

EQUIVARIANT SEMI-TOPOLOGICAL INVARIANTS, ATIYAH'S KR -THEORY, AND REAL ALGEBRAIC CYCLES

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ABSTRACT. We establish an Atiyah-Hirzebruch type spectral sequence relating real morphic cohomology and real semi-topological K -theory and prove it to be compatible with the Atiyah-Hirzebruch spectral sequence relating Bredon cohomology and Atiyah's KR -theory constructed by Dugger. An equivariant and a real version of Suslin's conjecture on morphic cohomology are formulated, proved to come from the complex version of Suslin conjecture and verified for certain real varieties. In conjunction with the spectral sequences constructed here this allows the computation of the real semi-topological K -theory of some real varieties. As another application of this spectral sequence we give an alternate proof of the Lichtenbaum-Quillen conjecture over \mathbb{R} , extending an earlier proof of Karoubi and Weibel.

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1. INTRODUCTION

A semi-topological Atiyah-Hirzebruch spectral sequence relating the morphic cohomology and the semi-topological K -theory of smooth complex variety X is constructed in [13]. There is a comparison map from the motivic Atiyah-Hirzebruch spectral sequence to the semi-topological spectral sequence and a map from the semi-topological spectral sequence to the topological Atiyah-Hirzebruch spectral sequence for connective complex K -theory. A first goal of this paper is to fill in this picture for real varieties. A second goal is to compute equivariant morphic cohomology and real semi-topological K -theory for some real varieties.

In Section 2 we establish the semi-topological Atiyah-Hirzebruch spectral sequence and show that it receives a map from the motivic spectral sequence and

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maps to the spectral sequence constructed by Dugger in [11, Corollary 1.3],

$$\begin{array}{c}
E_2^{p,q}(alg) = \mathbb{H}_{\mathcal{M}}^{p-q}(X; A(-q)) \implies K_{-p-q}(X; A) \\
\downarrow \\
E_2^{p,q}(sst) = L^{-q}H\mathbb{R}^{p,-q}(X; A) \implies K\mathbb{R}_{-p-q}^{sst}(X; A) \\
\downarrow \\
E_2^{p,q}(top) = H^{p,-q}(X(\mathbb{C}); \underline{A}) \implies kr^{p+q}(X(\mathbb{C}); A).
\end{array}$$

To construct the real semi-topological spectral sequence we apply Friedlander-Haesemeyer-Walker's method [13], with straightforward modifications for real varieties. As their method makes clear, in order to produce the semi-topological spectral sequence one may use as input any model for the motivic spectral sequence. However, constructing the companion topological spectral sequence and identifying the equivariant homotopy type of the spectra appearing in its construction requires more care. Presumably this can be carried out with any of the available models, however we find it convenient to make use of Grayson's model (as described in [51, 52]). Suslin [44] has proved that the motivic cohomology appears as the E_2 -term of Grayson's spectral sequence. Moreover it is widely believed that Friedlander-Suslin's spectral sequence [18], Levine's spectral sequence [34], and Grayson's spectral sequence are in fact all the same spectral sequence. The advantage that this model has for us is that the spectra in this tower arise from homotopy group completions of simplicial sets of algebraic maps to certain quot schemes. This geometric description makes it straightforward to prove Proposition B.1, which is one of the main technical tools we use to identify the form of our topological spectral sequence. The topological spectral sequence we construct here has the same form as the spectral sequence constructed by Dugger in [11, Corollary 1.3] and indeed in Proposition 2.35 we prove that these two spectral sequences are the same.

Suslin's conjecture on morphic cohomology of a smooth complex variety predicts that the cycle map $L^q H^n(X) \rightarrow H_{sing}^n(X(\mathbb{C}); \mathbb{Z})$ is an isomorphism for $n \leq q$ and an injection for $n = q + 1$. We formulate appropriate analogues for the real morphic cohomology and the equivariant morphic cohomology of a smooth real variety (see Section 3 for precise statements). The real morphic cohomology groups of real varieties and the morphic cohomology of complex varieties are particular instances of equivariant morphic cohomology, and so at first glance the equivariant conjecture would appear to be stronger than the rest of these conjectures. However, as we show in Theorem 3.4 all three of these conjectures are equivalent. As a consequence we verify the equivariant Suslin conjecture in codimension one for all smooth real varieties and give a complete computation of real morphic cohomology in codimension one (see Theorem 3.9), generalizing a result of Teh [46]. For certain real varieties, such as geometrically rationally connected threefolds and generic cubics of small dimension we verify the equivariant Suslin conjecture in all codimensions.

Moreover in Section 3, we discuss the real algebraic equivalence relation on real cycles (as defined by [22]). We prove that it satisfies a Bloch-Ogus formula for Bredon cohomology (and also Borel cohomology) as well as a number of other good properties satisfied by the usual algebraic equivalence over complex algebraic cycles (see Theorem 3.24 and Theorem 3.9). Together with other techniques, these allow

us to prove a stronger result than is predicted in general by the real Suslin's conjecture in some cases (e.g. geometrically rationally connected threefolds), extending results from the complex case of Voineagu [49]. In particular we obtain that all of the equivariant morphic cohomology groups are finitely generated in these cases.

Using the spectral sequences constructed in Section 2 in conjunction with our computations of the real morphic cohomology, we compute the real semi-topological K -theory of some real varieties. As an additional application of the spectral sequences constructed here, in conjunction with Voevodsky's verification of the Milnor conjecture, we give an alternate proof of the 2-adic Lichtenbaum-Quillen over \mathbb{R} . This was proved [31] (in a slightly weaker form) and in general (for any real closed field) in [42]. One would like to use Voevodsky's solution of Milnor conjecture [48] together with a comparison of Atiyah-Hirzebruch spectral sequences in order to prove this conjecture (indeed this method works just fine over non-real closed fields). The difficulty which has to be worked around in [31] and [42] is that if X has a real point then its étale 2-cohomological dimension is infinite and the étale Atiyah-Hirzebruch spectral sequence will not converge. The proof we give avoids this problem by using the Atiyah-Hirzebruch spectral sequence based on the Bredon cohomology of the $\mathbb{Z}/2$ -space $X(\mathbb{C})$ (which is convergent). We view this application as an example of the fact that Bredon cohomology is a more natural and useful target for Chow groups of real algebraic cycles than Borel cohomology.

Finally in Section 4 we verify the expected compatibility between the duality proved in [28] relating equivariant morphic cohomology and dos Santos' equivariant Lawson homology groups of a smooth projective variety X and the Poincaré duality relating the Bredon cohomology and homology of the space $X(\mathbb{C})$. An important consequence of this compatibility is Corollary 4.2 which asserts that in weights larger than $\dim U$, the cycle map from the equivariant morphic cohomology of U to the Bredon cohomology of U is an isomorphism for any smooth quasi-projective real variety U . This identification is an important ingredient in the previously mentioned applications of the spectral sequences. Another application of this compatibility was given in [28, Corollary 5.14].

The techniques and computations presented in this paper are a sequel to our previous work on real algebraic cycles [28] and to the paper [13] of Friedlander-Haesemeyer-Walker which had an important influence on this work.

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1.1. Notation. By a k -variety we mean a reduced, separated quasi-projective scheme of finite type over a field k . We write Sch/k for the category of k -varieties and Sm/k for the subcategory of smooth quasi-projective k -varieties. Unless otherwise specified, G denotes the group $Gal(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$.

If X is a real variety then the set $\text{Hom}_{\mathbb{R}}(\mathbb{C}, X)$ is denoted as $X(\mathbb{C})$. This set agrees with the set of closed points $X_{\mathbb{C}}(\mathbb{C})$ of the complexification and is given an analytic topology via this identification. Note that when X is a complex variety viewed as a real variety this notation is potentially ambiguous but it will always be

clear in such situations which is meant. Via the action of complex conjugation the space $X(\mathbb{C})$ is a G -space.

By a spectrum we mean a spectrum in the sense of Bousfield-Friedlander, which consists of a sequence $A = (A_1, A_2, \dots)$ of simplicial sets together with connecting maps $\sigma_i : S^1 \wedge A_i \rightarrow A_{i+1}$. A naive G -spectrum is a G -object in the category of spectra.

A map of G -spaces (or naive G -spectra) $f : A \rightarrow B$ is a G -weak equivalence provided both $A \rightarrow B$ and $A^G \rightarrow B^G$ are weak equivalences.

Let V be a real representation of G , write S^V for the one-point compactification of V . The V th homotopy group of a based G -space X is

$$\pi_V X = [S^V, X]_G,$$

(where $[-, -]_G$ denotes maps in the based G -homotopy category). When $V = \mathbb{R}^{p,q}$ we use the notation $\pi_{p,q} X = \pi_{\mathbb{R}^{p,q}} X$ where $\mathbb{R}^{p,q}$ is \mathbb{R}^{p+q} with trivial action on the first p -components and is multiplication by -1 on the last q -components. In particular $\pi_{k,0} X = \pi_k X^G$. We use a similar convention for the $RO(G)$ -graded Bredon cohomology with coefficients in a Mackey functor \underline{M} and write $H^{p,q}(X; \underline{M}) = H^{\mathbb{R}^{p,q}}(X; \underline{M})$. For more details on equivariant homotopy and cohomology see [38].

2. ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCES

In this section we construct the semi-topological Atiyah-Hirzebruch spectral sequence for real varieties following the construction in [13] for complex varieties. We show that this spectral sequence receives a map from the motivic Atiyah-Hirzebruch spectral sequence and maps to the Atiyah-Hirzebruch spectral sequence for Bredon cohomology and real K -theory constructed by Dugger.

2.1. Motivic spectral sequence. There are several available constructions of a motivic Atiyah-Hirzebruch spectral sequence. For our purposes we find it convenient to make use of Grayson's construction [25] (as reinterpreted by Walker [51, 52]).

Let f be a polynomial (integer valued with rational coefficients), \mathcal{E} a coherent sheaf on a quasi-projective variety Y , and $Y \rightarrow S$ a morphism of quasi-projective varieties. The quot scheme $\text{Quot}_{\mathcal{E}/Y/S}^f$ represents the functor which sends X to the collection of quotient objects $[\pi_X^* \mathcal{E} \twoheadrightarrow \mathcal{F}]$ (where $\pi_X : X \times_S Y \rightarrow Y$ is the projection) which are flat over X and have support proper over X with Hilbert polynomial f .

Definition 2.1. For an integer n define $G_Y(n) \subseteq \coprod_{\deg(f)=0} \text{Quot}_{\mathcal{O}_Y^n/Y/\text{Spec } k}^f$ to be the subfunctor whose value on X is the collection of quotient objects $[\mathcal{O}_{X \times Y}^n \twoheadrightarrow \mathcal{F}]$ which satisfy the conditions

- (1) \mathcal{F} is flat over X and $\text{Supp}(\mathcal{F})$ is finite over X ,
- (2) The induced map $\mathcal{O}_X^n \rightarrow \pi_* \mathcal{O}_{X \times Y}^n \rightarrow \pi_* \mathcal{F}$ is surjective.

By [53, Lemma 2.2] this functor is represented by a variety which we also denote $G_Y(n)$. In fact $G_Y(n) \subseteq \coprod_{\deg(f)=0}^n \text{Quot}_{\mathcal{O}_Y^n/Y/\text{Spec } k}^f$ is an open subvariety. In [26] Grayson-Walker use $K_Y(n) = \coprod_{\deg(f)=0}^n \text{Quot}_{\mathcal{O}_Y^n/Y/\text{Spec } k}^f$ rather than $G_Y(n)$. Using $G_Y(n)$ rather than $K_Y(n)$ does not change the resulting bivariant theory [53] and has several advantages such as making certain induced maps of spectra functorial rather than merely functorial up to homotopy.

The projection $\mathcal{O}^{n+1} \rightarrow \mathcal{O}^n$ onto the first n basis elements defines a map $G_Y(n) \rightarrow G_Y(n+1)$ and let G_Y denote the ind-scheme $G_Y = \text{colim}_n G_Y(n)$. The elements of $G_Y(X) = \text{colim}_n \text{Hom}(X, G_Y(n))$ are considered as quotient objects $[p : \mathcal{O}_{X \times Y}^\infty \rightarrow \mathcal{F}]$ such that all but finitely many of the standard basis elements are in the kernel of p .

Definition 2.2. An m -tuple $([p_1 : \mathcal{O}_{X \times Y}^\infty \rightarrow \mathcal{F}_1], \dots, [p_m : \mathcal{O}_{X \times Y}^\infty \rightarrow \mathcal{F}_m])$ is said to be in general position if the induced map

$$\sum_i p_i : \mathcal{O}_{X \times Y}^\infty \rightarrow \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_m$$

is again a surjection. Write $G_Y^{(m)}(X) \subseteq G_Y(X)^{\times m}$ for the subset of all m -tuples which are in general position.

By [26] the functor $X \mapsto G_Y^{(m)}(X)$ on Sch/k is represented by an open ind-scheme $G_Y^{(m)} \subseteq G_Y^{\times m}$. By convention $G_Y^{(0)} = \text{Spec } k$.

The collection $\{G_Y^{(m)}\}_{m \geq 0}$ gives a Γ -object in presheaves of sets on Sch/k (see Appendix A for a recollection of Γ -spaces). Given a map $\sigma : \underline{m} \rightarrow \underline{n}$ then $\sigma_* : G_Y^{(m)}(X) \rightarrow G_Y^{(n)}(X)$ is defined by

$$\begin{aligned} \sigma_*([p_1 : \mathcal{O}_{X \times Y}^\infty \rightarrow \mathcal{F}_1], \dots, [p_m : \mathcal{O}_{X \times Y}^\infty \rightarrow \mathcal{F}_m]) = \\ = ([q_1 : \mathcal{O}_{X \times Y}^\infty \rightarrow \mathcal{E}_1], \dots, [q_n : \mathcal{O}_{X \times Y}^\infty \rightarrow \mathcal{E}_n]), \end{aligned}$$

where $\mathcal{E}_k = \bigoplus_{i \in \sigma^{-1}(k)} \mathcal{F}_i$ and $q_k : \mathcal{O}_{X \times Y}^\infty \rightarrow \mathcal{E}_k$ is the sum over $i \in \sigma^{-1}(k)$ of the maps p_i . The condition of being in general-position implies that this definition is well-defined. This is functorial in X and so defines a Γ -object in the category of presheaves on Sch/k . The argument in [26, Lemma 2.2] shows that the Γ -space $m \mapsto G_Y^{(m)}(X \times \Delta^\bullet)$ is special for any X .

Definition 2.3. For quasi-projective k -varieties X and Y define $\mathcal{K}_{geom}(X, Y)$ to be the Ω -spectrum associated with the special Γ -space $m \mapsto G_Y^{(m)}(X \times \Delta^\bullet)$.

By [53] the Ω -spectrum $\mathcal{K}_{geom}(X, Y)$ defined here is equivalent to the spectrum obtained from the special Γ -space $m \mapsto K_Y^{(m)}(X \times \Delta^\bullet)$. Let $\mathcal{K}(X, Y)$ denote the K -theory spectrum of the exact category $\mathcal{P}(X, Y)$ of coherent $\mathcal{O}_{X \times Y}$ -modules which are flat over X and have finite support over X . By [26, Theorem 2.3] for any X and Y there is a natural map of Ω -spectra $\mathcal{K}_{geom}(X, Y) \rightarrow |\mathcal{K}(X \times \Delta^\bullet, Y)|$ which is always a weak equivalence. In particular whenever X is regular then $\pi_* \mathcal{K}_{geom}(X, Y) = K_*(X, Y)$.

Lemma 2.4. ([53, Lemma 2.3]) *A map $f : Y \rightarrow Y'$ of k -varieties induces a natural map of Γ -spaces $f_* : \mathcal{G}_Y^{(m)}(X \times \Delta^\bullet) \rightarrow \mathcal{G}_{Y'}^{(m)}(X \times \Delta^\bullet)$, by $f_*[\mathcal{O}_{X \times Y}^n \rightarrow \mathcal{M}] = [\mathcal{O}_{X \times Y'}^n \rightarrow (id \times f)_* \mathcal{M}]$. In particular we obtain a natural transformation of presheaves of Ω -spectra*

$$f_* : \mathcal{K}_{geom}(-, Y) \rightarrow \mathcal{K}_{geom}(-, Y').$$

Proof. Let $[\mathcal{O}_{X \times Y}^n \rightarrow \mathcal{M}]$ be an object of $G_Y(n)(X)$. The second condition on \mathcal{M} in Definition 2.1 guarantees that the induced map $[\mathcal{O}_{X \times Y'}^n \rightarrow (id \times f)_* \mathcal{M}]$ is again a surjection. Thus $f_* : G_Y(X \times \Delta^\bullet) \rightarrow G_{Y'}(X \times \Delta^\bullet)$ is well-defined. The second condition in Definition 2.1 also guarantees that if $([p_1 : \mathcal{O}_{X \times Y}^\infty \rightarrow \mathcal{M}_1], \dots, [p_m : \mathcal{O}_{X \times Y}^\infty \rightarrow \mathcal{M}_m])$ is an m -tuple in general position then the induced

m -tuple $((id \times f)_*(p_1), \dots, (id \times f)_*(p_m))$ is again in general position. Thus we have a natural map of Γ -spaces $f_* : G_Y^{(m)}(X \times \Delta^\bullet) \rightarrow G_{Y'}^{(m)}(X \times \Delta^\bullet)$. \square

Let $Cube(n)$ denote the category whose objects are the subsets of $\{1, \dots, n\}$ and morphisms are subset inclusions. An n -cube in Sch/k is a functor from $Cube(n)$ to Sch/k . For a subset $I \subseteq \{1, \dots, n\}$ define $\mathbb{P}_I^{\wedge n} = P_1 \times \dots \times P_n$ where $P_i = \mathbb{P}^1$ if $i \in I$ and otherwise $P_i = \{\infty\}$. By the above lemma we have natural transformations $\mathcal{K}_{geom}(-, \mathbb{P}_I^{\wedge n}) \rightarrow \mathcal{K}_{geom}(-, \mathbb{P}_J^{\wedge n})$ for $I \subseteq J$. Define $\mathcal{W}^{(n)}(X)$ to be homotopy colimit of the cube $I \mapsto \mathcal{K}_{geom}(-, \mathbb{P}_I^{\wedge n})$, this is natural in X and so defines the presheaf $\mathcal{W}^{(n)}(-)$. When X is regular, $\mathcal{W}^{(0)}(X) \sim \mathcal{K}(X)$.

There are functorial maps $\mathcal{W}^{(n+1)}(X) \rightarrow \mathcal{W}^{(n)}(X)$ [52, Section 2], from which we obtain a tower of presheaves

$$\dots \rightarrow \mathcal{W}^{(n+1)}(-) \rightarrow \mathcal{W}^{(n)}(-) \rightarrow \dots \rightarrow \mathcal{W}^{(0)}(-).$$

For a simplicial abelian group A_\bullet write $\mathcal{M}(A_\bullet)$ for the associated Ω -spectrum, $\mathcal{M}(A_\bullet) = (A_\bullet, BA_\bullet, B^2A_\bullet, \dots)$.

