mathrm{Hom}_{\mathrm{Sch}_{\mathbb{C}}}(V, U)$. Using Jouanolou's device [Jou73], we see that any map $\Delta^n_{\mathrm{top}} \to V(\mathbb{C})$ factors through $W(\mathbb{C}) \to V(\mathbb{C})$ where $W(\mathbb{C})$ is affine and $\mathrm{W}(\mathbb{C}) \to V(\mathbb{C})$ is a vector bundle torsor. Thus, by [FW02, Proposition 4.2], $r^*U(\Delta^n_{\mathrm{top}}) \cong \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(\Delta^n_{\mathrm{top}}, U(\mathbb{C})) \cong \mathrm{Sing}\, U(\mathbb{C})_n$ and hence $\widetilde{\mathrm{Re}}(U) \simeq \mathrm{Sing}\, U(\mathbb{C})$, as desired. $\square$

Remark 2.4. A theorem of Friedlander and Walker says that when X is smooth and quasi-projective, $\mathrm{K}^{\mathrm{sst}}(X)[\beta^{-1}] \simeq \mathrm{KU}(X(\mathbb{C}))$, the complex K-theory spectrum of the space of $\mathbb{C}$-points of X (see [FW05, Theorem 32]). It follows from the theorem that $\mathrm{K}^{\mathrm{top}}(\mathrm{Perf}_X) = \mathrm{K}^{\mathrm{st}}(\mathrm{Perf}_X)[\beta^{-1}] \simeq \mathrm{K}^{\mathrm{sst}}(X)[\beta^{-1}] \simeq \mathrm{KU}(X(\mathbb{C}))$ when X is smooth and quasi-projective, where $\mathrm{K}^{\mathrm{st}}(\mathrm{Perf}_X) \simeq \mathrm{K}^{\mathrm{cn,st}}(\mathrm{Perf}_X)$ because X is smooth and by [Bla16, Theorem 3.18]. This gives a new proof of one of the main theorems of Blanc's paper [Bla16, Theorem 1.1(b)] in the special case of X smooth and quasi-projective. Blanc's theorem says more generally that if X is separated and finite type over $\mathbb{C}$, then $\mathrm{K}^{\mathrm{top}}(\mathrm{Perf}_X) \simeq \mathrm{KU}(X(\mathbb{C}))$.

Remark 2.5. It is clear that one could have defined a nonconnective version $\mathrm{K}^{\mathrm{nc,sst}}(X)$ of Friedlander and Walker's $\mathrm{K}^{\mathrm{sst}}(X)$ simply by replacing connective K-theory with nonconnective K-theory in [FW01, Definition 1.1]. If this is done, then the proof of Theorem 2.3 goes through and shows that there are natural equivalences $\mathrm{K}^{\mathrm{nc,sst}}(X) \simeq \mathrm{K}^{\mathrm{st}}(X)$ for quasi-projective complex varieties X.

3 Blanc's conjecture

Let $\mathrm{C} \subseteq \mathrm{Sch}_{\mathbb{C}}$ be a full subcategory closed under taking products with $\mathbb{A}^1_{\mathbb{C}}$. Let $\mathcal{P}^{\mathbb{A}^1}_{\mathcal{S}\mathrm{p}}(\mathrm{C}) \subseteq \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{C})$ be the full subcategory of $\mathbb{A}^1$-invariant presheaves of spectra, i.e., those F such that the pullback maps $F(X) \to F(X \times_{\mathbb{C}} \mathbb{A}^1)$ are equivalences for all $X \in \mathrm{C}$. The inclusion has a left adjoint, $\mathrm{L}_{\mathbb{A}^1} : \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{C}) \to \mathcal{P}^{\mathbb{A}^1}_{\mathcal{S}\mathrm{p}}(\mathrm{C})$. A map $F \to G$ is an $\mathbb{A}^1$-equivalence if $\mathrm{L}_{\mathbb{A}^1}F \to \mathrm{L}_{\mathbb{A}^1}G$ is an equivalence. Given a presheaf of spectra F on C, we let $\mathrm{Sing}^{\mathbb{A}^1} F$ be the presheaf defined by