Write $z_{equi}(Y, 0)(-)$ for the presheaf of equidimensional cycles of relative dimension zero [45]. Using the localization property for the complex of equidimensional cycles [19] we see that the complex $z_{equi}(\mathbb{P}^{\wedge n}, 0)(X \times \Delta^\bullet)$ is equivalent to the complex $z_{equi}(\mathbb{A}^n, 0)(X \times \Delta^\bullet)$ which computes motivic cohomology (see [19, 39]). As explained in [52, Section 4] there is a natural sequence of spectra

$$\mathcal{W}^{(n+1)}(X) \rightarrow \mathcal{W}^{(n)}(X) \rightarrow \mathcal{M}(z_{equi}(\mathbb{P}^{\wedge n}, 0)(X \times \Delta^\bullet))$$

which is homotopic to the constant map. By [44] this sequence is a homotopy fiber sequence when $X = \text{Spec } \mathcal{O}_{Y,y}$ is the local ring of a smooth variety Y at a point $y \in Y$.

This tower is globalized as in [18] via Godement resolutions (for the Zariski topology) which are recalled in Appendix B. Write \mathcal{G} for the Godement resolution and define $\mathcal{K}^{(n)}(-) = \mathcal{G}\mathcal{W}^{(n)}(-)$ and $\mathcal{M}^{(n)}(-) = \mathcal{G}\mathcal{M}(z_{equi}(\mathbb{P}^{\wedge n}, 0)(- \times \Delta^\bullet))$. Observe that because the presheaf of simplicial abelian groups $z_{equi}(\mathbb{P}^{\wedge n}, 0)(- \times \Delta^\bullet)$ satisfies Zariski descent on smooth varieties, for any smooth k -variety X the natural map $\mathcal{M}(z_{equi}(\mathbb{P}^{\wedge n}, 0)(X \times \Delta^\bullet)) \rightarrow \mathcal{M}^{(n)}(X)$ is a weak equivalence.

Because $\mathcal{W}^{(n+1)}(X) \rightarrow \mathcal{W}^{(n)}(X) \rightarrow \mathcal{M}(z_{equi}(\mathbb{P}^{\wedge n}, 0)(X \times \Delta^\bullet))$ is a homotopy fiber sequence when $X = \text{Spec } \mathcal{O}_{Y,y}$ for a smooth variety Y and a point $y \in Y$, we see that

$$(2.5) \quad \mathcal{K}^{(n+1)}(X) \rightarrow \mathcal{K}^{(n)}(X) \rightarrow \mathcal{M}^{(n)}(X)$$

is a homotopy fiber sequence for any smooth X . In general when X is not smooth the composition is homotopic to the constant map.

This gives the tower of presheaves of spectra on Sch/k

$$(2.6) \quad \begin{array}{ccccccc} \dots & \rightarrow & \mathcal{K}^{(q+1)} & \rightarrow & \mathcal{K}^{(q)} & \rightarrow & \mathcal{K}^{(q-1)} & \rightarrow & \dots & \rightarrow & \mathcal{K}^{(0)} & \rightarrow & \mathcal{K} \\ & & & & \downarrow & & \downarrow & & & & \downarrow & & \\ & & & & \mathcal{M}^{(q)} & & \mathcal{M}^{(q-1)} & & & & \mathcal{M}^{(0)} & & \end{array}$$

We summarize the properties of this tower which we use.

Theorem 2.7. [25, 44] *Let X be a smooth k -variety.*

- (1) *Each of $\mathcal{K}^{(q)}(X)$ and $\mathcal{M}^{(q)}(X)$ are (-1) -connected Ω -spectra.*

(2) *The sequence*

$$\mathcal{K}^{(q+1)}(X) \rightarrow \mathcal{K}^{(q)}(X) \rightarrow \mathcal{M}^{(q)}(X)$$

is a homotopy fibration sequence of spectra (in general the sequence is homotopic to the constant map if X is not smooth).

(3) *For any abelian group A ,*

$$\pi_k(\mathcal{M}^{(q)}(X); A) = H_{\mathcal{M}}^{2q-k}(X; A(q))$$

(4) *The augmentation $\mathcal{K}^{(0)}(X) \rightarrow \mathcal{K}(X)$ is a weak equivalences of spectra.*

(5) *If $k < q - \dim X$ then $\pi_k \mathcal{K}^{(q)}(X) = 0$.*

Proof. The first four items have already been discussed above. Let $\tilde{\pi}_i \mathcal{W}^{(q)}$ denote the Zariski sheafification of the presheaf $U \mapsto \pi_i \mathcal{W}^{(q)}(U)$ on Sm/k . For $Y = \text{Spec } \mathcal{O}_{X,x}$, the localization of a smooth variety X at a point $x \in X$, there is a natural homotopy equivalence of spectra $\mathcal{K}_{geom}(Y, \mathbb{G}_m^{\wedge q}) \rightarrow \Omega^q \mathcal{W}^{(q)}(Y)$ by [51, Theorem 7.11]. Therefore $\tilde{\pi}_i \mathcal{W}^{(q)} = 0$ for $i < q$. That $\pi_k \mathcal{K}^{(q)}(X) = 0$ for $k < q - \dim X$ now follows from the descent spectral sequence [5]

$$E_2^{s,t} = H_{Zar}^s(X; \tilde{\pi}_t \mathcal{W}^{(q)}) \implies \pi_{t-s} \mathcal{K}^{(q)}(X)$$

since $E_2^{s,t} \neq 0$ means that $s \leq \dim X$ and $t \geq q$. \square

Corollary 2.8. [25, 44] *For any smooth k -variety X and any abelian group A , there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H_{\mathcal{M}}^{p-q}(X; A(-q)) \implies K_{-p-q}(X; A).$$

Proof. For any abelian group A , the homotopy fiber sequences (2.5) yield an exact couple,

$$D_2^{p,q} = \pi_{-p-q}(\mathcal{K}^{(-q)}(X); A) \quad \text{and} \quad E_2^{p,q} = \pi_{-p-q}(\mathcal{M}^{(-q)}(X); A).$$

By the previous theorem this exact couple is bounded below and thus the resulting spectral sequence converges to $\text{colim}_p D_2^{p,n-p} = \pi_{-n}(\mathcal{K}^{(p-n)}(X); A) = \pi_{-n} \mathcal{K}(X; A)$ (see for example [54, Corollary 5.9.7]). \square

We are primarily interested in the cases $k = \mathbb{R}$ and $k = \mathbb{C}$. Let L/k be a finite extension of fields. In the following lemma, in order to (temporarily) distinguish between the presheaf of spectra on Sch/k obtained using $G_{\mathbb{P}_k^n}$ and the presheaf on Sch/L using $G_{\mathbb{P}_L^n}$ we use the notation $\mathcal{K}_k^{(n)}(X)$ and $\mathcal{K}_L^{(n)}(X)$ (and similarly for the presheaves $\mathcal{M}^{(n)}$). Because of the following lemma we will not make careful distinction between the two in the rest of this paper.

Lemma 2.9. *Let L/k be a finite field extension and let $p : Sch/L \rightarrow Sch/k$ be the map of sites specified by $X \mapsto X_L$. Then for an L -variety Y we have that $p^* \mathcal{K}_k^{(n)}(Y) = \mathcal{K}_L^{(n)}(Y)$ and $p^* \mathcal{M}_k^{(n)}(Y) = \mathcal{M}_L^{(n)}(Y)$*

Proof. For a k -variety Y we have the agreement of presheaves of sets on Sch/L ,

$$p^* \text{Hom}_k(-, G_Y^{(m)}) = \text{Hom}_L(-, G_{Y_L}^{(m)}).$$

Thus for any L -variety X we have, $p^* G_Y^{(m)}(X \times_L \Delta_L^\bullet) = G_{Y_L}^{(m)}(X \times_L \Delta_L^\bullet)$. If $F(-)$ is a presheaf of Γ -spaces then $p^* BF = Bp^* F$ and so $p^* \mathcal{K}_{geom,k}(-, Y) = \mathcal{K}_{geom,L}(-, Y_L)$. This implies that

$$p^* \mathcal{W}_k^{(n)}(-) = \mathcal{W}_L^{(n)}(-).$$

Finally, because the Godement resolution commutes with p^* we have

$$p^* \mathcal{K}_k^{(n)}(-) = p^* \mathcal{G}\mathcal{W}_k^{(n)}(-) = \mathcal{G}\mathcal{W}_L^{(n)}(-) = \mathcal{K}_L^{(n)}(-).$$

A similar argument gives the result for the presheaves $\mathcal{M}^{(n)}(-)$. \square

2.2. Semi-topological Spectral Sequence. We now construct a semi-topological spectral sequence for real varieties, using the motivic spectral sequence above, and following the method of [13]. We begin by recalling a construction used by Friedlander-Walker.

Let $F(-)$ be a presheaf of abelian groups, simplicial sets, or spectra on Sch/\mathbb{R} . If T is a topological space and X is a real variety then define $F(T \times_{\mathbb{R}} X)$ by the filtered colimit

$$F(T \times_{\mathbb{R}} X) = \operatorname{colim}_{T \rightarrow V(\mathbb{R})} F(V \times_{\mathbb{R}} X).$$

The colimit is over continuous maps $f : T \rightarrow V(\mathbb{R})^{an}$, where V is a real variety, and a map from $(f : T \rightarrow V(\mathbb{R}))$ to $(g : T \rightarrow U(\mathbb{R}))$ is a map $\alpha : V \rightarrow U$ of real varieties such that $g = \alpha f$. Similarly define $F(T \times_{\mathbb{C}} X)$ by the colimit

$$F(T \times_{\mathbb{C}} X) = \operatorname{colim}_{T \rightarrow V(\mathbb{C})} F((V_{\mathbb{C}}) \times_{\mathbb{R}} X).$$

Via the action of $Gal(\mathbb{C}/\mathbb{R})$ on each $F((V_{\mathbb{C}}) \times_{\mathbb{R}} X)$ we obtain an action of $Gal(\mathbb{C}/\mathbb{R})$ on $F(T \times_{\mathbb{C}} X)$.

Remark 2.10. When H is a presheaf on Sch/\mathbb{C} , Friedlander-Walker [22, proof of Lemma 2.4] show that

$$\operatorname{colim}_{T \rightarrow U(\mathbb{R})} H(U_{\mathbb{C}}) \cong \operatorname{colim}_{T \rightarrow W(\mathbb{C})} H(W)$$

where the first colimit is over continuous maps $T \rightarrow U(\mathbb{R})$ with U a real variety and in the second $T \rightarrow W(\mathbb{C})$ with W a complex variety. In particular writing $p : Sch/\mathbb{R} \rightarrow Sch/\mathbb{C}$ for the map $X \mapsto X_{\mathbb{C}}$ we see that for a presheaf $F(-)$ on Sch/\mathbb{R} we have the agreement

$$F(T \times_{\mathbb{C}} \mathbb{R}) = (p_* p^* F)(T \times_{\mathbb{R}} \mathbb{R}) = (p^* F)(T \times_{\mathbb{C}} \mathbb{C}).$$

Let F be a presheaf of simplicial sets on Sch/\mathbb{R} . We obtain a bisimplicial set by the formula $d \mapsto F(\Delta_{top}^d \times_{\mathbb{R}} X)$ and define

$$F(\Delta_{top}^{\bullet} \times_{\mathbb{R}} X) = \operatorname{diag}(d \mapsto F(\Delta_{top}^d \times_{\mathbb{R}} X)).$$

Similarly, if \mathcal{F} is a presheaf of spectra on Sch/\mathbb{R} then $d \mapsto F(\Delta_{top}^d \times_{\mathbb{R}} X)$ is a simplicial spectrum. Define the spectrum $F(\Delta_{top}^{\bullet} \times_{\mathbb{R}} X)$ by the formula

$$F(\Delta_{top}^{\bullet} \times_{\mathbb{R}} X)^i = \operatorname{diag}(d \mapsto F(\Delta_{top}^d \times_{\mathbb{R}} X)^i)$$

and bonding maps given by

$$\begin{aligned} S^1 \wedge \operatorname{diag}(d \mapsto F(\Delta_{top}^d \times_{\mathbb{R}} X)^i) &\cong \operatorname{diag}(d \mapsto S^1 \wedge F(\Delta_{top}^d \times_{\mathbb{R}} X)^i) \rightarrow \\ &\rightarrow \operatorname{diag}(d \mapsto F(\Delta_{top}^d \times_{\mathbb{R}} X)^{i+1}) \end{aligned}$$

Definition 2.11. Let F be a presheaf of simplicial sets or spectra. Define a new presheaf F_{sst} on Sch/\mathbb{R} by

$$F_{sst}(X) = F(\Delta_{top}^{\bullet} \times_{\mathbb{R}} X).$$

Remark 2.12. For emphasis and to avoid confusion we will often write below $F(\Delta_{top}^\bullet \times_{\mathbb{R}} \mathbb{R})$ instead of $F(\Delta_{top}^\bullet)$ when F is a presheaf on Sch/\mathbb{R} and similarly we will write $G(\Delta_{top}^\bullet \times_{\mathbb{C}} \mathbb{C})$ instead of $G(\Delta_{top}^\bullet)$ when G is a presheaf on Sch/\mathbb{C} .

In [24] the *uad*-topology on Sch/\mathbb{C} is used to establish a recognition principle identifying when a map $F(-) \rightarrow G(-)$ of presheaves on Sch/\mathbb{C} induces a weak equivalence $F(\Delta_{top}^\bullet \times_{\mathbb{C}} \mathbb{C}) \rightarrow G(\Delta_{top}^\bullet \times_{\mathbb{C}} \mathbb{C})$. In [28] it is observed that their work carries over to Sch/\mathbb{R} , using a suitable real *uad*-topology. We briefly recall this topology, which is essentially due to Deligne.

Definition 2.13. (1) A continuous map of topological spaces $f : S \rightarrow T$ is said to satisfy *cohomological descent* if for any sheaf A of abelian groups on T the natural map

$$H^*(T, A) \rightarrow H^*(N_T(S), f^*A)$$

is an isomorphism. Here $N_T(S) \rightarrow T$ is the Čech nerve of f , i.e. $N_T(S)$ is the simplicial space which in degree n is the $n + 1$ -fold fiber product of S over T . A map $f : S \rightarrow T$ is said to be of *universal cohomological descent* provided the pullback $S \times_T T' \rightarrow T'$ along any continuous map $T' \rightarrow T$ is again of cohomological descent.

- (2) The *uad*-topology on Sch/\mathbb{C} is the Grothendieck topology associated to the pretopology generated by collections $\{U_i \rightarrow X\}$ such that $\coprod U_i(\mathbb{C})^{an} \rightarrow X(\mathbb{C})^{an}$ is a surjective map of universal cohomological descent.
- (3) The *uad*-topology on Sch/\mathbb{R} is the Grothendieck topology associated to the pretopology generated by collections $\{U_i \rightarrow X\}$ such that $\coprod U_i(\mathbb{R})^{an} \rightarrow X(\mathbb{R})^{an}$ is a surjective map of universal cohomological descent.

A useful feature of the *uad*-topologies is that a complex variety is locally smooth in the *uad*-topology and a real variety is locally smooth in the real *uad*-topology.

The following is the recognition principle proved by Friedlander-Walker [24] (for the real analogue see [28]).

Theorem 2.14 ([24, Theorem 2.2]). *Let $k = \mathbb{C}$ or \mathbb{R} . Suppose that $F \rightarrow G$ is a natural transformation of presheaves of abelian groups on Sch/k . If $F_{uad} \rightarrow G_{uad}$ is an isomorphism of *uad*-sheaves, then*

$$F(\Delta_{top}^\bullet \times_k k) \rightarrow G(\Delta_{top}^\bullet \times_k k)$$

is a homotopy equivalence of simplicial abelian groups.

The following two propositions were proved in the complex case in [13].

Proposition 2.15. (c.f. [13, Theorem 2.6]) *Let $k = \mathbb{C}$ or \mathbb{R} . Suppose that $f : \mathcal{F} \rightarrow \mathcal{G}$ is a natural transformation of presheaves of spectra on Sch/k . Suppose that there is an integer N with the property that the maps $(\pi_q \mathcal{F})_{uad} \rightarrow (\pi_q \mathcal{G})_{uad}$ are isomorphisms of *uad*-sheaves for $q \leq N$ and a surjection for $q = N + 1$. Then*

$$\pi_q \mathcal{F}(\Delta_{top}^\bullet \times_k k) \rightarrow \pi_q \mathcal{G}(\Delta_{top}^\bullet \times_k k)$$

is an isomorphism for $q \leq N$ and a surjection for $q = N + 1$. In particular if $\mathcal{F}(X) \rightarrow \mathcal{G}(X)$ is a weak equivalence of spectra for all smooth X then

$$\mathcal{F}(\Delta_{top}^\bullet \times_k k) \rightarrow \mathcal{G}(\Delta_{top}^\bullet \times_k k)$$

is also a weak equivalence of spectra.

Proof. Define the presheaf of spectra $\mathcal{Q}(-)$ by $\mathcal{Q}(X) = \text{hocofib}(\mathcal{F}(X) \rightarrow \mathcal{G}(X))$. Then $\pi_q \mathcal{Q}_{uad} = 0$ for all $q \leq N + 1$. By the recognition principle the simplicial abelian group $(d \mapsto \pi_q \mathcal{Q}(\Delta_{top}^d \times_k k))$ is contractible for any $q \leq N + 1$.

From the strongly convergent spectral sequence [30, Corollary 4.22]

$$E_{s,t}^2 = \pi_s(d \mapsto \pi_t \mathcal{Q}(\Delta_{top}^d \times_k k)) \implies \pi_{s+t} \mathcal{Q}(\Delta_{top}^\bullet \times_k k)$$

we conclude that $\pi_q \mathcal{Q}(\Delta_{top}^\bullet \times_k k) = 0$ for $q \leq N + 1$. Finally observe that

$$\mathcal{Q}(\Delta_{top}^d \times_k k) \simeq \text{hocofib}(\mathcal{F}(\Delta_{top}^d \times_k k) \rightarrow \mathcal{G}(\Delta_{top}^d \times_k k))$$

because filtered colimits preserve homotopy cofiber sequences. Since the diagonal converts degreewise homotopy cofiber sequences of simplicial spectra into homotopy cofiber sequences of spectra, we have that

$$\mathcal{Q}(\Delta_{top}^\bullet \times_k k) \simeq \text{hocofib}(\mathcal{F}(\Delta_{top}^\bullet \times_k k) \rightarrow \mathcal{G}(\Delta_{top}^\bullet \times_k k)).$$

Finally from the long exact sequence for homotopy cofiber sequences of spectra we conclude that $\pi_k \mathcal{F}(\Delta_{top}^\bullet \times_k k) \rightarrow \pi_k \mathcal{G}(\Delta_{top}^\bullet \times_k k)$ is an isomorphism for $k \leq N$ and a surjection for $k = N + 1$. \square

Proposition 2.16. ([13, Corollary 2.7]) *Let $k = \mathbb{C}$ or \mathbb{R} . Suppose that*

$$\mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H}$$

is a sequence of presheaves of spectra on Sch/k such that the composition is homotopic to the constant map and the sequence

$$\mathcal{F}(X) \rightarrow \mathcal{G}(X) \rightarrow \mathcal{H}(X)$$

is a homotopy fiber sequence for any smooth X . Then

$$(2.17) \quad \mathcal{F}(\Delta_{top}^\bullet \times_k X) \rightarrow \mathcal{G}(\Delta_{top}^\bullet \times_k X) \rightarrow \mathcal{H}(\Delta_{top}^\bullet \times_k X)$$

is a homotopy fiber sequence of spectra.

Proof. A sequence of spectra is a homotopy fiber sequence if and only if it is a homotopy cofiber sequence. Write $\mathcal{Q}(X) = \text{hocofib}(\mathcal{F}(X) \rightarrow \mathcal{G}(X))$. Filtered colimits preserve homotopy cofiber sequences of spectra. Therefore for each d the sequence

$$\mathcal{F}(\Delta_{top}^d \times_k k) \rightarrow \mathcal{G}(\Delta_{top}^d \times_k k) \rightarrow \mathcal{Q}(\Delta_{top}^d \times_k k)$$

is a homotopy cofiber sequence. The diagonal converts degreewise homotopy cofiber sequences of simplicial spectra into homotopy cofiber sequences of spectra and thus

$$\mathcal{F}(\Delta_{top}^\bullet \times_k k) \rightarrow \mathcal{G}(\Delta_{top}^\bullet \times_k k) \rightarrow \mathcal{Q}(\Delta_{top}^\bullet \times_k k)$$

is a homotopy cofiber sequence. The natural map $\mathcal{Q} \rightarrow \mathcal{H}$ is a weak equivalence on smooth varieties which implies that $(\pi_q \mathcal{Q})_{uad} \rightarrow (\pi_q \mathcal{H})_{uad}$ is an isomorphism for all q . Therefore $\mathcal{Q}(\Delta_{top}^\bullet \times_k k) \rightarrow \mathcal{H}(\Delta_{top}^\bullet \times_k k)$ is also a weak equivalence by Proposition 2.15. We conclude that the sequence (2.17) is a homotopy cofiber (and thus a homotopy fiber) sequence of spectra. \square

Write $\mathcal{KR}_{sst}^{(q)}(X) = \mathcal{K}^{(q)}(X \times_{\mathbb{R}} \Delta_{top}^\bullet)$ and $\mathcal{MR}_{sst}^{(q)}(X) = \mathcal{M}^{(q)}(X \times_{\mathbb{R}} \Delta_{top}^\bullet)$. Also write $\mathcal{KR}_{sst}(X) = \mathcal{K}(X \times_{\mathbb{R}} \Delta_{top}^\bullet)$, which is the real semi-topological K -theory spectrum defined in [22, Definition 2.1]. We obtain from the motivic tower (2.6) the

semi-topological tower of presheaves of spectra,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{K}\mathbb{R}_{sst}^{(q+1)} & \longrightarrow & \mathcal{K}\mathbb{R}_{sst}^{(q)} & \longrightarrow & \mathcal{K}\mathbb{R}_{sst}^{(q-1)} & \longrightarrow & \cdots & \longrightarrow & \mathcal{K}\mathbb{R}_{sst}^{(0)} & \longrightarrow & \mathcal{K}\mathbb{R}_{sst}. \\ & & & & \downarrow & & \downarrow & & & & \downarrow & & & \\ & & & & \mathcal{M}\mathbb{R}_{sst}^{(q)} & & \mathcal{M}\mathbb{R}_{sst}^{(q-1)} & & & & \mathcal{M}\mathbb{R}_{sst}^{(0)} & & & \end{array}$$

Theorem 2.18. *Let X be a smooth real variety.*

- (1) *The spectra $\mathcal{K}\mathbb{R}_{sst}^{(q)}(X)$ and $\mathcal{M}\mathbb{R}_{sst}^{(q)}(X)$ are (-1) -connected spectra. Moreover $\pi_k \mathcal{K}\mathbb{R}_{sst}^{(q)}(X) = 0$ for $k < q - \dim X$.*
- (2) *The sequence*

$$(2.19) \quad \mathcal{K}\mathbb{R}_{sst}^{(q+1)}(X) \rightarrow \mathcal{K}\mathbb{R}_{sst}^{(q)}(X) \rightarrow \mathcal{M}\mathbb{R}_{sst}^{(q)}(X)$$

is a homotopy fiber sequence of spectra.

- (3) *For any abelian group A ,*

$$\pi_k(\mathcal{M}\mathbb{R}_{sst}^{(q)}(X); A) = L^q H\mathbb{R}^{q-k, q}(X; A),$$

where $L^q H\mathbb{R}^{q-k, q}(X; A)$ denotes the real morphic cohomology defined in [22].

- (4) *The augmentation*

$$\mathcal{K}\mathbb{R}_{sst}^{(0)}(X) \rightarrow \mathcal{K}\mathbb{R}_{sst}(X)$$

is a weak equivalence of spectra.

Proof. (1) This follows from Theorem 2.7 and Proposition 2.15.

- (2) This follows from Theorem 2.7 and Proposition 2.16.

- (3) The spectrum $\mathcal{M}^{(q)}(X)$ is the spectrum associated to the simplicial abelian group $z_{\text{equi}}(\mathbb{P}_{\mathbb{R}}^{\wedge q}, 0)(X_{\mathbb{R}} \times_{\mathbb{R}} \Delta_{\mathbb{R}}^{\bullet})$. Consequently we have that $\mathcal{M}_{sst}^{(q)}(X)$ is naturally equivalent to the spectrum associated to the simplicial abelian group $z_{\text{equi}}(\mathbb{A}_{\mathbb{R}}^q, 0)(X_{\mathbb{R}} \times \Delta_{\mathbb{R}}^{\bullet} \times \Delta_{\text{top}}^{\bullet})$. By [28] the homotopy groups of this simplicial abelian group compute real morphic cohomology.

- (4) Follows from Theorem 2.14 and Theorem 2.7. □

The following is the Atiyah-Hirzebruch type spectral sequence relating the real morphic cohomology and the real semi-topological algebraic K -theory, of a smooth real variety X .

Corollary 2.20. *For any smooth real variety X and any abelian group A , there is a strongly convergent spectral sequence*

$$E_2^{p, q} = L^{-q} H\mathbb{R}^{p, -q}(X; A) \implies K\mathbb{R}_{-p-q}^{sst}(X; A).$$

Proof. For any abelian group A , the homotopy fiber sequences (2.19) yield an exact couple,

$$D_2^{p, q}(sst) = \pi_{-p-q}(\mathcal{K}\mathbb{R}_{sst}^{(-q)}(X); A) \quad \text{and} \quad E_2^{p, q}(sst) = \pi_{-p-q}(\mathcal{M}\mathbb{R}_{sst}^{(-q)}(X); A).$$

This exact couple is bounded below by the previous theorem and therefore the resulting spectral sequence converges to $\text{colim}_p D_2^{p, n-p} = \pi_{-n} \mathcal{K}\mathbb{R}_{sst}(X; A)$ (see for example [54, Corollary 5.9.7]). □

Remark 2.21. More generally we could use the equivariant homotopy groups $\pi_{s,t}$ rather than $\pi_{s,0}$ in order to produce other spectral sequences relating equivariant morphic cohomology and equivariant semi-topological K -theory $\mathcal{K}\mathbb{R}_{s,t}(X) \stackrel{\text{def}}{=} \pi_{s,t}\mathcal{K}(X_{\mathbb{C}} \times_{\mathbb{C}} \Delta_{top}^{\bullet})$.

2.3. Topological spectral sequence. We continue to write $G = Gal(\mathbb{C}/\mathbb{R})$. Let F be a presheaf of simplicial sets or spectra on Sch/\mathbb{R} . We remind the reader that for a topological space T we have a natural G -action on $F(T \times_{\mathbb{C}} X) = \text{colim}_{T \rightarrow U(\mathbb{R})} F(U_{\mathbb{C}} \times_{\mathbb{R}} X)$ and that $F(T \times_{\mathbb{C}} X) = p^*F(T \times_{\mathbb{C}} X_{\mathbb{C}})$ where $p : Sch/\mathbb{R} \rightarrow Sch/\mathbb{C}$ is the map of sites given by $X \mapsto X_{\mathbb{C}}$ (see Remark 2.10).

The construction of [13, Lemma 3.2] yields an equivariant natural transformation of presheaves of G -simplicial sets

$$p^*F(\Delta_{top}^{\bullet} \times_{\mathbb{C}} X_{\mathbb{C}}) \rightarrow \underline{\text{HOM}}(\text{Sing}_{\bullet} X(\mathbb{C}), p^*F(\Delta_{top}^{\bullet})),$$

where $\underline{\text{HOM}}(-, -)$ denotes the simplicial function complex equipped with the usual G -action. If F is a presheaf of spectra then this is an equivariant natural transformation of naive G -spectra.

The projections $U_{\mathbb{C}} \rightarrow U$ induce the natural map

$$F(T \times_{\mathbb{R}} X) = \text{colim}_{T \rightarrow U(\mathbb{R})} F(U \times_{\mathbb{R}} X) \rightarrow \left(\text{colim}_{T \rightarrow U(\mathbb{R})} F(U_{\mathbb{C}} \times_{\mathbb{R}} X) \right)^G = p^*F(T \times_{\mathbb{C}} X_{\mathbb{C}})^G.$$

Composing with the map from [13, Lemma 3.2] yields the natural transformation

$$F(\Delta_{top}^{\bullet} \times_{\mathbb{R}} X) \rightarrow \underline{\text{HOM}}_G(\text{Sing}_{\bullet} X(\mathbb{C}), p^*F(\Delta_{top}^{\bullet})),$$

where $\underline{\text{HOM}}_G(-, -)$ is the simplicial complex of equivariant maps.

Definition 2.22. If F is a presheaf of simplicial sets on Sch/\mathbb{R} define a presheaf of G -simplicial sets on Sch/\mathbb{R} by

$$F_{top}(X) = \underline{\text{HOM}}(\text{Sing}_{\bullet} X(\mathbb{C}), p^*F(\Delta_{top}^{\bullet})).$$

If F is a presheaf of spectra on Sch/\mathbb{R} define F_{top} by the same formula. In this case F_{top} is a presheaf of naive G -spectra.

The preceding discussion shows that we have the sequence of natural transformations

$$F \rightarrow F_{sst} \rightarrow F_{top}^G$$

of presheaves on Sch/\mathbb{R} .

Applying the functor $(-)_top$ to the motivic tower on Sch/\mathbb{R} gives rise to the tower of presheaves of naive G -spectra on Sch/\mathbb{R}

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{K}_{top}^{(q+1)} & \longrightarrow & \mathcal{K}_{top}^{(q)} & \longrightarrow & \mathcal{K}_{top}^{(q-1)} & \longrightarrow & \cdots & \longrightarrow & \mathcal{K}_{top}^{(0)} & \longrightarrow & \mathcal{K}_{top} \\ & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\ & & \mathcal{M}_{top}^{(q)} & & \mathcal{M}_{top}^{(q-1)} & & \mathcal{M}_{top}^{(0)} & & & & & & \end{array}$$

We have taken the Grayson spectral sequence as our model for the motivic spectral sequence because with this choice it is straightforward to see that $\mathcal{K}^{(n)}(X_{\mathbb{C}})^G = \mathcal{K}^{(n)}(X)$. This observation is crucial in the following proposition, which together with the two propositions after it, form the basis of the companion topological spectral sequence to the previously constructed semi-topological spectral sequence.

Proposition 2.23. *Let X be a smooth real variety. Then*

$$(2.24) \quad \mathcal{K}_{top}^{(q+1)}(X) \rightarrow \mathcal{K}_{top}^{(q)}(X) \rightarrow \mathcal{M}_{top}^{(q)}(X)$$

is an equivariant homotopy fiber sequence.

Proof. It is enough to verify the claim for $X = \text{Spec } \mathbb{R}$ and to verify that the sequence is a homotopy fiber sequence upon forgetting the G -action and upon taking fixed points. Evaluating this sequence on $\text{Spec } \mathbb{R}$ and using the observation at the end of Remark 2.10 and Lemma 2.9, the sequence becomes

$$\mathcal{K}^{(q+1)}(\Delta_{top}^\bullet \times_{\mathbb{C}} \mathbb{C}) \rightarrow \mathcal{K}^{(q)}(\Delta_{top}^\bullet \times_{\mathbb{C}} \mathbb{C}) \rightarrow \mathcal{M}^{(q)}(\Delta_{top}^\bullet \times_{\mathbb{C}} \mathbb{C}).$$

By Proposition B.1, upon taking fixed points this sequence becomes

$$\mathcal{K}^{(q+1)}(\Delta_{top}^\bullet \times_{\mathbb{R}} \mathbb{R}) \rightarrow \mathcal{K}^{(q)}(\Delta_{top}^\bullet \times_{\mathbb{R}} \mathbb{R}) \rightarrow \mathcal{M}^{(q)}(\Delta_{top}^\bullet \times_{\mathbb{R}} \mathbb{R}).$$

That both of these are homotopy fiber sequences of spectra now follows from Proposition 2.16 and Theorem 2.7. \square

Recall that Atiyah's real K -theory $KR(-)$ [1] is represented by the G -space $\mathbb{Z} \times BU$ (where BU has G -action given by complex conjugation). For a special Γ -space F we write $\mathbb{S}F$ for the associated Ω -spectrum. If F has a G -action compatible with the Γ -space structure then $\mathbb{S}F$ is a naive G -spectrum (see Appendix A for a recollection of these matters). We define the naive G -spectrum spectrum kr to be the the spectrum $kr = \mathbb{S}Grass_{\mathbb{C}}(\mathbb{C})^{an}$ (where $Grass_{\mathbb{C}}(\mathbb{C})^{an}$ is given the structure of a Γ -space in a manner analogous to the Γ -space structure on $G_Y(- \times \Delta^\bullet)$ in Section 2.1). For a G -space W we write $kr^i(W) = kr^{i,0}(W)$. Note that for $n \geq 0$ we have that $kr^{-n}(W) = \pi_n \text{Map}_G(W, kr_0)$ where $\text{Map}_G(-, -)$ denotes the space of equivariant maps. Furthermore we have that $\mathbb{Z} \times BU = kr_0$ and therefore $kr^{-n}(W) = KR^{-n}(W)$ for $n \geq 0$.

Proposition 2.25. *The naive G -spectrum $\mathcal{K}_{top}(\mathbb{R})$ is equivariantly equivalent to kr . In particular $\mathcal{K}_{top}(\mathbb{R})_0 \simeq \mathbb{Z} \times BU$.*

Proof. The map

$$\mathcal{K}_{top}^{(0)}(\mathbb{R}) = \mathbb{S}\text{Hom}_{\mathbb{C}}(\Delta_{top}^\bullet \times_{\mathbb{C}} \Delta_{\mathbb{C}}^\bullet, Grass_{\mathbb{C}}) \leftarrow \mathbb{S}\text{Hom}_{\mathbb{C}}(\Delta_{top}^\bullet, Grass_{\mathbb{C}})$$

is a homotopy equivalence by [20, Lemma 1.2]. On fixed points

$$\mathcal{K}_{top}^{(0)}(\mathbb{R})^G = \mathbb{S}\text{Hom}_{\mathbb{R}}(\Delta_{top}^\bullet \times_{\mathbb{R}} \Delta_{\mathbb{R}}^\bullet, Grass_{\mathbb{R}}) \leftarrow \mathbb{S}\text{Hom}_{\mathbb{R}}(\Delta_{top}^\bullet, Grass_{\mathbb{R}})$$

is a homotopy equivalence by [22]. Thus $\mathbb{S}\text{Hom}_{\mathbb{C}}(\Delta_{top}^\bullet, Grass_{\mathbb{C}}) \rightarrow \mathcal{K}_{top}^{(0)}(\mathbb{R})$ is an equivariant equivalence. In general if T is a compact Hausdorff topological space and Y is a real variety then $\text{Hom}_{\mathbb{C}}(T, Y_{\mathbb{C}}) \cong \text{Hom}_{cts}(T, Y(\mathbb{C}))$ is an equivariant isomorphism (by an argument similar to [23, Proposition 4.2]). Therefore $\text{Hom}_{\mathbb{C}}(\Delta_{top}^\bullet, Grass_{\mathbb{C}}^{(n)}) \cong \text{Sing}_\bullet Grass_{\mathbb{C}}^{(n)}(\mathbb{C})^{an}$ is an isomorphism of equivariant Γ -spaces and so $\mathcal{K}_{top}^{(0)}(\mathbb{R})$ is equivariantly equivalent to the the Ω -spectrum $\mathbb{S}Grass_{\mathbb{C}}(\mathbb{C})^{an} = kr$. \square

Proposition 2.26. *The 0th space of the Ω -spectrum $\mathcal{M}_{top}^{(q)}(\mathbb{R})$ is naturally weakly equivalent to $\text{Sing}_\bullet \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q)$ for $g \geq 0$. In particular for any real variety X ,*

$$\pi_k \mathcal{M}_{top}^{(q)}(X) = H^{q-k,q}(X(\mathbb{C}); \mathbb{Z}).$$

Proof. The spectrum $\mathcal{M}(\mathbb{R})_{top}^{(q)}$ is equivalent to the spectrum associated to the simplicial G -module $z_{equi}(\mathbb{A}_{\mathbb{C}}^q, 0)(\Delta_{top}^{\bullet})$. By [24, Corollary 3.5] and [28, Proposition 4.16] this is equivariantly homotopic to $\text{Sing}_{\bullet} \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q)$. By [9] the space $\mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q)$ is the Eilenberg-MacLane space $K(\mathbb{Z}, \mathbb{R}^{q,q})$. \square

That this spectral sequence associated to the fiber sequences (2.24) is strongly convergent spectral sequence will follow from the following proposition.

Proposition 2.27. *Let W be a finite-dimensional G -CW complex of dimension d . There are natural isomorphisms, for any abelian group A*

$$H^{n,m}(W; \underline{A}) \rightarrow H^{n,m+1}(W; \underline{A})$$

for $n + m > d$ and for $n + m + 1 < 0$. In particular

- (1) if $m \geq 0$, then $H^{n,m}(X, \underline{A}) = 0$ for $n > d$ and for $n + 1 < -m$,
- (2) if $m \leq 0$ then $H^{n,m}(X, \underline{A}) = 0$ for $n + m > d$ and for $n < 0$.