$$(\mathrm{Sing}^{\mathbb{A}^1} F)(X) = \operatorname*{colim}_{\Delta} F(X \times_{\mathbb{C}} \Delta^\bullet_{\mathbb{C}}),$$

where $\Delta^\bullet_{\mathbb{C}}$ is the standard cosimplicial affine scheme. It is a standard fact that $\mathrm{Sing}^{\mathbb{A}^1} F$ is $\mathbb{A}^1$-invariant in the sense that for every $X \in \mathrm{C}$, the pullback map $(\mathrm{Sing}^{\mathbb{A}^1} F)(X) \to (\mathrm{Sing}^{\mathbb{A}^1} F)(X \times_{\mathbb{C}} \mathbb{A}^1)$ induced by the projection $X \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \to X$ is an equivalence. Moreover, $F \to \mathrm{Sing}^{\mathbb{A}^1} F$ is an $\mathbb{A}^1$-equivalence. It follows that $\mathrm{Sing}^{\mathbb{A}^1} F \simeq \mathrm{L}_{\mathbb{A}^1} F$ for all F and that if F is $\mathbb{A}^1$-invariant, then the natural transformation $F \to \mathrm{L}_{\mathbb{A}^1} F$ is an equivalence. For proofs of these facts, see [MV99, Section 2.3]. From the natural equivalences $\mathrm{L}_{\mathbb{A}^1} F \simeq \mathrm{Sing}^{\mathbb{A}^1} F$, we see that if $i : \mathrm{C}' \subseteq \mathrm{C}$ is a subcategory (also closed under taking products with $\mathbb{A}^1_{\mathbb{C}}$) and if F is a presheaf on C, then $i^*\mathrm{L}_{\mathbb{A}^1} F = \mathrm{L}_{\mathbb{A}^1} i^* F$.

Let $\mathcal{C}$ be a $\mathbb{C}$-linear dg category. Then, $\mathrm{L}_{\mathbb{A}^1}\underline{\mathrm{K}}(\mathcal{C}) \simeq \underline{\mathrm{KH}}(\mathcal{C})$, where $\mathrm{KH} : \mathrm{Cat}_{\mathbb{C}} \to \mathcal{S}\mathrm{p}$ is the homotopy K-theory of dg categories, as defined for example in [Tab15]. If $F : \mathrm{Sch}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ is a presheaf of spectra, write F^{st} for the presheaf with value at $f : X \to \operatorname{Spec} \mathbb{C}$ in $\mathrm{Sch}_{\mathbb{C}}$ the spectrum $F^{\mathrm{st}}(X) \simeq \mathrm{Re}(f_* f^* F)$.

Lemma 3.1. *If $F : \mathrm{Sch}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ is a presheaf of spectra, then*

$$F^{\mathrm{st}} \simeq (\mathrm{L}_{\mathbb{A}^1} F)^{\mathrm{st}} \simeq \mathrm{L}_{\mathbb{A}^1}(F^{\mathrm{st}}).$$

In particular, if $\mathcal{C}$ is a $\mathbb{C}$-linear dg category, then

$$\mathrm{K}^{\mathrm{st}}(\mathcal{C}) \simeq \mathrm{KH}^{\mathrm{st}}(\mathcal{C}),$$

where $\mathrm{KH}^{\mathrm{st}}(\mathcal{C}) = \mathrm{Re}(\underline{\mathrm{KH}}(\mathcal{C}))$.

Proof. Since $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec}\mathbb{C}$ realizes to an equivalence in $\mathcal{S}\mathrm{p}$, the realization functor $\mathrm{Re} : \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_{\mathbb{C}}) \to \mathcal{S}\mathrm{p}$ factors through the $\mathbb{A}^1$-localization $\mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_{\mathbb{C}}) \to \mathcal{P}^{\mathbb{A}^1}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_{\mathbb{C}})$, which is modeled concretely by $\mathrm{L}_{\mathbb{A}^1}$. This proves that $F^{\mathrm{st}} \simeq (\mathrm{L}_{\mathbb{A}^1}F)^{\mathrm{st}}$. If we prove that $(\mathrm{L}_{\mathbb{A}^1}F)^{\mathrm{st}}$ is $\mathbb{A}^1$-invariant, then we will have proved that $\mathrm{L}_{\mathbb{A}^1}(F^{\mathrm{st}}) \simeq F^{\mathrm{st}}$. It is enough to prove that $\mathrm{st} : \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_{\mathbb{C}}) \to \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sch}_{\mathbb{C}})$ preserves $\mathbb{A}^1$-invariant presheaves. Let G be $\mathbb{A}^1$-invariant. If $f : X \to \operatorname{Spec}\mathbb{C}$, then $f_*f^*G \simeq g_*g^*G$ where $g : X \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec}\mathbb{C}$ since G is $\mathbb{A}^1$-invariant. Thus, $G^{\mathrm{st}}(X) \simeq \mathrm{Re}(f_*f^*G) \simeq \mathrm{Re}(g_*g^*G) \simeq G^{\mathrm{st}}(X \times_{\mathbb{C}} \mathbb{A}^1)$, as desired. The second claim follows from the equivalence $\underline{\mathrm{K}}^{\mathrm{st}}(\mathcal{C}) \simeq \underline{\mathrm{KH}}^{\mathrm{st}}(\mathcal{C})$ of presheaves evaluated at $\operatorname{Spec}\mathbb{C}$. ☐

Write $\mathcal{S}\mathrm{p}(\mathbb{C})$ for the ∞-category of motivic $\mathbb{P}^1$-spectra over $\mathbb{C}$, $\underline{\mathrm{Map}}_{\mathcal{S}\mathrm{p}(\mathbb{C})}(-,-)$ for the internal mapping object, and $\mathrm{Map}_{\mathcal{S}\mathrm{p}(\mathbb{C})}(-,-)$ for the (classical) mapping spectrum. A good reference for $\mathcal{S}\mathrm{p}(\mathbb{C})$ in the language of ∞-categories is [Rob15].

Proposition 3.2. *Let $\mathcal{C}$ be a $\mathbb{C}$-linear dg category. There is a motivic spectrum $\mathrm{KGL}(\mathcal{C}) \in \mathcal{S}\mathrm{p}(\mathbb{C})$ such that*

$$\mathrm{Map}_{\mathcal{S}\mathrm{p}(\mathbb{C})}(\Sigma^{\infty}_{\mathbb{P}^1}X_+, \mathrm{KGL}(\mathcal{C})) \simeq \underline{\mathrm{KH}}(\mathcal{C})(X)$$

for any $X \in \mathrm{Sm}_{\mathbb{C}}$.

Proof. Below, we use the (nonstandard) notation $T = (\mathbb{P}^1, \infty)$ for the based scheme and for the duration of this proof write $\mathbb{P}^1$ only to denote the unbased scheme.

Write $\mathcal{S}\mathrm{p}_{S^1}(\mathbb{C})$ for the ∞-category of $\mathbb{A}^1$-invariant, Nisnevich sheaves of spectra on $\mathrm{Sm}_{\mathbb{C}}$. Note that $\underline{\mathrm{KH}}(\mathcal{C}) \in \mathcal{S}\mathrm{p}_{S^1}(\mathbb{C})$. Indeed, it is $\mathbb{A}^1$-invariant by definition and it is a Nisnevich sheaf, because it is the restriction of a localizing invariant to $\mathrm{Sm}_{\mathbb{C}}$ (see [Bla16, Theorem 1.1(c)]).

By [Rob15, Corollary 2.22] for example, we have an equivalence

$$\mathcal{S}\mathrm{p}(\mathbb{C}) \simeq \mathrm{Stab}_T(\mathcal{S}\mathrm{p}_{S^1}(\mathbb{C})) := \lim(\mathcal{S}\mathrm{p}_{S^1}(\mathbb{C}) \xleftarrow{\Omega_T} \mathcal{S}\mathrm{p}_{S^1}(\mathbb{C}) \leftarrow \cdots).$$