Proof. Consider the homotopy cofibration sequence

$$W_+ \wedge \mathbb{Z}/2_+ \rightarrow W_+ \rightarrow W_+ \wedge S^{0,1}.$$

This induces a long exact sequence in reduced Bredon cohomology

$$\cdots \rightarrow \tilde{H}^{n-1,m+1}(W_+ \wedge \mathbb{Z}/2_+) \rightarrow \tilde{H}^{n,m+1}(W_+ \wedge S^{0,1}) \rightarrow \tilde{H}^{n,m+1}(W_+) \rightarrow \cdots,$$

which may be rewritten as

$$(2.28) \quad \rightarrow H_{sing}^{(n-1)+(m+1)}(W; A) \rightarrow H^{n,m}(W; \underline{A}) \rightarrow H^{n,m+1}(W; \underline{A}) \rightarrow H_{sing}^{n+m+1}(W; A) \rightarrow \cdot$$

This implies the first statement because for n, m as stated $H_{sing}^{(n-1)+(m+1)}(W; A) = 0 = H_{sing}^{n+m+1}(W; A)$.

If $m \geq 0$ and $n > d$ or $n + 1 < -m \leq 0$ then we have $0 = H_{sing}^n(W/G) = H^{n,0}(W) = H^{n,1}(W) = \cdots = H^{n,m}(W)$. Similarly if $m \leq 0$ and $d < n + m \leq n$ or $n < 0$ then we have $H^{n,m}(W) = H^{n,m+1}(W) = \cdots = H^{n,-1}(W) = H^{n,0}(W) = H_{sing}^n(W/G) = 0$. \square

Theorem 2.29. *If X is a smooth real variety then for any abelian group A , there is a strongly convergent spectral sequence*

$$(2.30) \quad E_2^{p,q}(top) \implies kr^{p+q}(X; A).$$

where $E_2^{p,q}(top) = H^{p,-q}(X; \underline{A})$ for $q \leq 0$ and $E_2^{p,q}(top) = 0$ for $q > 0$.

Proof. The homotopy fiber sequences (2.24) give the exact couple,

$$D_2^{p,q}(top) = \pi_{-p-q,0}(\mathcal{K}_{top}^{(-q)}(X); A) \quad \text{and} \quad E_2^{p,q}(top) = \pi_{-p-q,0}(\mathcal{M}_{top}^{(-q)}(X); A),$$

which gives rise to a half-plane spectral sequence with exiting differentials in the terminology of [4, Section 6]. We have $\pi_s \mathcal{K}_{top}^{(n)}(\mathbb{R}) = 0 = \pi_s \mathcal{K}_{top}^{(n)}(\mathbb{R})^G$ for $s < n$ by Theorem 2.7(5) together with Proposition 2.15. Therefore $\lim_n \pi_s \mathcal{K}_{top}^{(n)}(\mathbb{R}) = 0 = \lim_n \pi_s \mathcal{K}_{top}^{(n)}(\mathbb{R})^G$ for all s . These towers satisfy the trivial Mittag-Leffler condition and so $\lim_n^1 \pi_s \mathcal{K}_{top}^{(n)}(\mathbb{R}) = 0 = \lim_n^1 \pi_s \mathcal{K}_{top}^{(n)}(\mathbb{R})^G$ for all s (see for example [54, Section 3.5]). From the Milnor exact sequence [47, Lemma 5.41] we conclude that $\text{holim} \mathcal{K}_{top}^{(n)}(\mathbb{R})$ and $\text{holim} \mathcal{K}_{top}^{(n)}(\mathbb{R})^G$ are both weakly contractible. The homotopy limit $\text{holim} \mathcal{K}_{top}^{(n)}(\mathbb{R})$ is computed by first replacing the maps of this tower with

G -fibrations between fibrant objects and then taking an inverse limit and so we see that $(\operatorname{holim} \mathcal{K}_{top}^{(n)}(\mathbb{R}))^G \simeq \operatorname{holim} \mathcal{K}_{top}^{(n)}(\mathbb{R})^G$. We conclude that $\operatorname{holim} \mathcal{K}_{top}^{(n)}(\mathbb{R})$ is equivariantly weakly contractible, and therefore $\operatorname{holim} \underline{\mathbf{HOM}}(\operatorname{Sing}_\bullet X(\mathbb{C}), \mathcal{K}_{top}^{(n)}(\mathbb{R}))$ is weakly contractible as well. In turn this implies that $\lim_p D_2^{p,n-p}(top) = 0$. Finally, because $E_2^{p,q}(top) = 0$ when $p > 2 \dim X$ by Proposition 2.27(1) we conclude by [4, Theorem 6.1] that the spectral sequence is strongly convergent and converges to $\operatorname{colim}_p D_2^{p,n-p}(top) = \pi_{-n,0}(\mathcal{K}_{top}(X); A)$. \square

To summarize, we have three strongly convergent spectral sequences and natural maps between them for a smooth real variety X and abelian group A ,

$$(2.31) \quad \begin{array}{c} E_2^{p,q}(alg) = \mathbb{H}_{\mathcal{M}}^{p-q}(X; A(-q)) \implies K_{-p-q}(X; A) \\ \downarrow \\ E_2^{p,q}(sst) = L^{-q} H\mathbb{R}^{p,-q}(X; A) \implies K\mathbb{R}_{-p-q}^{sst}(X; A) \\ \downarrow \\ E_2^{p,q}(top) = H^{p,-q}(X(\mathbb{C}); \underline{A}) \implies kr^{p+q}(X(\mathbb{C}); A). \end{array}$$

We finish by comparing our construction with Dugger's [11]. We warn the reader that the indexing conventions used there are different from ours.

For $n \geq 0$, define $\mathcal{K}_{(n-1)}^{top}(\mathbb{R})$ so that it fits into the homotopy fiber sequence of naive G -spectra

$$(2.32) \quad \mathcal{K}_{top}^{(n)}(\mathbb{R}) \rightarrow \mathcal{K}_{top}(\mathbb{R}) \rightarrow \mathcal{K}_{(n-1)}^{top}(\mathbb{R}).$$

There are homotopy fiber sequences of naive G -spectra

$$(2.33) \quad \mathcal{M}_{top}^{(n)}(\mathbb{R}) \rightarrow \mathcal{K}_{(n)}^{top}(\mathbb{R}) \rightarrow \mathcal{K}_{(n-1)}^{top}(\mathbb{R})$$

and the towers $\{\mathcal{K}_{(n-1)}^{top}(\mathbb{R})\}$ and $\{\mathcal{K}_{top}^{(n)}(\mathbb{R})\}$ give rise to the same spectral sequence (see for example [27, Appendix B]).

Write $\Omega^\infty E$ for the infinite loop space associated to a naive G -spectrum E . By Proposition 2.25 and Proposition 2.26 we have that $\Omega^\infty \mathcal{M}_{top}^{(n)}(\mathbb{R}) \simeq K(\mathbb{Z}, \mathbb{R}^{n,n})$ and $\Omega^\infty \mathcal{K}^{top}(\mathbb{R}) \simeq \mathbb{Z} \times BU$. Taking associated infinite loop spaces of the homotopy fiber sequences (2.33) gives the homotopy fiber sequences of G -spaces

$$(2.34) \quad K(\mathbb{Z}, \mathbb{R}^{n,n}) \rightarrow \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R}) \rightarrow \Omega^\infty \mathcal{K}_{(n-1)}^{top}(\mathbb{R}).$$

We briefly recall Dugger's construction, for complete details see [11, Section 3]. For a based G -space X , define $P_{2n}X$ to be the nullification of X at the set of G -spaces $\mathcal{A}_{\mathbb{R}^{n,n}} \stackrel{\text{def}}{=} \{S^W \mid \mathbb{R}^{n,n} \subsetneq W\} \cup \{S^W \wedge G_+ \mid \mathbb{R}^{n,n} \subsetneq W\}$ (where the W are G -representations). Equivalently, this may be viewed as the Bousfield localization at the set of maps $\{S^W \rightarrow * \mid \mathbb{R}^{n,n} \subsetneq W\} \cup \{S^W \wedge G_+ \rightarrow * \mid \mathbb{R}^{n,n} \subsetneq W\}$. Dugger defines his spectral sequence [11, Corollary 1.3] via the tower $\{P_{2n}(\mathbb{Z} \times BU)\}$.

Proposition 2.35. *There is a map of towers of G -spaces*

$$\{P_{2n}\Omega^\infty \mathcal{K}^{top}(\mathbb{R})\} \rightarrow \{\Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R})\}$$

such that $P_{2n}\Omega^\infty \mathcal{K}^{top}(\mathbb{R}) \rightarrow \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R})$ is an equivariant weak equivalence.

Proof. From the homotopy fiber sequences (2.34) and the fact that $\pi_{a,b}K(\mathbb{Z}, \mathbb{R}^{k,k}) = H^{k-a,k-b}(pt, \mathbb{Z}) = 0$ whenever $a \geq k$, $b \geq k$ and $a + b > 2k$, we see that if W is a G -representation with $\mathbb{R}^{n,n} \subsetneq W$ then $\pi_W \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R}) \subseteq \pi_W \Omega^\infty \mathcal{K}_{(-1)}^{top}(\mathbb{R}) = 0$. Similarly if $\mathbb{R}^{n,n} \subsetneq W$, then we also have that $[S^W \wedge G_+, \Omega^\infty \mathcal{K}_{(k)}^{top}(\mathbb{R})]_G = \pi_{|W|} \Omega^\infty \mathcal{K}_{(k)}^{top}(\mathbb{R}) = 0$. Therefore by the universal property of the functors P_{2n} , the map $\Omega^\infty \mathcal{K}^{top}(\mathbb{R}) \rightarrow \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R})$ factors through the localization $\Omega^\infty \mathcal{K}^{top}(\mathbb{R}) \rightarrow P_{2n} \Omega^\infty \mathcal{K}^{top}(\mathbb{R})$. Thus we obtain unique up to homotopy maps $P_{2n} \Omega^\infty \mathcal{K}^{top}(\mathbb{R}) \rightarrow \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R})$ from Dugger's Postnikov tower so that

$$\begin{array}{ccccc} \Omega^\infty \mathcal{K}^{top}(\mathbb{R}) & \longrightarrow & \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R}) & \longrightarrow & \Omega^\infty \mathcal{K}_{(n-1)}^{top}(\mathbb{R}) \\ & \searrow & \uparrow & & \uparrow \\ & & P_{2n} \Omega^\infty \mathcal{K}^{top}(\mathbb{R}) & \longrightarrow & P_{2n-2} P_{2n} \Omega^\infty \mathcal{K}^{top}(\mathbb{R}) = P_{2n-2} \Omega^\infty \mathcal{K}^{top}(\mathbb{R}) \end{array}$$

commutes up to homotopy. Thus we obtain the homotopy commutative comparison diagram of homotopy fiber sequences

$$\begin{array}{ccccc} K(\mathbb{Z}, \mathbb{R}^{n,n}) & \longrightarrow & P_{2n} \Omega^\infty \mathcal{K}^{top}(\mathbb{R}) & \longrightarrow & P_{2n-2} \Omega^\infty \mathcal{K}^{top}(\mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathbb{Z}, \mathbb{R}^{n,n}) & \longrightarrow & \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R}) & \longrightarrow & \Omega^\infty \mathcal{K}_{(n-1)}^{top}(\mathbb{R}). \end{array}$$

To see that the map $K(\mathbb{Z}, \mathbb{R}^{n,n}) \rightarrow K(\mathbb{Z}, \mathbb{R}^{n,n})$ is an equivariant weak equivalence it is enough to verify that it is an isomorphism on $\pi_{n,n}$ and an isomorphism on π_{2n} upon forgetting the equivariant structure [36, Definition 1.4, Lemma 3.7]. It is an isomorphism on π_{2n} because the tower $\{\mathcal{K}^{(n)}(\Delta_{top}^\bullet \times_{\mathbb{C}} \mathbb{C})\}$ has the nonequivariant homotopy type of the usual Postnikov tower [13, Theorem 3.4]. The forgetful map $\pi_{n,n} K(\mathbb{Z}, \mathbb{R}^{n,n}) \rightarrow \pi_{2n} K(\mathbb{Z}, \mathbb{R}^{n,n})$ is seen to be an isomorphism using the sequence (2.28) because $H^{0,-1}(pt; \mathbb{Z}) = 0 = H^{1,-1}(pt; \mathbb{Z})$. We conclude that all the maps $K(\mathbb{Z}, \mathbb{R}^{n,n}) \rightarrow K(\mathbb{Z}, \mathbb{R}^{n,n})$ appearing in the comparison diagram above are equivariant weak equivalences.

By induction we see that the induced maps $\pi_k P_{2n} \Omega^\infty \mathcal{K}^{top}(\mathbb{R}) \rightarrow \pi_k \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R})$ and $\pi_k P_{2n} \Omega^\infty \mathcal{K}^{top}(\mathbb{R})^G \rightarrow \pi_k \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R})^G$ are isomorphisms for all $k \geq 1$. Now we address the case $k = 0$. By Theorem 2.7 together with Proposition 2.15 we see that $\pi_s \mathcal{K}_{top}^{(n)}(\mathbb{R}) = 0 = \pi_s \mathcal{K}_{top}^{(n)}(\mathbb{R})^G$ for $s < n$. This together with the homotopy fiber sequence (2.32) of naive G -spectra shows that the maps $\pi_s \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R}) \rightarrow \pi_s \Omega^\infty \mathcal{K}_{(n-1)}^{top}(\mathbb{R})$ and $\pi_s \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R})^G \rightarrow \pi_s \Omega^\infty \mathcal{K}_{(n-1)}^{top}(\mathbb{R})^G$ are isomorphisms for $s \leq n - 1$. Together with the corresponding fact for the tower $P_{2n} \Omega^\infty \mathcal{K}^{top}(\mathbb{R})$ (which follows from [11, Proposition 3.8]) allows us to conclude the case $k = 0$. Thus the maps $P_{2n} \Omega^\infty \mathcal{K}^{top}(\mathbb{R}) \rightarrow \Omega^\infty \mathcal{K}_{(n)}^{top}(\mathbb{R})$ are equivariant weak equivalences and we are done. \square

Remark 2.36. As a consequence the spectral sequence $\{E_2^{p,q}(top)\}$ agrees with Dugger's for $p + q \leq 0$, with the caveat that his indexing conventions differ from ours. (Note that Dugger's spectral sequence is fringed and confined to $p + q \leq 0$ while we can have non-zero $E_2^{p,q}$ terms for $p + q > 0$, as long as $q \leq 0$, because our spectral sequence is constructed using naive G -spectra).

3. CONJECTURES AND COMPUTATIONS

In this section we introduce an equivariant version of Suslin's conjecture on morphic cohomology and we show that the complex, real, and equivariant versions of these conjectures are all equivalent. Based on this we compute real morphic cohomology in weight one, generalizing a partial computation of [46]. We also prove a real version of the Bloch-Ogus formula [3, Corollary 7.4]. This allows us to prove that real algebraic equivalence coincides with homological equivalence in codimension two for a large class of real varieties. Using these techniques, we compute real and equivariant morphic cohomology of certain real varieties which together with the spectral sequences constructed in the previous section allows us to make new computations of real semi-topological K -theory in terms of Atiyah's KR -theory. As another application of the spectral sequence we give a simple proof of the mod-2 Lichtenbaum-Quillen conjecture over \mathbb{R} generalizing a proof of Karoubi-Weibel [31].

Recall Suslin's conjecture on morphic cohomology:

Conjecture 3.1 (Suslin's Conjecture in weight q). *Let X be a smooth, quasi-projective complex variety. The cycle map*

$$L^q H^p(X) \rightarrow H_{sing}^p(X(\mathbb{C}); \mathbb{Z})$$

is an isomorphism for $p \leq q$ and an injection for $p \leq q + 1$.

We introduce the analogues of this conjecture for real morphic cohomology and for equivariant morphic cohomology. Real morphic cohomology is introduced by Friedlander-Walker in [22] (for equivariant morphic cohomology see [28]).

Conjecture 3.2 (Real Suslin's Conjecture in weight q). *Let X be a smooth, quasi-projective real variety. The cycle map*

$$L^q H\mathbb{R}^{a+q}(X) = L^q H\mathbb{R}^{a,q}(X) \rightarrow H^{a,q}(X(\mathbb{C}); \mathbb{Z})$$

is an isomorphism for $a \leq 0$ and an injection for $a \leq 1$.

Conjecture 3.3 (Equivariant Suslin's Conjecture in weight q). *Let X be a smooth, quasi-projective real variety. Let $a \leq q$ and $b \leq q$. The cycle map*

$$L^q H\mathbb{R}^{a,b}(X) \rightarrow H^{a,b}(X(\mathbb{C}); \mathbb{Z})$$

is an isomorphism for $a \leq 0$ and an injection for $a = 1$.

As a consequence of Milnor's Conjecture proved by Voevodsky [48] all three of these conjectures are true with $\mathbb{Z}/2^k$ -coefficients (see [28, Corollary 5.11]). This allows us to conclude the following integral equivalence of these conjectures.

Theorem 3.4. *The following are all equivalent statements:*

- (1) *Suslin's conjecture is valid in weight q ,*
- (2) *Real Suslin's conjecture is valid in weight q ,*
- (3) *Equivariant Suslin's conjecture is valid in weight q .*

Proof. Recall that if X is a complex variety considered as a real variety, then $Z^q(X_{\mathbb{C}}) = Z^q(X) \times Z^q(X)$, where G acts by switching the factors. Consequently $L^q H\mathbb{R}^{a,q}(X) = L^q H^{a+q}(X)$ when X is a complex variety viewed as a real variety. Thus if the real Suslin's conjecture is valid in weight q then Suslin's conjecture is valid as well. The real Suslin's conjecture is a special case of the equivariant Suslin's

conjecture. We show that the validity of Suslin's conjecture in weight q implies the real Suslin's conjecture in weight q and that the validity of the real Suslin's conjecture in weight q implies the validity of the equivariant Suslin's conjecture in weight q .

Assume that Suslin's conjecture is true in weight q . For any $\ell \geq 1$, by [28, Corollary 5.11] the map $L^q H\mathbb{R}^{a,q}(X; \mathbb{Z}/2^\ell) \rightarrow H^{a,q}(X(\mathbb{C}); \mathbb{Z}/2^\ell)$ is an isomorphism for $a \leq 0$ and an injection for $a = 1$. Therefore it is enough to verify the real Suslin's conjecture with $\mathbb{Z}[1/2]$ -coefficients. For a smooth real variety X , let $\pi : X_{\mathbb{C}} \rightarrow X$ be the projection and consider the commutative diagram

$$(3.5) \quad \begin{array}{ccccc} L^k H\mathbb{R}^{a,q}(X; \mathbb{Z}[1/2]) & \xrightarrow{\pi^*} & L^k H^{a+q}(X_{\mathbb{C}}; \mathbb{Z}[1/2]) & \xrightarrow{\pi_*} & L^k H\mathbb{R}^{a,q}(X; \mathbb{Z}[1/2]) \\ \downarrow & & \downarrow & & \downarrow \\ H^{a,q}(X(\mathbb{C}); \mathbb{Z}[1/2]) & \xrightarrow{\pi^*} & H_{sing}^{a+q}(X(\mathbb{C}); \mathbb{Z}[1/2]) & \xrightarrow{\pi_*} & H^{a,q}(X(\mathbb{C}); \mathbb{Z}[1/2]). \end{array}$$

The compositions $\pi_* \pi^* = 2$ and thus the horizontal compositions are isomorphisms. It follows that Suslin's conjecture implies the real Suslin's conjecture.

Now we assume the validity of the real Suslin's conjecture and conclude that the equivariant Suslin's conjecture is valid. Let $F_q = \text{hofib}(\mathcal{Z}^q(X_{\mathbb{C}}) \rightarrow \mathcal{Z}_{top}^q(X(\mathbb{C})))$ be the homotopy fiber of the cycle map. The homotopy fiber construction is equivariant and so we obtain the equivariant homotopy fiber sequence

$$F_q \rightarrow \mathcal{Z}^q(X_{\mathbb{C}}) \rightarrow \mathcal{Z}_{top}^q(X(\mathbb{C})).$$

The real Suslin's conjecture implies that $\pi_k F_q^G = 0$ for $k \geq q - 1$ and also $\pi_k F = 0$ for $k \geq q - 1$ (because the real Suslin's conjecture implies Suslin's conjecture). We see therefore that $\Omega^{q-1} F_q$ is equivariantly contractible for $q \geq 1$ and for $q = 0$, F_0 is equivariantly contractible as well. It follows that the equivariant Suslin's conjecture holds. \square

Remark 3.6. In general, from Diagram 3.5 we see that if Suslin's conjecture is valid in weight k then

$$L^k H\mathbb{R}^{a,b}(X; \mathbb{Z}[1/2]) \rightarrow H^{a,b}(X(\mathbb{C}); \mathbb{Z}[1/2])$$

is an isomorphism for $a + b \leq k$ and an injection for $a + b \leq k + 1$. A consequence of this observation is the fact that the cycle map

$$(3.7) \quad L^k H\mathbb{R}^{a,b}(X; \mathbb{Z}) \rightarrow H^{a,b}(X(\mathbb{C}); \mathbb{Z})$$

has 2-torsion kernel $a + b \leq k + 1$ and 2-torsion cokernel for any $a + b \leq k$, whenever Suslin's conjecture is valid in weight k . Moreover if the above isomorphism extends for $a + b \leq k + 1$ then the map 3.7 has a 2-torsion cokernel for any $a + b \leq k + 1$. This remark will be used below in Theorem 3.17.

Corollary 3.8. *Let X be a smooth, quasi-projective real variety. Then the cycle map*

$$L^1 H\mathbb{R}^{a,b}(X) \rightarrow H^{a,b}(X(\mathbb{C}); \mathbb{Z})$$

is an isomorphism for $a \leq 0$ and an monomorphism for $a \leq 1$.

Proof. By [15] Suslin's conjecture holds in weight 1. According to Theorem 3.4, this implies that the equivariant Suslin's conjecture holds in weight 1. \square

Friedlander-Walker introduce in [22] the equivalence relation of *real algebraic equivalence* for algebraic cycles on a real variety. Briefly two cycles α, β on a real variety X are real algebraically equivalent provided there is a smooth real curve C , two real points c_0, c_1 in the same connected component (in the analytic topology) of $C(\mathbb{R})$, and a cycle γ on $X \times C$ such that $\alpha = \gamma|_{c_0}$ and $\beta = \gamma|_{c_1}$. We write $A_{rae}^q(X)$ for the group of codimension- q cycles modulo real algebraic equivalence. If X is smooth then $A_{rae}^q(X) = L^q H\mathbb{R}^{q,q}(X)$ by [9, Proposition 4.6] and [28, Corollary 4.20].

We obtain an improvement of the computation of real Lawson homology of the space of divisors given in [46].

Corollary 3.9. *Let X be a smooth, quasi-projective real variety. Then*

$$L^1 H\mathbb{R}^{a,1}(X) \cong H^{a,1}(X(\mathbb{C}); \mathbb{Z}) \cong H_G^{a+1}(X(\mathbb{C}); \mathbb{Z}(1))$$

for $a \leq 0$ and

$$L^1 H\mathbb{R}^{1,1}(X) = A_{rae}^1(X) \hookrightarrow H^{1,1}(X(\mathbb{C}); \mathbb{Z}) \hookrightarrow H_G^2(X(\mathbb{C}), \mathbb{Z}(1)).$$

Here $H_G^*(X(\mathbb{C}); \mathbb{Z}(q))$ denotes Borel cohomology, and $\mathbb{Z}(q)$ is the sheaf \mathbb{Z} where G acts by $(-1)^q$.

In particular, real algebraic equivalence coincides with homological equivalence for divisors on real smooth varieties.

Proof. This follows by setting $b = 1$ in Corollary 3.8. The relation between Bredon cohomology and Borel cohomology is a particular case of [28, Proposition 5.17]. The last statement is just a reformulation of the monomorphism $L^1 H\mathbb{R}^{1,1}(X) = A_{rae}^1(X) \hookrightarrow H^{1,1}(X(\mathbb{C}); \mathbb{Z})$ and the equality $L^1 H\mathbb{R}^{1,1}(X) = A_{rae}^1(X)$ is proved in [8] and in [28]. \square

Remark 3.10. Notice that $L^1 H\mathbb{R}^{a,1}(X) = 0$ for any $a \leq -1$ because $H_G^r(X, \mathbb{Z}(1)) = 0$ for any $r \leq 0$. In particular, the above computation gives exactly the vanishing range obtained in [46].

Conjecture 3.11. (*2-adic Lichtenbaum-Quillen conjecture*) *Let X be a smooth real variety. For any $k \geq 1$ the map*

$$\pi_n(K(X); \mathbb{Z}/2^k) \rightarrow \pi_n(K(X_{\mathbb{C}})^{hG}; \mathbb{Z}/2^k)$$

is an isomorphism for $n \geq \dim X - 1$ and an injection for $n \geq \dim X - 2$.

By [22, Proposition 4.7, Corollary 3.10] this is equivalent to the statement that for any smooth real variety X and $k \geq 1$, the map

$$K\mathbb{R}_n^{sst}(X; \mathbb{Z}/2^k) \rightarrow kr^{-n}(X(\mathbb{C}); \mathbb{Z}/2^k)$$

is an isomorphism for $n \geq \dim X - 1$ and an injection for $n \geq \dim X - 2$.

This conjecture was verified in some cases by Friedlander-Walker [22], by Karoubi-Weibel [31] (in slightly weaker form for varieties with real points), and in general by Rosenschon-Ostvaer [42]. We provide a simple proof of this result using Voevodsky's verification of the Milnor conjecture [48] in conjunction with the comparison of spectral sequences (2.31).

Theorem 3.12. *Let X be a smooth, quasi-projective real variety. Then for any $k \geq 1$,*

$$K\mathbb{R}_n^{sst}(X; \mathbb{Z}/2^k) \rightarrow kr^{-n}(X(\mathbb{C}); \mathbb{Z}/2^k)$$

is an isomorphism for $n \geq \dim X - 1$ and a monomorphism for $n = \dim X - 2$

Proof. We write $E(sst)$ (respectively $E(top)$) for the semi-topological (respectively the topological) spectral sequence constructed in Section 2.2 and 2.3. We show that the map $E_r^{p,q}(sst) \rightarrow E_r^{p,q}(top)$ from (2.31) is an isomorphism for $p+q \leq 1 - \dim X$ and a monomorphism for $p+q = 2 - \dim X$. Since these spectral sequences are strongly convergent, this will yield the result.

First the case $r = 2$. Suppose that $p+q \leq 1 - \dim X$ then either $p \leq 0$ or $q \leq -d$. If $p \leq 0$ then by [28, Corollary 5.11]

$$E_2^{p,q}(sst) = L^{-q}H^{p,-q}(X; \mathbb{Z}/2^k) \rightarrow H^{p,-q}(X(\mathbb{C}); \mathbb{Z}/2^k) = E_2^{p,q}(top)$$

is an isomorphism (note that if both $p, -q \leq 0$ then $E_2^{p,q}(sst) = 0 = E_2^{p,q}(top)$). If $q \leq -d$ then by Corollary 4.2 then $E_2^{p,q}(sst) \rightarrow E_2^{p,q}(top)$ is an isomorphism. Similarly if $p+q \leq 2 - \dim X$ then either $p \leq 1$ or $q \leq -d$ and we see by [28, Corollary 5.11] and Corollary 4.2 that $E_2^{p,q}(sst) \rightarrow E_2^{p,q}(top)$ is a monomorphism. Proceeding by induction as in [13, Theorem 6.1] shows that $E_r^{p,q}(sst) \rightarrow E_r^{p,q}(top)$ is an isomorphism for all r and $p+q \leq 1 - \dim X$ and a monomorphism for all r and $p+q \leq 2 - \dim X$. \square

Remark 3.13. The above proof is along the same lines as in [31], a key difference being that there they use a spectral sequence based on the Borel cohomology of $X(\mathbb{C})$. If $X(\mathbb{R}) \neq \emptyset$ (i.e. the G -action on $X(\mathbb{C})$ is not free) then this spectral sequence will not converge and in this way the use of Bredon cohomology simplifies the situation.

If we know that Suslin's conjecture is valid for a particular variety then we can deduce an integral agreement of real semi-topological K -theory and kr -theory.

Theorem 3.14. *Suppose that X is a smooth, quasi-projective real variety such that the cycle maps $L^q H\mathbb{R}^{a,q}(X) \rightarrow H^{a,q}(X(\mathbb{C}); \mathbb{Z})$ are isomorphisms for any q , any $a \leq 0$ and injections for $a \leq 1$ (in other words X satisfies the real Suslin's conjecture). Then*

$$K\mathbb{R}_n^{sst}(X) \rightarrow kr^{-n}(X(\mathbb{C}))$$

is an isomorphism for $n \geq \dim X - 1$ and a monomorphism for $n \geq \dim X - 2$.

Proof. Follows from the comparison of spectral sequences by the same argument as in Theorem 3.12. \square

Corollary 3.15. *Let X be a smooth, quasi-projective real surface. Then*

$$K\mathbb{R}_n^{sst}(X; \mathbb{Z}) \rightarrow kr^{-n}(X(\mathbb{C}); \mathbb{Z})$$

is an isomorphism for $n \geq 1$ and a monomorphism for $n = 0$.

Proof. This follows from Corollary 3.9 and Theorem 3.14. \square

The following corollary was proved in [22, Corollary 6.9] using different methods.

Corollary 3.16. *Let X be a smooth, quasi-projective real curve. Then*

$$K\mathbb{R}_n^{sst}(X; \mathbb{Z}) \rightarrow kr^{-n}(X(\mathbb{C}); \mathbb{Z})$$

is an isomorphism for $n \geq 0$.

Proof. It follows again from Corollary 3.9 and Theorem 3.14. \square

In certain special examples of real smooth projective varieties, some of the semi-topological invariants can be computed. For example, suppose that X is a smooth, projective geometrically rationally connected threefold defined over real numbers or that X is generic smooth, projective real cubic of dimension d where $3 \leq d \leq 8$, $d \neq 7$. Then:

Theorem 3.17. *Let X be one of the varieties from the above list. Then*

$$L^q H\mathbb{R}^{a,b}(X) \rightarrow H^{a,b}(X(\mathbb{C}); \mathbb{Z})$$

is an isomorphism for $a \leq 1$ (and $b \leq q$).

Proof. By [49] and [50], for any X as in the hypothesis, the complexification $X_{\mathbb{C}}$ satisfies

$$L^q H^{a+b}(X_{\mathbb{C}}) \cong H^{a+b}(X_{\mathbb{C}})$$

for $a+b \leq q+1$. By [28, Corollary 5.11] $L^q H\mathbb{R}^{a,q}(X; \mathbb{Z}/2^\ell) \rightarrow H^{a,q}(X(\mathbb{C}); \mathbb{Z}/2^\ell)$ is an isomorphism for $a \leq 0$ and an injection for $a = 1$ and any $\ell > 0$. In particular, we can conclude that the kernel of the cycle map $L^q H\mathbb{R}^{a,q}(X; \mathbb{Z}) \rightarrow H^{a,q}(X(\mathbb{C}); \mathbb{Z})$ is 2-divisible for any $a \leq 1$ and the cokernel is 2-torsion free for any $a \leq 1$ (see [49, Proposition 3.1]). According to the Remark 3.6, the kernel and cokernel of the cycle map for $a \leq 1$ and $b \leq q$ are 2-torsion. This implies that the cycle map $L^q H\mathbb{R}^{a,q}(X; \mathbb{Z}) \rightarrow H^{a,q}(X(\mathbb{C}); \mathbb{Z})$ is an isomorphism for any $a \leq 1$.

Recall that for a based G -space W and fixed value of $q \geq 0$, there is a long exact sequence relating the non-equivariant and equivariant homotopy groups

$$(3.18) \quad \cdots \rightarrow \pi_{p+q+1} W \xrightarrow{\delta^{(q)}} \pi_{p,q+1} W \rightarrow \pi_{p,q} W \xrightarrow{\phi^{(q)}} \pi_{p+q} W \xrightarrow{\delta^{(q)}} \cdots \rightarrow \pi_{0,q} W \xrightarrow{\phi^{(q)}} \pi_q W,$$

(the map $\phi^{(q)}$ is the forgetful map). This sequence arises by considering the homotopy cofiber sequence

$$\mathbb{Z}/2_+ \rightarrow S^0 \rightarrow S^{0,1}$$

and the above long-exact sequence is the one induced by the homotopy fiber sequence

$$F(S^{0,1}, W) \rightarrow F(S^0, W) \rightarrow F(\mathbb{Z}/2_+, W).$$

Applying this homotopy fiber sequence to $Z^q(X_{\mathbb{C}})$ and $Z_{top}^q(X(\mathbb{C}))$ we obtain the following diagram

$$(3.19) \quad \begin{array}{ccccccccc} L^q H\mathbb{R}^{a-1,b} & \longrightarrow & L^q H^{a-1+b} & \longrightarrow & L^q H\mathbb{R}^{a,b-1} & \longrightarrow & L^q H\mathbb{R}^{a,b} & \longrightarrow & L^q H^{a+b} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{a-1,b} & \longrightarrow & H^{a-1+b} & \longrightarrow & H^{a,b-1} & \longrightarrow & H^{a,b} & \longrightarrow & H^{a+b} \end{array}$$

The theorem now follows from the above diagram and reverse induction on $b \leq q$. \square

Remark 3.20. Theorem 3.17 can be extended, using the same ideas, to real smooth threefolds and fourfolds with zero cycles on the complexification supported on a proper subvariety of dimension less or equal than two. We also remark that equivariant Suslin's conjecture holds for these varieties and those from Theorem 3.17.

Our next results will involve the coniveau spectral sequences for real morphic cohomology and for Bredon cohomology. We make use of [6] for these coniveau spectral sequences. Let $Y \subseteq X$ be a closed subvariety, with open complement $U = X - Y$. Let $\mathcal{Z}^q(X_{\mathbb{C}})_{Y_{\mathbb{C}}} = \text{hofib}(\mathcal{Z}^q(X_{\mathbb{C}}) \rightarrow \mathcal{Z}^q(U_{\mathbb{C}}))$ and define the equivariant morphic cohomology of X with support on Y by $L^q H\mathbb{R}_Y^{q-a, q-b}(X) = \pi_{a,b} \mathcal{Z}^q(X_{\mathbb{C}})_{Y_{\mathbb{C}}}$. Similarly, $H_{Y(\mathbb{C})}^{q-a, q-b}(X(\mathbb{C}); \mathbb{Z}) = H^{q-a, q-b}(X(\mathbb{C}), U(\mathbb{C}); \mathbb{Z})$ denotes the Bredon cohomology with support on $Y(\mathbb{C})$. The generalized cycle map extends to a map on cohomology theories with support $\Phi : L^q H\mathbb{R}_Y^{s,t}(X) \rightarrow H_{Y(\mathbb{C})}^{s,t}(X(\mathbb{C}); \mathbb{Z})$. The following lemma makes explicit that both theories satisfy all the properties required in order to make use of the results of [6].

Lemma 3.21. *Real morphic cohomology and Bredon cohomology are homotopy invariant cohomologies that satisfy Nisnevich excision.*

Proof. Both theories are homotopy invariant. That real morphic cohomology satisfies Nisnevich excision follows from the fact that motivic cohomology does. To be explicit, let

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U^{\subset} & \longrightarrow & X \end{array}$$

be a distinguished Nisnevich square. Let $\mathcal{M}^{(q)}(-)$ be the presheaf of spectra on Sch/\mathbb{R} from section 2.1 which computes motivic cohomology (so that $\mathcal{M}^{(q)}(- \times \Delta_{top}^{\bullet})$ computes real morphic cohomology). The commutative square

$$\begin{array}{ccc} \mathcal{M}^{(q)}(X \times W) & \longrightarrow & \mathcal{M}^{(q)}(Y \times W) \\ \downarrow & & \downarrow \\ \mathcal{M}^{(q)}(U \times W)^{\subset} & \longrightarrow & \mathcal{M}^{(q)}(V \times W) \end{array}$$

is homotopy cartesian whenever W is smooth. Let $\mathcal{F}(W)$ and $\mathcal{G}(W)$ denote the homotopy fibers of the top and bottom horizontal arrows. By Proposition 2.15 $\mathcal{F}(\Delta_{top}^{\bullet}) \rightarrow \mathcal{G}(\Delta_{top}^{\bullet})$ is a weak equivalence. By Proposition 2.16 we see that these are the homotopy fibers of the horizontal arrows in the diagram

$$\begin{array}{ccc} \mathcal{M}^{(q)}(X \times \Delta_{top}^{\bullet}) & \longrightarrow & \mathcal{M}^{(q)}(Y \times \Delta_{top}^{\bullet}) \\ \downarrow & & \downarrow \\ \mathcal{M}^{(q)}(U \times \Delta_{top}^{\bullet})^{\subset} & \longrightarrow & \mathcal{M}^{(q)}(V \times \Delta_{top}^{\bullet}). \end{array}$$

It follows that real morphic cohomology satisfies Nisnevich descent.

That Bredon cohomology satisfies Nisnevich descent follows from the fact that

$$\begin{array}{ccc} V(\mathbb{C}) & \longrightarrow & Y(\mathbb{C}) \\ \downarrow & & \downarrow \\ U(\mathbb{C})^{\subset} & \longrightarrow & X(\mathbb{C}) \end{array}$$

is an equivariant homotopy pushout diagram of G -spaces (see for example [12]). \square

We will use below the following lemma that proves the local vanishing for equivariant motivic cohomology.