Let $\beta \in \pi_0\mathrm{Map}_{\mathcal{S}\mathrm{p}_{S^1}(\mathbb{C})}(T, \underline{\mathrm{KH}}(\mathrm{Perf}_{\mathbb{C}})) \cong \mathrm{KH}_0((\mathbb{P}^1,\infty))$ be the usual Bott element. Write as well $\beta : \underline{\mathrm{KH}}(\mathcal{C}) \to \mathrm{Map}_{\mathcal{S}\mathrm{p}_{S^1}(\mathbb{C})}(T, \underline{\mathrm{KH}}(\mathcal{C}))$ for the "multiplication by β" map, obtained from the $\underline{\mathrm{KH}}(\mathrm{Perf}_{\mathbb{C}})$)-module structure on $\underline{\mathrm{KH}}(\mathcal{C})$. Now define $\mathrm{KGL}(\mathcal{C}) \in \mathcal{S}\mathrm{p}(\mathbb{C})$ to be the "constant" spectrum whose value is $\underline{\mathrm{KH}}(\mathcal{C})$ and structure maps $\beta : \underline{\mathrm{KH}}(\mathcal{C}) \to \mathrm{Map}_{\mathcal{S}\mathrm{p}_{S^1}(\mathbb{C})}(T, \underline{\mathrm{KH}}(\mathcal{C}))$.

Since $\underline{\mathrm{KH}}(\mathcal{C})$ is a localizing invariant and $\mathrm{Perf}_{\mathbb{P}^1_X} \simeq \mathrm{Perf}_X \oplus \mathrm{Perf}_X$, the projective bundle formula holds in $\underline{\mathrm{KH}}(\mathcal{C})$:

$$\underline{\mathrm{KH}}(\mathcal{C})(\mathbb{P}^1_X) \simeq \underline{\mathrm{KH}}(\mathcal{C})(X) \oplus \underline{\mathrm{KH}}(\mathcal{C})(X)$$

for $X \in \mathrm{Sch}_{\mathbb{C}}$. This splitting identifies $\mathrm{Map}_{\mathcal{S}\mathrm{p}_{S^1}(\mathbb{C})}(T, \underline{\mathrm{KH}}(\mathcal{C}))$ with $\underline{\mathrm{KH}}(\mathcal{C})$ via the map β defined above. In particular, we see that $\mathrm{KGL}(\mathcal{C})$ is a periodic motivic spectrum and $\Omega^{\infty}_T(\mathrm{KGL}(\mathcal{C})) \simeq \underline{\mathrm{KH}}(\mathcal{C})$. It is now immediate that

$$\mathrm{Map}_{\mathcal{S}\mathrm{p}(\mathbb{C})}(\Sigma^{\infty}_T X_+, \mathrm{KGL}(\mathcal{C})) \simeq \mathrm{Map}_{\mathcal{S}\mathrm{p}_{S^1}(\mathbb{C})}(\Sigma^{\infty}_{S^1}X_+, \underline{\mathrm{KH}}(\mathcal{C})) \simeq \underline{\mathrm{KH}}(\mathcal{C})(X),$$

for any $X \in \mathrm{Sm}_{\mathbb{C}}$.

☐

Now, we prove Blanc's conjecture. Blanc has told us that Horel was exploring a similar argument.

Theorem 3.3. *If $\mathcal{C}$ is a $\mathbb{C}$-linear dg category, then the natural map $\mathrm{K}(\mathcal{C})/n \to \mathrm{K}^{\mathrm{st}}(\mathcal{C})/n$ is an equivalence for any $n \geqslant 1$.*

Proof. By [Bla16, Theorem 3.18], we may compute the semi-topological K-theory of $\mathcal{C}$ using only smooth $\mathbb{C}$-schemes $\mathrm{Sm}_{\mathbb{C}} \subseteq \mathrm{Aff}_{\mathbb{C}}$. In fact, if $F : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ is any presheaf of spectra, we can compute $\mathrm{Re}(F)$ by first restricting F to $\mathrm{Sm}_{\mathbb{C}}$ and then using the realization $\mathrm{Re} : \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sm}_{\mathbb{C}}) \to \mathcal{S}\mathrm{p}$ given as the left Kan extension

$$\begin{tikzcd} \mathrm{Sm}_{\mathbb{C}} \arrow[r, "X \mapsto \Sigma^\infty \mathrm{Sing}\, X(\mathbb{C})_+"] \arrow[d] & \mathcal{S}\mathrm{p} \\ \mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sm}_{\mathbb{C}}). \arrow[ur, "\mathrm{Re}"'] & \end{tikzcd}$$