Lemma 3.22. *Let $R = \mathcal{O}_{X, x_1, \dots, x_n}$ be a semi-local ring of a smooth real variety X at the points $x_1, \dots, x_n \in X$. Then $L^s H\mathbb{R}^{a,s}(\mathrm{Spec} R) = 0$ if $a > 0$.*

Proof. Let $\mathcal{M}^{(s)}$ be a presheaf computing weight- s motivic cohomology (for example the presheaf from Section 2.1). If R is as in the statement of the lemma, then $\pi_i \mathcal{M}^{(s)}(\mathrm{Spec} R) = H_{\mathcal{M}}^{2s-i}(\mathrm{Spec} R; \mathbb{Z}(s)) = 0$ for $i < s$. By Theorem 2.14 we see that the simplicial abelian group $(d \mapsto \pi_i \mathcal{M}^{(s)}(\mathrm{Spec} R \times_{\mathbb{R}} \Delta_{top}^d))$ is contractible for all $i < s$. Applying the spectral sequence for a simplicial spectrum [30, Corollary 4.22] shows that $\pi_i \mathcal{M}_{sst}^{(s)}(\mathrm{Spec} R) = L^s H\mathbb{R}^{s-i,s}(\mathrm{Spec} R) = 0$ for $i < s$. \square

We now prove a real analogue of the Bloch-Ogus formula [3, Corollary 7.4]. Namely for a smooth quasi-projective real variety X we show that $H_{Zar}^p(X, \mathcal{H}^{0,p}) \cong A_{rae}^p(X)$, where $\mathcal{H}^{a,b}$ is the Zariski sheafification of the presheaf $U \mapsto H^{a,b}(U(\mathbb{C}); \mathbb{Z})$ and $A_{rae}^p(X)$ is the group of codimension- p cycles modulo real algebraic equivalence.

Observe that by [28, Proposition 5.17] and [10, Proposition 1.15] we have that the sheaf $\mathcal{H}^{0,p}$ agrees with the Zariski sheafification of the Borel cohomology presheaf $U \mapsto H_G^p(U(\mathbb{C}); \mathbb{Z}(p))$ (where $\mathbb{Z}(p)$ is the G -module \mathbb{Z} with action given by multiplication by $(-1)^p$). Thus the following theorem may be seen as an integral extension, for real varieties, of the Bloch-Ogus formula for etale cohomology with finite coefficients [3, Theorem 7.7], because Borel cohomology with finite coefficients agrees with etale cohomology with finite coefficients for real varieties [7].

Theorem 3.23. *Let X be a smooth quasi-projective real variety. There is a natural isomorphism*

$$H_{Zar}^p(X, \mathcal{H}^{0,p}) \cong A_{rae}^p(X).$$

Proof. Let $X^{(p)}$ denote the set of points $x \in X$ whose closure has codimension p . If H^* is a cohomology theory with supports, define $H_x^i(X) = \mathrm{colim}_{U \subseteq X} H_{\bar{x} \cap U}^i(U)$ and $H^i(k(x)) = \mathrm{colim}_{U \subseteq \bar{x}} H^i(U)$ (where in both colimits, U ranges over nonempty opens). By [6, Corollary 5.1.11] if H is homotopy invariant and satisfies Nisnevich excision then for a smooth quasi-projective X , the coniveau spectral sequence $E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H_x^{p+q}(X) \implies H^{p+q}(X)$ has E_2 -term given by $H_{Zar}^p(X; \mathcal{H}^q)$, where \mathcal{H}^q is the sheafification of the presheaf $U \mapsto H^q(U)$. The associated filtration is $F^p H^n(X) = \bigcup_Z \mathrm{Im}(H_Z^n(X) \rightarrow H(X))$, where the union is over closed subvarieties $Z \subseteq X$ of codimension p .

Equivariant morphic cohomology and Bredon cohomology satisfy these properties by Lemma 3.21 and using Lemma 4.3, the coniveau spectral sequences can be written as

$$E_1^{p,q}(k, b) = \bigoplus_{x \in X^{(p)}} L^{k-p} H\mathbb{R}^{q, b-p}(k(x)) \implies L^k H\mathbb{R}^{p+q, b}(X)$$

and

$$\tilde{E}_1^{p,q}(b) = \bigoplus_{x \in X^{(p)}} H^{q, b-p}(k(x); \mathbb{Z}) \implies H^{p+q, b}(X(\mathbb{C}); \mathbb{Z}),$$

and the E_2 terms are respectively given by $E_2^{p,q}(k, b) = H_{Zar}^p(X; \mathcal{L}^k \mathcal{H}^{q, b})$ and $\tilde{E}_2^{p,q}(b) = H_{Zar}^p(X; \mathcal{H}^{q, b})$. Observe that $F^p L^p H\mathbb{R}^{p,p}(X) = L^p H\mathbb{R}^{p,p}(X)$ (and $F^{p+1} L^p H\mathbb{R}^{p,p}(X) = 0$) and so $E_{\infty}^{p,0}(p, p) = L^p H\mathbb{R}^{p,p}(X)$.

In fact we show that we have $E_2^{p,0}(p,p) = L^p H\mathbb{R}^{p,p}(X)$. By Lemma 3.22 we see that $E_1^{p-r,r-1}(p,p) = \bigoplus_{x \in X^{(p-r)}} L^r H\mathbb{R}^{r-1,r}(k(x)) = 0$ for $r > 1$. Also $E_1^{p+r,-r+1}(p,p) = 0$ if $r \geq 1$. We conclude that $E_2^{p,0}(p,p) = E_\infty^{p,0}(p,p)$, and therefore $H_{Zar}^p(X, \mathcal{L}^p \mathcal{H}\mathbb{R}^{0,p}) = A_{rae}^p(X)$.

Consider the commutative diagram

$$\begin{array}{ccccc} \bigoplus_{x \in X^{(p-1)}} L^1 H\mathbb{R}^{0,1}(k(x)) & \longrightarrow & \bigoplus_{x \in X^{(p)}} L^0 H\mathbb{R}^{0,0}(k(x)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \bigoplus_{x \in X^{(p-1)}} H^{0,1}(k(x); \mathbb{Z}) & \longrightarrow & \bigoplus_{x \in X^{(p)}} H^{0,0}(k(x)) & \longrightarrow & 0, \end{array}$$

the cohomology of the top row computes $H_{Zar}^p(X, \mathcal{L}^p \mathcal{H}\mathbb{R}^{0,p})$ and the bottom computes $H_{Zar}^p(X, \mathcal{H}\mathbb{R}^{0,p})$. The right vertical arrow is an isomorphism and by Corollary 3.9 the left vertical arrow is also an isomorphism. Therefore

$$A_{rae}^p(X) \cong H_{Zar}^p(X; \mathcal{L}^p \mathcal{H}\mathbb{R}^{0,p}) \rightarrow H_{Zar}^p(X; \mathcal{H}^{0,p})$$

is an isomorphism. \square

We can now prove a real analogue of a result due to Bloch-Srinivas [2].

Theorem 3.24. *Let X be a smooth projective real variety of dimension d . Suppose that there is a closed subvariety $V \subseteq X_{\mathbb{C}}$ with $\dim V \leq 2$ such that $CH_0(X_{\mathbb{C}} \setminus V) = 0$. Then*

$$L^2 H\mathbb{R}^{2,2}(X) = A_{rae}^2(X) \hookrightarrow H^{2,2}(X(\mathbb{C}); \mathbb{Z}),$$

so that real algebraic equivalence on X in codimension 2 agrees with homological equivalence.

Proof. We use the same notations as in the previous theorem for the coniveau spectral sequence for Bredon cohomology. We know that $\tilde{E}_\infty^{2,0}(2) = F^2 H^{2,2}(X(\mathbb{C}); \mathbb{Z})$ because $F^3 H^{2,2}(X(\mathbb{C}); \mathbb{Z}) = 0$. Because $\tilde{E}_1^{p,q}(2) = 0$ if $q \leq 1$ and $p > 2$ or if $p < 0$ we see that $\tilde{E}_3^{2,0}(2) = \tilde{E}_\infty^{2,0}(2) = F^2 H^{2,2}(X(\mathbb{C}); \mathbb{Z}) \subseteq H^{2,2}(X(\mathbb{C}); \mathbb{Z})$. We thus have the exact sequence $\tilde{E}_2^{0,1}(2) \xrightarrow{d_2} \tilde{E}_2^{2,0}(2) \rightarrow \tilde{E}_3^{2,0}(2) \subseteq H^{2,2}(X(\mathbb{C}); \mathbb{Z})$ which is the exact sequence

$$H_{Zar}^0(X, \mathcal{H}^{1,2}) \rightarrow H_{Zar}^2(X, \mathcal{H}^{0,2}) \xrightarrow{\alpha} H^{2,2}(X(\mathbb{C}), \mathbb{Z}).$$

The cycle map Φ induces the comparison of coniveau spectral sequences which together with the preceding theorem and its proof gives us the commutative diagram

$$\begin{array}{ccc} E_2^{2,0}(2,2) = H_{Zar}^2(X, \mathcal{L}^2 \mathcal{H}\mathbb{R}^{0,2}) & \xrightarrow{\cong} & E_\infty^{2,0}(2,2) = L^2 H\mathbb{R}^{2,2}(X) \\ \downarrow \cong & & \downarrow \Phi \\ \tilde{E}_2^{2,0}(2) = H_{Zar}^2(X, \mathcal{H}^{0,2}) & \longrightarrow & \tilde{E}_\infty^{2,0}(2) \subseteq H^{2,2}(X(\mathbb{C}); \mathbb{Z}). \end{array}$$

We see that under the identification $H_{Zar}^2(X, \mathcal{H}^{0,2}) = A^2(X)_{rae}$ the map α is identified with the cycle map and so the result will follow from showing that $H_{Zar}^0(X, \mathcal{H}^{1,2}) = 0$. Let \mathcal{H}_{csing}^k denote the sheaf $U \mapsto H_{sing}^k(U(\mathbb{C}); \mathbb{Z})$. With $\mathbb{Z}[1/2]$ -coefficients the composition induced by the map $\pi : X_{\mathbb{C}} \rightarrow X$

$$H_{Zar}^0(X, \mathcal{H}^{1,2}[1/2]) \xrightarrow{\pi^*} H_{Zar}^0(X_{\mathbb{C}}, (\mathcal{H}_{csing}^3)[1/2]) \xrightarrow{\pi_*} H_{Zar}^0(X, \mathcal{H}^{1,2}[1/2])$$

is an isomorphism. Since $H_{Zar}^0(X_{\mathbb{C}}, (\mathcal{H}_{csing}^3)[1/2]) = 0$ for a variety X satisfying the condition from hypothesis (by the proof of [2, Theorem 1(ii)]) it is enough to show that $H_{Zar}^0(X, \mathcal{H}^{1,2})$ is 2-torsion free. Write $\mathcal{L}^q \mathcal{H}\mathbb{R}_{\mathbb{Z}/2^k}^{a,b}$ for the sheafification of $U \mapsto L^q H\mathbb{R}^{a,b}(U; \mathbb{Z}/2^k)$ and similarly for $\mathcal{H}_{\mathbb{Z}/2^k}^{a,b}$. Consider the comparison of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}^2 \mathcal{H}\mathbb{R}_{\mathbb{Z}/2^k}^{0,2} \otimes \mathbb{Z}/2^k & \longrightarrow & \mathcal{L}^2 \mathcal{H}\mathbb{R}_{\mathbb{Z}/2^k}^{0,2} & \longrightarrow & Tor(\mathcal{L}^2 \mathcal{H}\mathbb{R}^{1,2}, \mathbb{Z}/2^k) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{H}^{0,2} \otimes \mathbb{Z}/2^k & \longrightarrow & \mathcal{H}\mathbb{R}_{\mathbb{Z}/2^k}^{0,2} & \longrightarrow & Tor(\mathcal{H}^{1,2}, \mathbb{Z}/2^k) \longrightarrow 0. \end{array}$$

The map $\mathcal{L}^2 \mathcal{H}\mathbb{R}_{\mathbb{Z}/2^k}^{0,2} \rightarrow \mathcal{H}\mathbb{R}_{\mathbb{Z}/2^k}^{0,2}$ is an isomorphism by [28, Corollary 5.13] and $\mathcal{L}^2 \mathcal{H}\mathbb{R}^{1,2} = 0$ by Lemma 3.22. We conclude that $Tor(\mathcal{H}^{1,2}, \mathbb{Z}/2^k) = 0$ and we are done. \square

Because all the real varieties from Theorem 3.17 fulfill the condition of Theorem 3.24, we can use Theorem 3.24 to improve the range of indexes in Theorem 3.17. We will restrict to the case of real geometrically rationally connected varieties of dimension 3, although the same type of arguments and computations can be done for more general varieties that fulfill the condition from Theorem 3.24.

Corollary 3.25. *Let X be a smooth projective real geometrically rationally connected threefold. Then*

$$L^q H\mathbb{R}^{a,b}(X) \rightarrow H^{a,b}(X(\mathbb{C}); \underline{\mathbb{Z}})$$

is an isomorphism for $a < q$ (and $b \leq q$) and a monomorphism if $a = q$ (and $b \leq q$). In particular, all equivariant morphic cohomology groups of X are finitely generated.

Proof. Write $F_q = \text{hofib}(\mathcal{Z}^q(X_{\mathbb{C}}) \rightarrow \text{Map}(X_{\mathbb{C}}(\mathbb{C}), \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q)))$ for the homotopy fiber of the cycle map. The homotopy fiber construction is equivariant and yields an equivariant homotopy fiber sequence

$$F_q \rightarrow \mathcal{Z}^q(X_{\mathbb{C}}) \rightarrow \text{Map}(X_{\mathbb{C}}(\mathbb{C}), \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q)).$$

Applying Theorem 3.24 and Theorem 3.17 we obtain that $\pi_*(F_q^G) = 0$. From [49] we have that $\pi_*(F_q) = 0$. We conclude that F is an equivariantly contractible space. Applying the long exact sequences for the above equivariant homotopy fiber sequence we obtain that $\pi_{0,p}(\mathcal{Z}^q(X_{\mathbb{C}})) \hookrightarrow \pi_{0,p}(\text{Map}(X_{\mathbb{C}}(\mathbb{C}), \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q)))$ for any $p \geq 0$ and that $\pi_{q,p}(\mathcal{Z}^q(X_{\mathbb{C}})) \simeq \pi_{q,p}(\text{Map}(X_{\mathbb{C}}(\mathbb{C}), \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q)))$ for any $q > 0$ and $p \geq 0$. This concludes the statement of the corollary. \square

The following corollary computes the real semi-topological K-theory of a real geometrically connected variety.

Corollary 3.26. *Let X be a smooth projective real geometrically rationally connected threefold. Then*

$$K\mathbb{R}_0^{sst}(X; \mathbb{Z}) \hookrightarrow kr^0(X; \mathbb{Z})$$

and

$$K\mathbb{R}_p^{sst}(X; \mathbb{Z}) \xrightarrow{\cong} kr^{-p}(X; \mathbb{Z})$$

for any $p \geq 1$.

Proof. This follows from the previous corollary together with a comparison of spectral sequences constructed in the previous section (see 2.31). \square

4. COMPATIBILITY OF POINCARÉ DUALITIES

The main result of this section asserts that the Poincaré duality proved in [28] between the equivariant morphic cohomology and Dos Santos's equivariant Lawson homology for a smooth, projective real variety is compatible with the Poincaré duality in Bredon homology and cohomology of a compact real manifold (see Theorem 4.1). The method used here is similar to that of [14] where Friedlander-Lawson show that the duality between morphic cohomology and Lawson homology for smooth projective complex varieties is compatible with Poincaré duality between singular cohomology and homology. An important consequence of this compatibility is Corollary 4.2 which is proved below and is needed for our applications in section 3 of the spectral sequence constructed in this paper.

Let X be a smooth real variety of dimension d . There are generalized cycle maps $\Phi : L^q H\mathbb{R}^{a,b}(X) \rightarrow H^{a,b}(X(\mathbb{C}); \mathbb{Z})$ relating equivariant morphic cohomology and Bredon cohomology (see [22, 28]). The equivariant Lawson homology defined by dos Santos [8] comes equipped with operations $s : L_q H\mathbb{R}_{a,b}(X) \rightarrow L_{q-1} H\mathbb{R}_{a,b}(X)$. In particular, iterating the s -map together with the equivariant Dold-Thom theorem [9, 37] defines the generalized cycle map $\Psi = s^q : L_q H\mathbb{R}_{a,b}(X) \rightarrow L_0 H\mathbb{R}_{a,b}(X) = H_{a,b}(X(\mathbb{C}); \mathbb{Z})$.

A *real n -bundle* over a G -space W is a complex n -plane bundle $\pi : E \rightarrow W$ together with an involution $\tau : E \rightarrow E$ covering the involution $\sigma : W \rightarrow W$. A G -manifold is called a *real n -manifold* if the tangent bundle is a real n -bundle over W , with involution given by $d\sigma$. A real n -manifold W has a fundamental class $[W] \in H_{n,n}(W; \mathbb{Z})$ and satisfies Poincaré duality ([10],[35]),

$$[W] \cap - : H^{p,q}(W; \mathbb{Z}) \xrightarrow{\cong} H_{n-p,n-q}(W; \mathbb{Z}).$$

The following is the main theorem of this section, whose proof we postpone until the end of the section.

Theorem 4.1. *Let X be a smooth, projective real variety of dimension d . The following square commutes for all $k, l \leq q$, where the horizontal arrows are the Poincaré duality isomorphisms*

$$\begin{array}{ccc} L^q H\mathbb{R}^{k,l}(X) & \xrightarrow{\mathcal{D}} & L_{d-q} H\mathbb{R}_{d-k,d-l}(X) \\ \Phi \downarrow & & \downarrow \Psi \\ H^{k,l}(X(\mathbb{C}), \mathbb{Z}) & \xrightarrow{\mathcal{P}} & H_{d-k,d-l}(X(\mathbb{C}), \mathbb{Z}). \end{array}$$

We make use of cohomology with supports in the proof of Corollary 4.2 below. The definitions of these are recalled prior to Theorem 3.23.

Corollary 4.2. *Let U be a smooth, quasi-projective real variety of dimension d . For $a \leq q$ and $b \leq q$, the cycle map*

$$\Phi : L^q H\mathbb{R}^{a,b}(U) \rightarrow H^{a,b}(U(\mathbb{C}); \mathbb{Z})$$

is an isomorphism for $q \geq d$.

Proof. If U is smooth and projective this follows immediately from Theorem 4.1 together with the fact that the map $L_{d-q}H\mathbb{R}_{d-a,d-b}(U) \rightarrow H_{d-a,d-b}(U(\mathbb{C}), \mathbb{Z})$ is an isomorphism for any $q \geq d$.

Reverse induction on b , together with the diagram (3.19) and the corresponding result for morphic cohomology of complex varieties, shows that in order to prove the general result we may assume that $b = q$. We proceed by induction on $\dim U$ to show that Φ is an isomorphism whenever $q \geq \dim U$ and U is smooth and quasi-projective.