Let E denote the constant presheaf on $\mathrm{Sm}_{\mathbb{C}}$ with value $\mathrm{K}(\mathcal{C})/n$. There is a natural map $\mathrm{E} \to \underline{\mathrm{K}}(\mathcal{C})/n$. The topological realization of E is $\mathrm{K}(\mathcal{C})/n$ since it is a constant sheaf. As topological realization factors through Nisnevich hypersheaves, it suffices to check that $\mathrm{E} \to \underline{\mathrm{K}}(\mathcal{C})/n$ induces an equivalence after Nisnevich hypersheafification. For this, it suffices to see that the natural map $\mathrm{K}(\mathcal{C})/n \to \mathrm{K}(R \otimes_{\mathbb{C}} \mathcal{C})/n$ is an equivalence for every essentially smooth hensel local ring R over $\mathbb{C}$.

By the proposition above, $\underline{\mathrm{KH}}(\mathcal{C})$ is represented by a motivic spectrum, $\mathrm{KGL}(\mathcal{C})$. Thus, $\underline{\mathrm{KH}}(\mathcal{C})/n$ is represented by a motivic spectrum denoted $\mathrm{KGL}(\mathcal{C})/n$. Gabber-Suslin rigidity is valid for $\mathrm{KGL}(\mathcal{C})/n$ by [HY07, Corollary 0.4]. (As noted in loc. cit., the normalization property of that result holds for any motivic spectrum over an algebraically closed field.) In particular, $\mathrm{KH}(R \otimes_{\mathbb{C}} \mathcal{C})/n \to \mathrm{KH}(R/\mathfrak{m} \otimes_{\mathbb{C}} \mathcal{C})/n \simeq \mathrm{K}(\mathcal{C})/n$ is an equivalence for any essentially smooth hensel local ring R, where $\mathfrak{m} \subseteq R$ is the maximal ideal. But by a result of Tabuada [Tab17, Theorem 1.2(i)], whose proof essentially follows the argument of Weibel in the case of associative rings [Wei89, Proposition 1.6], $\underline{\mathrm{K}}(\mathcal{C})/n$ is $\mathbb{A}^1$-homotopy invariant so that $\underline{\mathrm{K}}(\mathcal{C})/n \simeq \underline{\mathrm{KH}}(\mathcal{C})/n$. It follows that $\mathrm{K}(R \otimes_{\mathbb{C}} \mathcal{C})/n \to \mathrm{K}(R/\mathfrak{m} \otimes_{\mathbb{C}} \mathcal{C})/n \simeq \mathrm{K}(\mathcal{C})/n$ is an equivalence so that $\mathrm{E} \to \underline{\mathrm{K}}(\mathcal{C})/n$ is an equivalence, which is what we wanted to prove. □

4 Descent for topological K-theory of dg catgories

In this section we prove the following result.

Theorem 4.1. *Let $\mathcal{C}$ be a $\mathbb{C}$-linear dg category.*

(i) *The presheaf $\underline{\mathrm{K}}^{\mathrm{top}}(\mathcal{C}) : \mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ satisfies étale hyperdescent.*

(ii) *The presheaf $\underline{\mathrm{K}}(\mathcal{C})/n[\beta^{-1}] : \mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ satisfies étale hyperdescent.*