The case $\dim U = 0$ is done and so we may assume that the result is valid for all smooth quasi-projective U , with $\dim U < d$. Let U be a quasi-projective real variety with $\dim U = d$ and let $U \subseteq X$ be a smooth projective closure with closed complement $Z \subseteq X$ (we may assume that U is connected so that $\dim Z < \dim U$). First we argue that $L^q H\mathbb{R}_W^{a,q}(X) \rightarrow H_{W(\mathbb{C})}^{a,q}(X(\mathbb{C}); \mathbb{Z})$ is an isomorphism for $q \geq d$ and $a \leq q$ whenever $W \subseteq X$ is a proper closed subvariety. Using Lemma 4.3 below we find that it is an isomorphism whenever W is smooth. In particular the case $\dim W = 0$ is done and we proceed by induction on $\dim W$. Let $W_s \subseteq W$ be the singular locus. Since $\dim W_s < \dim W$ we have by induction that $L^q H\mathbb{R}_{W_s}^{a,q}(X) \rightarrow H_{W_s(\mathbb{C})}^{a,q}(X(\mathbb{C}))$ is an isomorphism for $q \geq d$ and $a \leq q$. Considering the comparison of long exact sequence

$$\begin{array}{ccccccc} \longrightarrow & L^q H\mathbb{R}_{W_s}^{a,q}(X) & \longrightarrow & L^q H\mathbb{R}_W^{a,q}(X) & \longrightarrow & L^q H\mathbb{R}_{W \setminus W_s}^{a,q}(X \setminus W_s) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & H_{W_s(\mathbb{C})}^{a,q}(X(\mathbb{C})) & \longrightarrow & H_{W(\mathbb{C})}^{a,q}(X(\mathbb{C})) & \longrightarrow & H_{(W \setminus W_s)(\mathbb{C})}^{a,q}((X \setminus W_s)(\mathbb{C})) & \longrightarrow, \end{array}$$

together with Lemma 4.3 below and the inductive hypothesis, we see that for $q \geq d$ and $a \leq q$ the map $L^q H\mathbb{R}_{W \setminus W_s}^{a,q}(X \setminus W_s) \rightarrow H_{(W \setminus W_s)(\mathbb{C})}^{a,q}((X \setminus W_s)(\mathbb{C}))$ is also an isomorphism. We conclude that $L^q H\mathbb{R}_W^{a,q}(X) \rightarrow H_{W(\mathbb{C})}^{a,q}(X(\mathbb{C}))$ is an isomorphism for $q \geq d$ and $a \leq q$ and any closed, proper subset $W \subseteq X$ (for the case $a = q$ observe that $L^q H\mathbb{R}_{W \setminus W_s}^{a,q}(X \setminus W_s) = 0 = H_{(W \setminus W_s)(\mathbb{C})}^{a,q}((X \setminus W_s)(\mathbb{C}))$ because $q \geq d > \dim W \setminus W_s$).

Now from the comparison of long exact sequences

$$\begin{array}{ccccccc} \longrightarrow & L^q H\mathbb{R}_Z^{a,q}(X) & \longrightarrow & L^q H\mathbb{R}^{a,q}(X) & \longrightarrow & L^q H\mathbb{R}^{a,q}(U) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & H_{Z(\mathbb{C})}^{a,q}(X(\mathbb{C})) & \longrightarrow & H^{a,q}(X(\mathbb{C})) & \longrightarrow & H^{a,q}(U(\mathbb{C})) & \longrightarrow \end{array}$$

we see that $L^q H\mathbb{R}^{a,q}(U) \rightarrow H^{a,q}(U(\mathbb{C}); \mathbb{Z})$ is also an isomorphism for $q \geq d$ and $a \leq q$ (for the case $a = q$, note that $L^q H\mathbb{R}^{q,q}(X) \rightarrow L^q H\mathbb{R}^{q,q}(U)$ is surjective and $H_{Z(\mathbb{C})}^{q+1,q}(X(\mathbb{C})) = 0$ by Proposition 2.27 and the long exact sequence for cohomology with supports, because $2q + 1 \geq \dim X(\mathbb{C})$). \square

Lemma 4.3. *Let X be a smooth, quasi-projective real variety and let $Z \subseteq X$ be a smooth closed subvariety of codimension c . There are natural isomorphisms $L^{q-c}H\mathbb{R}^{a-c,b-c}(Z) \xrightarrow{\cong} L^q H\mathbb{R}_Z^{a,b}(X)$ and $H^{a-c,b-c}(Z(\mathbb{C}); \mathbb{Z}) \xrightarrow{\cong} H_{Z(\mathbb{C})}^{a,b}(X(\mathbb{C}); \mathbb{Z})$*

which fit into the commutative square

$$\begin{array}{ccc} L^{q-c}H\mathbb{R}^{a-c,b-c}(Z) & \xrightarrow{\cong} & L^qH\mathbb{R}_Z^{a,b}(X) \\ \downarrow \Phi & & \downarrow \Phi \\ H^{a-c,b-c}(Z(\mathbb{C});\mathbb{Z}) & \xrightarrow{\cong} & H_{Z(\mathbb{C})}^{a,b}(X(\mathbb{C});\mathbb{Z}). \end{array}$$

Proof. Let N denote the normal bundle of $Z \subseteq X$. We consider Z as a closed subvariety of N via the zero-section. The lemma will follow by seeing that we have the commutative diagram

$$\begin{array}{ccccc} L^{q-c}H\mathbb{R}^{a-c,b-c}(Z) & \xrightarrow{\cong} & L^qH\mathbb{R}_Z^{a,b}(N) & \xrightarrow{\cong} & L^qH\mathbb{R}_Z^{a,b}(X) \\ \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi \\ H^{a-c,b-c}(Z(\mathbb{C});\mathbb{Z}) & \xrightarrow{\cong} & H_{Z(\mathbb{C})}^{a,b}(N(\mathbb{C});\mathbb{Z}) & \xrightarrow{\cong} & H_{Z(\mathbb{C})}^{a,b}(X(\mathbb{C});\mathbb{Z}). \end{array}$$

The first horizontal arrows are Thom isomorphisms and the second horizontal arrows arise from the deformation to the normal cone.

We apply the results of [41]. To put ourselves in the context of this paper we consider the total equivariant morphic cohomology and Bredon cohomology: $\oplus_{q,a,b}L^qH\mathbb{R}_Z^{a,b}(X)$ and $\oplus_{a,b}H_{Z(\mathbb{C})}^{a,b}(X(\mathbb{C});\mathbb{Z})$. These are cohomology theories in the sense of [41, Definition 2.1], which amounts to saying these are homotopy invariant, satisfy Nisnevich excision, and have localization exact sequences. Both equivariant morphic cohomology and Bredon cohomology clearly satisfy the first and third property, and the excision property is made explicit in Lemma 3.21. As shown in [41, Theorem 2.2] the deformation to the normal cone yields natural isomorphisms $h_Z(N) \xrightarrow{\cong} h_Z(X)$ for any cohomology theory $h(-)$. The maps involved in this isomorphism preserve the gradings and therefore we obtain the right-hand commutative square in the diagram above.

Now we need to see that Φ is compatible with Thom isomorphisms (which Panin calls an orientation [41, Definition 3.1]) in both theories. By [41, Theorem 3.35, Theorem 3.5] in order for $L^*H\mathbb{R}^{*,*}(-)$ to have Thom isomorphisms it is enough to see that it has a Chern structure [41, Definition 3.2] which amounts to having first Chern classes for line bundles which satisfy a naturality condition, a projective bundle formula for $X \times \mathbb{P}^1$, and vanishing on trivial line bundles. These are easily deduced from the corresponding properties for motivic cohomology (over \mathbb{C} and over \mathbb{R}), Theorem 2.14, and the fact that equivariant morphic cohomology may be computed using the complex $z_{equi}(\mathbb{A}_{\mathbb{C}}^q, 0)(X_{\mathbb{C}} \times_{\mathbb{C}} \Delta_{top}^{\bullet})$. Moreover the Thom isomorphism for E is given by cupping with a Thom class $th(E)$ [41, Lemma 3.33] and keeping track of gradings we see that $th(E) \in L^nH\mathbb{R}_X^{n,n}(E)$ where E is an n -plane bundle.

In [10, Proposition 1.13] it is verified that Bredon cohomology has Thom isomorphisms for real bundles (in the above sense). In particular Bredon cohomology viewed as a theory on Sm/\mathbb{R} has Thom isomorphisms. To see that the natural transformation Φ is compatible with the Thom isomorphisms it suffices to see that $\Phi(\mu) \in H_{X(\mathbb{C})}^{n,n}(E(\mathbb{C}))$ is a Thom class whenever $\mu \in L^nH\mathbb{R}_X^{n,n}(E)$ is a Thom class. By the description of the Thom class given in [10, Proposition 1.13], it suffices to see that for any closed point $i : x \hookrightarrow X$ the restriction $i^*\Phi(\mu)$ is a generator of

$H_x^{*,*}(i^*E(\mathbb{C}); \underline{\mathbb{Z}})$. Note that $i^*E = \mathbb{A}_{\mathbb{R}}^n$ or $\mathbb{A}_{\mathbb{C}}^n$ depending on whether $k(x) = \mathbb{R}$, or \mathbb{C} . In both cases $\Phi : L^*H\mathbb{R}_x^{*,*}(i^*E) \rightarrow H_x^{*,*}(i^*E(\mathbb{C}); \underline{\mathbb{Z}})$ is an isomorphism and we are done. \square

Let X be a quasi-projective real variety and Y a projective real variety. The cap-product pairing ([21, Proposition 1.6], [16, Theorem 7.2])

$$\mathcal{C}_s(X_{\mathbb{C}}) \times \text{Mor}(X_{\mathbb{C}}, \mathcal{C}_r(Y_{\mathbb{C}})) \rightarrow \mathcal{C}_{r+s}(X_{\mathbb{C}} \times_{\mathbb{C}} Y_{\mathbb{C}})$$

is defined by sending a pair (f, Z) to the graph of the composite function $Z \rightarrow X_{\mathbb{C}} \rightarrow \mathcal{C}_r(Y_{\mathbb{C}})$. This pairing is G -equivariant and by [21, Theorem 2.6] it induces a continuous pairing

$$\mathcal{C}_s(X_{\mathbb{C}})^{an} \times \text{Mor}(X_{\mathbb{C}}, \mathcal{C}_r(Y_{\mathbb{C}}))^{an} \rightarrow \mathcal{C}_{r+s}(X_{\mathbb{C}} \times_{\mathbb{C}} Y_{\mathbb{C}})^{an},$$

which in turn induces the continuous equivariant pairing

$$\cap : \mathcal{Z}_r(X_{\mathbb{C}}) \wedge \text{Mor}(X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q)) \rightarrow \mathcal{Z}_r(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^q),$$

where for typographical convenience here and below we write

$$\text{Mor}(X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q)) = \frac{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_0(\mathbb{P}_{\mathbb{C}}^q))^+}{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_0(\mathbb{P}_{\mathbb{C}}^{q-1}))^+}.$$

We also denote by \cap the induced pairing,

$$\cap : L_p H\mathbb{R}_{s,t}(X) \otimes L^q H\mathbb{R}^{k,l}(X) \rightarrow L_{p-q} H\mathbb{R}_{s-k,t-l}(X).$$

When X is irreducible of dimension d , the d -cycle $X_{\mathbb{C}} \subseteq X_{\mathbb{C}}$ defines the fundamental class

$$\eta_X \in \pi_{0,0} \mathcal{Z}_d(X_{\mathbb{C}}) = L_d H\mathbb{R}_{d,d}(X_{\mathbb{C}}),$$

and cap-product with the fundamental class gives a map

$$\eta_X \cap - : L^q H\mathbb{R}^{k,l}(X) \rightarrow L_{d-q} H\mathbb{R}_{d-k,d-l}(X),$$

which is the Poincaré duality isomorphism.

For a G -space W , define the equivariant pairing \sqcap to be the composition

$$\sqcap : \mathcal{Z}_0(W) \wedge F(W, \mathcal{Z}_0(S^{d,d})_0) \rightarrow \mathcal{Z}_0(W) \wedge \mathcal{Z}_0(S^{d,d})_0 \xrightarrow{\mu} \mathcal{Z}_0(W \wedge S^{d,d})_0$$

where the first arrow is defined by $(x, f) \mapsto (x, f(x))$ and μ is the map induced by $(x, y) \mapsto x \wedge y$. Together with the suspension isomorphism in Bredon homology this defines a pairing

$$\sqcap : H_{s,t}(W; \underline{\mathbb{Z}}) \otimes H^{k,l}(W; \underline{\mathbb{Z}}) \rightarrow H_{s+d-k,t+d-l}(W \wedge S^{d,d}; \underline{\mathbb{Z}}) \cong H_{s-k,t-k}(W; \underline{\mathbb{Z}}).$$

Remark 4.4. The pairing

$$\mathcal{Z}_0(X_{\mathbb{C}}) \wedge \text{Mor}(X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q)) \rightarrow \mathcal{Z}_0(X_{\mathbb{C}}) \wedge F(X(\mathbb{C}), \mathcal{Z}_0(S^{q,q})_0) \xrightarrow{\sqcap} \mathcal{Z}_0(X(\mathbb{C})_+ \wedge S^{d,d})_0$$

agrees with the pairings \cap under the isomorphism $\mathcal{Z}_0(X(\mathbb{C})_+ \wedge S^{d,d})_0 \cong \mathcal{Z}_0(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d)$.

The following is merely a rephrasing of [10, Proposition 1.14].

Proposition 4.5. [10, Proposition 1.14] *Let W be a smooth compact G -manifold of dimension d and let $[W] \in H_{d,d}(W; \underline{\mathbb{Z}})$ denote its fundamental class. The map $[W] \sqcap -$ defined above agrees with cap-product (in Bredon cohomology and homology) with $[W]$*

$$[W] \cap - = [W] \sqcap - : H^{k,l}(W; \underline{\mathbb{Z}}) \rightarrow H_{d-k,d-l}(W; \underline{\mathbb{Z}}).$$

We finish our discussion of pairings by showing that the pairings \cap and \sqcap are compatible under the natural transformations Φ and Ψ .

Proposition 4.6. *Let X be a smooth projective real variety. The diagram*

$$\begin{array}{ccc} \pi_{0,0}\mathcal{Z}_d(X_{\mathbb{C}}) \otimes \pi_{k,l}\mathrm{Mor}(X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d)) & \xrightarrow{\cap} & \pi_{k,l}\mathcal{Z}_d(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d) \\ \downarrow \Psi \otimes \Phi & & \downarrow \Psi \\ \pi_{0,0}\Omega^{d,d}\mathcal{Z}_0(X(\mathbb{C})) \otimes \pi_{k,l}\mathrm{Map}(X(\mathbb{C}), \mathcal{Z}_0(S^{d,d})_0) & \xrightarrow{\sqcap} & \pi_{k,l}\Omega^{d,d}\mathcal{Z}_0(X(\mathbb{C})_+ \wedge S^{d,d})_0 \end{array}$$

commutes.

Proof. Consider the following diagram

$$\begin{array}{ccc} \mathbb{P}_{\mathbb{C}}^d/\mathbb{P}_{\mathbb{C}}^{d-1} \wedge \mathcal{Z}_d(X_{\mathbb{C}}) \wedge \mathrm{Mor}(X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d)) & \xrightarrow{1 \wedge \cap} & \mathbb{P}_{\mathbb{C}}^d/\mathbb{P}_{\mathbb{C}}^{d-1} \wedge \mathcal{Z}_d(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d) \\ \downarrow u \wedge 1 & & \downarrow u \wedge 1 \\ \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d) \wedge \mathcal{Z}_d(X_{\mathbb{C}}) \wedge \mathrm{Mor}(X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d)) & \xrightarrow{1 \wedge \cap} & \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d) \wedge \mathcal{Z}_d(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d) \\ \downarrow \mu \wedge 1 & & \downarrow \mu \\ \mathcal{Z}_d(\mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}}) \wedge \mathrm{Mor}(X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d)) & & \downarrow \mu \\ \downarrow 1 \wedge \pi^* & & \downarrow \mu \\ \mathcal{Z}_d(\mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}}) \wedge \mathrm{Mor}(\mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d)) & \xrightarrow{\cap} & \mathcal{Z}_d(\mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d) \\ \uparrow \pi^* \wedge \pi^* & & \uparrow \pi^* \\ \mathcal{Z}_0(X_{\mathbb{C}}) \wedge \mathrm{Mor}(X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d)) & \xrightarrow{\cap} & \mathcal{Z}_0(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d) \\ \downarrow 1 \wedge \Psi & & \downarrow \cong \\ \mathcal{Z}_0(X(\mathbb{C})) \wedge \mathrm{Map}(X(\mathbb{C}), \mathcal{Z}_0(S^{d,d})_0) & \xrightarrow{\sqcap} & \mathcal{Z}_0(X(\mathbb{C})_+ \wedge S^{d,d})_0. \end{array}$$

Here u is the map induced by sending $x \in \mathbb{P}_{\mathbb{C}}^d$ to $x - \infty \in \mathcal{Z}_p(\mathbb{P}_{\mathbb{C}}^d)_0$ and μ is induced by $(x, V) \mapsto \{x\} \times V$. The composition of the left vertical maps induces (by taking an adjoint) the map $\Psi \otimes \Phi$ and the adjoint of the right vertical compositions induces the map $\Psi = s^d$. Thus to establish the proposition it is enough to establish the commutativity of this diagram.

The first square of this diagram is evidently commutative. Denote the graph of a morphism $g : W_{\mathbb{C}} \rightarrow \mathcal{C}_0(\mathbb{P}_{\mathbb{C}}^k)$ by $\Gamma(g) \in \mathcal{Z}_d(W_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}_{\mathbb{C}}^k)$. Let $a \in \mathbb{P}_{\mathbb{C}}^d$ and $f : X_{\mathbb{C}} \rightarrow \mathcal{C}_0(\mathbb{P}_{\mathbb{C}}^d)$. The equality of graphs $\{a\} \times \Gamma(f) = \Gamma(f\pi|_{\{a\} \times X}) \in \mathcal{Z}_d(\mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d)$ establishes the commutativity of the second square. Similarly for the third square.