Part (ii) of the theorem is a noncommutative generalization of the main theorem of Thomason [Tho85]. Indeed, if $\mathcal{C} \simeq \mathrm{Perf}_X$ where X is an essentially smooth separated $\mathbb{C}$-scheme, then $\underline{\mathrm{K}}(\mathrm{Perf}_X)/n[\beta^{-1}]$ is equivalent to the presheaf

$$Y \mapsto \mathrm{K}(Y \times_{\mathbb{C}} X)/n[\beta^{-1}],$$

which satisfies étale hyperdescent by [Tho85, Theorem 4.1]. In general, we cannot improve the result to semi-topological K-theory. Indeed, it is well-known that $\underline{\mathrm{K}}^{\mathrm{st}}(\mathrm{Perf}_X)/n \simeq \underline{\mathrm{K}}(\mathrm{Perf}_X)/n$ does not satisfy étale descent. The Quillen–Lichtenbaum conjectures (which follow from the, now proved, Bloch–Kato conjecture) give a bound on the failure of étale hyperdescent for K-theory with finite coefficients. For example, if X is an essentially smooth separated $\mathbb{C}$-scheme of Krull dimension d, then

$$\mathrm{K}(X)/\ell \to \mathrm{K}(X)/\ell[\beta^{-1}]$$

is $2d$-coconnective. (See [Tho86, Section 5] for a discussion of the bound $2d$.) Recall that a map of spectra $M \to N$ is r-coconnected if the induced map $\pi_r M \to \pi_r N$ is an injection and $\pi_s M \to \pi_s N$ is an isomorphism for $s > r$. Following the tradition of proposing noncommutative versions of theorems known for $\mathbb{C}$-linear dg categories of the form Perf_X, we ask the following question.

Question 4.2 (Noncommutative Quillen-Lichtenbaum)**.** If $\mathcal{C}$ is a nice (probably smooth and proper) $\mathbb{C}$-linear dg category, is

$$\mathrm{K}(\mathcal{C})/n \to \mathrm{K}^{\mathrm{top}}(\mathcal{C})/n$$

is r-coconnective for some r.

To prove Theorem 4.1, we make use of the topological realization functor $\mathrm{Re} : \mathcal{S}\mathrm{p}(\mathbb{C}) \to \mathcal{S}\mathrm{p}$, extending the functor of taking complex points of a $\mathbb{C}$-scheme. This functor factors through the localization $\mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sm}_{\mathbb{C}}) \to \mathrm{Shv}^{\mathrm{Nis},\mathbb{A}^1}_{\mathcal{S}\mathrm{p}}(\mathrm{Sm}_{\mathbb{C}})$, which is equivalent to $\mathcal{S}\mathrm{p}_{S^1}(\mathbb{C})$, the category of motivic S^1-spectra. To see that it factors through the T-stabilization functor $\mathcal{S}\mathrm{p}_{S^1}(\mathbb{C}) \to \mathcal{S}\mathrm{p}(\mathbb{C})$, it is enough to note that the realization of T is S^2, which is already tensor invertible in $\mathcal{S}\mathrm{p}$. Thus, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Sm}_{\mathbb{C}} \arrow[rr, "X \mapsto \Sigma^\infty_+ X(\mathbb{C})"] \arrow[d] & & \mathcal{S}\mathrm{p} & & \\
\mathcal{P}_{\mathcal{S}\mathrm{p}}(\mathrm{Sm}_{\mathbb{C}}) \arrow[rr] \arrow[urr, dotted, "\mathrm{Re}"] & & \mathcal{S}\mathrm{p}_{S^1}(\mathbb{C}) \arrow[rr] \arrow[u, dotted, "\mathrm{Re}"] & & \mathcal{S}\mathrm{p}(\mathbb{C}) \arrow[ull, dotted, "\mathrm{Re}"']
\end{tikzcd}$$

of realization functors, and we will abuse notation by not distinguishing them.

Lemma 4.3. *Let $\mathcal{C}$ be a $\mathbb{C}$-linear dg category. Then, there is an equivalence* $\mathrm{Re}(\mathrm{KGL}(\mathcal{C})) \simeq \mathrm{K}^{\mathrm{top}}(\mathcal{C})$.

Proof. By definition, $\mathrm{KGL}(\mathcal{C}) \in \lim(\mathcal{S}\mathrm{p}_{S^1}(\mathbb{C}) \xleftarrow{\Omega_T} \mathcal{S}\mathrm{p}_{S^1}(\mathbb{C}) \leftarrow \cdots)$ is the periodic T-spectrum with value $\underline{\mathrm{KH}}(\mathcal{C})$ and structure maps induced by β:

$$\underline{\mathrm{KH}}(\mathcal{C}) \xrightarrow{\beta} \Omega_T \underline{\mathrm{KH}}(\mathcal{C}) \xrightarrow{\beta} \Omega_T^2 \underline{\mathrm{KH}}(\mathcal{C}) \xrightarrow{\beta} \cdots .$$

The realization functor $\mathrm{Re} : \mathcal{S}\mathrm{p}(\mathbb{C}) \to \mathcal{S}\mathrm{p}$ factors through the equivalence $\mathcal{S}\mathrm{p}_{S^2}\mathcal{S}\mathrm{p} \simeq \mathcal{S}\mathrm{p}$, where $\mathcal{S}\mathrm{p}_{S^2}\mathcal{S}\mathrm{p}$ is the ∞-category of S^2-spectra in spectra. The realization functor sends $\mathrm{KGL}(\mathcal{C})$ to the S^2-spectrum

$$\mathrm{K}^{\mathrm{st}}(\mathcal{C}) \xrightarrow{\beta} \Omega^2 \mathrm{K}^{\mathrm{st}}(\mathcal{C}) \xrightarrow{\beta} \Omega^4 \mathrm{K}^{\mathrm{st}} \xrightarrow{\beta} \cdots .$$

The underlying spectrum of this S^2-spectrum in spectra is "$\Omega^{2\infty}$", the colimit of the diagram, which is by definition $\mathrm{K}^{\mathrm{top}}(\mathcal{C})$. ☐

Lemma 4.4. *If $X \in \mathrm{Sm}_{\mathbb{C}}$, then* $\mathrm{Re}(\underline{\mathrm{Map}}_{\mathcal{S}\mathrm{p}(\mathbb{C})}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathrm{KGL}(\mathcal{C}))) \simeq \underline{\mathrm{K}}^{\mathrm{top}}(\mathcal{C})(X)$.

Proof. By adjunction,

$$\underline{\mathrm{Map}}_{\mathcal{S}\mathrm{p}(\mathbb{C})}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathrm{KGL}(\mathcal{C})) \simeq \mathrm{KGL}(\mathrm{Perf}_X \otimes_{\mathbb{C}} \mathcal{C}).$$

The claim follows from the fact that $\mathrm{Re}(\mathrm{KGL}(\mathrm{Perf}_X \otimes_{\mathbb{C}} \mathcal{C})) \simeq \mathrm{K}^{\mathrm{top}}(\mathrm{Perf}_X \otimes_{\mathbb{C}} \mathcal{C}) \simeq \underline{\mathrm{K}}^{\mathrm{top}}(\mathcal{C})(X)$.

☐

Lemma 4.5. *If $X \in \mathrm{Sm}_{\mathbb{C}}$, then* $\underline{\mathrm{K}}^{\mathrm{top}}(\mathcal{C})(X) \simeq \mathrm{Map}_{\mathcal{S}\mathrm{p}}(\Sigma^\infty X(\mathbb{C})_+, \mathrm{K}^{\mathrm{top}}(\mathcal{C}))$.

Proof. Because the functor Re is symmetric monoidal, it commutes with internal mapping objects. Since $\Sigma^\infty_{\mathbb{P}^1} X_+$ is dualizable in $\mathcal{S}\mathrm{p}(\mathbb{C})$, the statement of the lemma follows from the previous lemma. ☐

Proof of Theorem 4.1. It follows from the equivalence $\underline{\mathrm{K}}(\mathcal{C})/n \simeq \underline{\mathrm{K}}^{\mathrm{st}}(\mathcal{C})/n$ of Theorem 3.3 that the second part follows from the first part. On the other hand, Lemma 4.5 shows that $\underline{\mathrm{K}}^{\mathrm{top}}(\mathcal{C})$ is the restriction of the cohomology theory on spaces represented by $\mathrm{K}^{\mathrm{top}}(\mathcal{C})$ to $\mathrm{Sch}_{\mathbb{C}}$. It follows that it satisfies étale hyperdescent since any cohomology theory does (see for example [DI04, Theorem 5.2]). ☐

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