The fourth square is commutative by remark 4.4. \square

The first step towards proving Theorem 4.1 will be to reduce to the case $q = d$. For this we need to introduce the equivariant analogue of the s -map in morphic

cohomology. Consider the commutative diagram

$$\begin{array}{ccc} \frac{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_0(\mathbb{P}_{\mathbb{C}}^q))^+}{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_0(\mathbb{P}_{\mathbb{C}}^{q-1}))^+} & \xrightarrow{\Sigma} & \frac{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_1(\mathbb{P}_{\mathbb{C}}^{q+1}))^+}{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_1(\mathbb{P}_{\mathbb{C}}^q))^+} \\ \downarrow \mathcal{D} & & \downarrow \mathcal{D} \\ \mathcal{Z}_d(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^q) & \xrightarrow{\Sigma} & \mathcal{Z}_{d+1}(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^{q+1}) \end{array}$$

where Σ denotes the algebraic suspension map. By [28, Corollary 4.20] the vertical maps are equivariant homotopy equivalences, by the real analogue of Lawson's suspension theorem proved by Lam [32] (see [33]) the bottom horizontal map is an equivariant homotopy equivalence. We conclude that the upper horizontal arrow is a homotopy equivalence. Let Σ^{-1} denote a choice of homotopy inverse to this map and consider the following composition

$$\text{Mor}(X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q)) \wedge \mathcal{Z}_0(\mathbb{P}_{\mathbb{C}}^1)_0 \xrightarrow{\#} \text{Mor}(X_{\mathbb{C}}, \mathcal{Z}_1(\mathbb{A}_{\mathbb{C}}^{q+2})) \xrightarrow{\Sigma^{-1}} \text{Mor}(X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^{q+1})).$$

Define the s -map in equivariant morphic cohomology

$$s : L^q H\mathbb{R}^{a,b}(X) \rightarrow L^{q+1} H\mathbb{R}^{a,b}(X)$$

to be the map induced by this composition together with $\alpha \in \pi_{1,1}\mathbb{P}_{\mathbb{C}}^1$ where the map $\alpha : S^{1,1} = \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathcal{Z}_0(\mathbb{P}_{\mathbb{C}}^1)_0$ is defined by $x \mapsto \{x\} - \{\infty\}$. Observe that under the duality morphism, the s -map in equivariant morphic cohomology is compatible with the s -map in dos Santos' equivariant Lawson homology in the sense that we have a commutative diagram

$$(4.7) \quad \begin{array}{ccc} L^q H\mathbb{R}^{a,b}(X) & \xrightarrow{s} & L^{q+1} H\mathbb{R}^{a,b}(X) \\ \downarrow \mathcal{D} & & \downarrow \mathcal{D} \\ L_{d-q} H\mathbb{R}_{d-a,d-b}(X) & \xrightarrow{s} & L_{d-q-q} H\mathbb{R}_{d-a,d-b}(X). \end{array}$$

Proposition 4.8. *The transformations Φ are compatible with the s -maps in the sense that*

$$\begin{array}{ccc} L^q H^{p,r}(X) & \xrightarrow{s} & L^{q+1} H^{p,r}(X) \\ \downarrow \Phi & \swarrow \Phi & \\ H^{p,r}(X(\mathbb{C}); \mathbb{Z}) & & \end{array}$$

commutes.

Proof. The space $\mathcal{Z}_k(\mathbb{A}_{\mathbb{C}}^q)$ has the equivariant homotopy type of an equivariant Eilenberg-MacLane space $K(\mathbb{Z}, \mathbb{R}^{q-k, q-k})$. By [8] the join pairing

$$\mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^p) \wedge \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^q) \xrightarrow{\#} \mathcal{Z}_1(\mathbb{A}_{\mathbb{C}}^{p+q+1}) \xleftarrow{\Sigma} \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^{p+q})$$

represents the cup product pairing in Bredon cohomology.

Consider the homotopy commutative square

$$\begin{array}{ccccc} \frac{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_0(\mathbb{P}_{\mathbb{C}}^q))^+}{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_0(\mathbb{P}_{\mathbb{C}}^{q-1}))^+} \wedge \mathcal{Z}_0(\mathbb{P}_{\mathbb{C}}^1)_0 & \xrightarrow{\#} & \frac{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_1(\mathbb{P}_{\mathbb{C}}^{q+2}))^+}{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_1(\mathbb{P}_{\mathbb{C}}^{q+1}))^+} & \xrightarrow{\Sigma^{-1}} & \frac{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_0(\mathbb{P}_{\mathbb{C}}^{q+1}))^+}{\text{Mor}(X_{\mathbb{C}}, \mathcal{C}_0(\mathbb{P}_{\mathbb{C}}^q))^+} \\ \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi \\ \text{Map}(X(\mathbb{C}), \mathcal{Z}_0(\mathbb{A}^q)) \wedge \mathcal{Z}_0(\mathbb{P}^1)_0 & \xrightarrow{\#} & \text{Map}(X(\mathbb{C}), \mathcal{Z}_1(\mathbb{A}^{q+2})) & \xrightarrow{\Sigma^{-1}} & \text{Map}(X(\mathbb{C}), \mathcal{Z}_0(\mathbb{A}^{q+1})) \end{array}$$

Let $\alpha : S^{1,1} = \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathcal{Z}_0(\mathbb{P}_{\mathbb{C}}^1)_0$ be the map defined by $x \mapsto \{x\} - \{\infty\}$. The map α represents the canonical generator of $\pi_{1,1}\mathcal{Z}_0(\mathbb{P}^1)_0 \cong \mathbb{Z}$. The upper horizontal composition restricted to α induces the s -map on equivariant morphic cohomology. On the other hand the bottom horizontal composition restricted to α induces the identity because it represents the product with $\alpha \in H^{0,0}(pt; \mathbb{Z}) = \mathbb{Z}$ and α is the canonical generator of this group. \square

The Poincare dualities \mathcal{D} and \mathcal{P} are both given by capping with a fundamental class. We verify that the transformation Ψ takes fundamental classes in equivariant Lawson homology to fundamental classes in Bredon homology. The cycle map $cyc : \mathcal{Z}_d(X) \rightarrow H_{d,d}(X(\mathbb{C}); \mathbb{Z})$ sends an irreducible closed subvariety $i : V \subseteq X$ to $cyc(V) = i_*[V(\mathbb{C})] \in H_{d,d}(X(\mathbb{C}); \mathbb{Z})$.

Lemma 4.9. *Let X be a projective real variety. The composition of s -maps together with the Dold-Thom isomorphism*

$$\Psi : L_k H\mathbb{R}_{k,k}(X) \rightarrow H_{k,k}(X(\mathbb{C}); \mathbb{Z}),$$

coincides with the cycle-map $V \mapsto [V(\mathbb{C})]$.

Proof. The proof is similar to [17, Proposition 6.4]. Let $V \subseteq X$ be a closed irreducible subvariety of dimension k and $i : \tilde{V} \rightarrow V \subseteq X$ be a resolution of singularities. Then $i_*\eta_{\tilde{V}} = [V_{\mathbb{C}} \subseteq X_{\mathbb{C}}] \in L_k H\mathbb{R}_{k,k}(X)$. By Noether normalization there is a dominant map $f : V \rightarrow \mathbb{P}_{\mathbb{R}}^k$ of some degree d . Consider the commutative diagram

$$\begin{array}{ccccc} L_k H\mathbb{R}_{k,k}(\mathbb{P}^k) & \xleftarrow{f_*} & L_k H\mathbb{R}_{k,k}(\tilde{V}) & \xrightarrow{i_*} & L_k H\mathbb{R}_{k,k}(X) \\ \cong \downarrow \Psi & & \downarrow \Psi & & \downarrow \Psi \\ H_{k,k}(\mathbb{P}^k; \mathbb{Z}) & \xleftarrow{f_*} & H_{k,k}(\tilde{V}(\mathbb{C}); \mathbb{Z}) & \xrightarrow{i_*} & H_{k,k}(X(\mathbb{C}); \mathbb{Z}). \end{array}$$

The map $\Psi : L_k H\mathbb{R}_{k,k}(\mathbb{P}^k) \rightarrow H_{k,k}(\mathbb{P}^k(\mathbb{C}); \mathbb{Z})$ is an isomorphism [8], $\Psi(\eta_{\mathbb{P}^k}) = [\mathbb{P}^k(\mathbb{C})]$, and $f_*(\eta_V) = d \cdot \eta_{\mathbb{P}^k} \in L_k H\mathbb{R}_{k,k}(\mathbb{P}^k)$. Note that $H^{0,0}(\tilde{V}(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}$ because

$$H^{0,0}(\tilde{V}(\mathbb{C}); \mathbb{Z}) \cong H_{sing}^0(\tilde{V}(\mathbb{C})/G; \mathbb{Z}) \cong \text{Hom}(H_0(\tilde{V}(\mathbb{C})/G; \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}$$

and so by equivariant Poincare duality we have $H_{k,k}(\tilde{V}(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}$ generated by the fundamental class $[\tilde{V}(\mathbb{C})]$.

If we see that $f_* : H_{k,k}(\tilde{V}(\mathbb{C}); \mathbb{Z}) \rightarrow H_{k,k}(\mathbb{P}^k(\mathbb{C}); \mathbb{Z})$ is multiplication by d we are done. Indeed if f_* is multiplication by d then it follows that Ψ takes the natural generator $\eta_{\tilde{V}}$ of $L_k H_{k,k}(\tilde{V}(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}$ to the generator $[\tilde{V}(\mathbb{C})]$ of $H_{k,k}(\tilde{V}(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}$.

The fact that this map is multiplication by d follows from the commutative square

$$\begin{array}{ccc} H_{k,k}(\tilde{V}(\mathbb{C}); \mathbb{Z}) & \longrightarrow & H_{2k}(\tilde{V}(\mathbb{C}); \mathbb{Z}) \\ \downarrow & & \downarrow \cdot d \\ H_{k,k}(\mathbb{P}^k(\mathbb{C}); \mathbb{Z}) & \xrightarrow{\cong} & H_{2k}(\mathbb{P}^k(\mathbb{C}); \mathbb{Z}), \end{array}$$

where the horizontal maps are the forgetful maps and send the equivariant fundamental class to the non-equivariant fundamental class. \square

We also need to know that the homotopy invariance in equivariant Lawson homology and the suspension isomorphism in Bredon homology are suitably related. First recall that the suspension isomorphism is given by the following explicit map of spaces.

Lemma 4.10. ([37, Remark 2.14]) *The suspension isomorphism ϕ is induced by the adjoint of the map*

$$S^{d,d} \wedge \mathcal{Z}_0(W) \xrightarrow{u \times 1} \mathcal{Z}_0(S^{d,d}) \wedge \mathcal{Z}_0(W) \xrightarrow{\mu} \mathcal{Z}_0(S^{d,d} \wedge W_+) \xrightarrow{\tau} \mathcal{Z}_0(W_+ \wedge S^{d,d})_0,$$

where the first map is the map $u(x) = \{x\} - \{\infty\}$.

Proposition 4.11. *Let X be a smooth, projective real variety. Then*

$$\begin{array}{ccc} \mathcal{Z}_d(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d) & \xleftarrow{\pi^*} & \mathcal{Z}_0(X_{\mathbb{C}}) \\ \downarrow \Psi & \swarrow \phi & \\ \Omega^{d,d} \mathcal{Z}_0(X(\mathbb{C})_+ \wedge S^{d,d})_0 & & \end{array}$$

is homotopy commutative.

Proof. By adjointness, the homotopy commutativity of the triangle follows from the homotopy commutativity of the following diagram:

$$\begin{array}{ccc} \mathbb{P}_{\mathbb{C}}^d / \mathbb{P}_{\mathbb{C}}^{d-1} \wedge \mathcal{Z}_d(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d) & \xleftarrow{1 \wedge \pi^*} & \mathbb{P}_{\mathbb{C}}^d / \mathbb{P}_{\mathbb{C}}^{d-1} \wedge \mathcal{Z}_0(X_{\mathbb{C}}) \\ \downarrow u \times 1 & & \downarrow u \times 1 \\ \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d) \wedge \mathcal{Z}_d(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d) & \xleftarrow{1 \wedge \pi^*} & \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d) \wedge \mathcal{Z}_0(X_{\mathbb{C}}) \\ \downarrow \mu & & \downarrow \mu \\ \mathcal{Z}_d(\mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d) & \xleftarrow{\pi_1^*} & \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}}) \\ \uparrow \pi_2^* & \swarrow \tau^* & \\ \mathcal{Z}_0(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d) & & \end{array}$$

Here $\pi_1 : \mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d \rightarrow \mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}}$ and $\pi_2 : \mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d \rightarrow X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d$ are the projections, μ is multiplication, and τ is the twist isomorphism. The left vertical composition induces the s -map and the right vertical composition induces the suspension isomorphism ϕ . The first two squares evidently commute. The two maps $\tau\pi_1$ and $\pi_2 : \mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d \rightarrow \mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}}$ are homotopic via

$$F : \mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^d \times_{\mathbb{C}} X_{\mathbb{C}},$$

where $F(a, x, b, t) = (ta + (1-t)b, x)$, and therefore $\pi_2^* \tau^*$ and π_1^* are homotopic. \square

We now prove the compatibility

(Theorem 4.1). *Let X be a smooth, projective real variety of dimension d . The following square commutes, where the horizontal arrows are the Poincare duality isomorphisms*

$$\begin{array}{ccc} L^q H\mathbb{R}^{k,l}(X) & \xrightarrow{\mathcal{D}} & L_{d-q} H\mathbb{R}_{d-k,d-l}(X) \\ \Phi \downarrow & & \downarrow \Psi \\ H^{k,l}(X(\mathbb{C}), \mathbb{Z}) & \xrightarrow{\mathcal{P}} & H_{d-k,d-l}(X(\mathbb{C}), \mathbb{Z}). \end{array}$$

Proof. Proposition together with the commutativity of diagram (4.7) shows that in order to prove the theorem it suffices to consider the case $q = d$ and to prove that the following square commutes

$$\begin{array}{ccc} L^d H\mathbb{R}^{d-k,d-l}(X) = \pi_{k,l} \mathcal{Z}^d(X_{\mathbb{C}}) & \xrightarrow{\mathcal{D}} & \pi_{k,l} \mathcal{Z}_0(X_{\mathbb{C}}) = L_0 H\mathbb{R}_{k,l}(X) \\ \Phi \downarrow & & \downarrow \Psi \\ H^{d-k,d-l}(X(\mathbb{C}), \mathbb{Z}) & \xrightarrow{\mathcal{P}} & H_{k,l}(X(\mathbb{C}), \mathbb{Z}). \end{array}$$

In this case the map Ψ is simply the Dold-Thom isomorphism.

Consider the following diagram

$$\begin{array}{ccccc} \pi_{k,l} \text{Mor}(X_{\mathbb{C}}, \mathcal{Z}_0(\mathbb{A}_{\mathbb{C}}^d)) & \xrightarrow{\eta_X \cap -} & \pi_{k,l} \mathcal{Z}_d(X_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^d) & \xleftarrow{\cong} & \pi_{k,l} \mathcal{Z}_0(X_{\mathbb{C}}) \\ \downarrow \Phi & & \downarrow \Psi & & \downarrow = \\ \pi_{k,l} \text{Map}(X(\mathbb{C}), \mathcal{Z}_0(S^{d,d})_0) & \xrightarrow{[X(\mathbb{C})] \cap -} & \pi_{k,l} \Omega^{d,d} \mathcal{Z}_0(X(\mathbb{C})_+ \wedge S^{d,d})_0 & \xleftarrow{\phi} & \pi_{k,l} \mathcal{Z}_0(X(\mathbb{C})). \end{array}$$

The upper-composition is the Poincare duality isomorphism between equivariant morphic cohomology and equivariant Lawson homology and by Proposition 4.5 the composition along the bottom row is the Poincare duality isomorphism between Bredon cohomology and homology.

The first square commutes by Proposition 4.6 together with Lemma 4.9 and the second square commutes by Proposition 4.11 \square

APPENDIX A. Γ -SPACES

We recall Segal's notion of Γ -space, introduced by Segal in [43], where he showed that such objects give rise infinite loop spaces. Let \underline{n} be the pointed set $\underline{n} = \{0, 1, 2, \dots, n\}$, pointed by 0. The category Γ^{op} is the category whose objects are \underline{n} for $n \geq 0$ with pointed set maps (Γ^{op} is equivalent to the opposite of the category Γ considered by Segal). A Γ -object in a category \mathcal{C} is a functor $F : \Gamma^{op} \rightarrow \mathcal{C}$. Generally \mathcal{C} will be a pointed category in which case we additionally require that $F(0) = *$. When \mathcal{C} is the category of spaces or of simplicial sets then we refer to a Γ -object F as a Γ -space. A Γ -space is said to be *special* if

$$(p_{1*}, \dots, p_{n*}) : F(\underline{n}) \rightarrow F(\underline{1})^{\times n}$$

is a weak equivalence, where $p_i : \underline{n} \rightarrow \underline{1}$ is the function $p_i(i) = 1$ and $p_i(j) = 0$ if $i \neq j$. If \mathcal{F} is a presheaf of special Γ -spaces then we will say that \mathcal{F} is special.

Given a special Γ -space F one can associate a spectrum, which we denote by $\mathbb{S}F$. First note that a Γ -space F is in particular a bisimplicial set (or simplicial space)

$n \mapsto F(\underline{n})$. Write BF for the classifying Γ -space defined by

$$BF(\underline{m}) = |n \mapsto F(\underline{m} \wedge \underline{n})|,$$

here $\underline{m} \wedge \underline{n}$ is identified with \underline{mn} via $(i, j) \mapsto n(i-1) + j$. For a Γ -space A there is a natural map $S^1 \wedge A(\underline{1}) \rightarrow BA(\underline{1})$. The spectrum associated to F is obtained by evaluating iterated classifying spaces of F :

$$\mathbb{S}F = (F(\underline{1}), BF(\underline{1}), B^2F(\underline{1}), \dots).$$

By [43] the fact that F is special implies that BF is special, that $B^kF(\underline{1}) \rightarrow \Omega B^kF(\underline{1})$ is a weak equivalence for $k \geq 1$, and that $F(\underline{1}) \rightarrow \Omega BF(\underline{1})$ is a homotopy group completion.

Now suppose that $G = \mathbb{Z}/2$ acts on the Γ -space F and that both $\underline{m} \mapsto F(\underline{m})$ and $\underline{m} \mapsto F(\underline{m})^G$ are special. Say that such a Γ -space is *equivariantly special*. Realization of a simplicial space commutes with taking fixed points and therefore we immediately conclude the following proposition.

Proposition A.1. *Let F be an equivariantly special Γ -space and $\mathbb{S}F$ the associated Ω -spectrum. Then the naive fixed point spectrum $(\mathbb{S}F)^G$ is the Ω -spectrum associated to the fixed point Γ -space $\underline{n} \mapsto F(\underline{n})^G$.*

APPENDIX B. GLOBALIZATION

We recall the Godement resolution which gives an explicit model for computing the Zariski hypercohomology of X with coefficients in a presheaf of spectra [47]. A particularly useful property of this resolution is that it behaves appropriately with respect to the Galois action of $Gal(\mathbb{C}/\mathbb{R})$.

Suppose that $F(-)$ is a presheaf of fibrant spectra on Sch/\mathbb{R} . The Godement complex $F(-) \rightarrow G^\bullet F(-)$ is defined as follows. Let

$$GF(U) = \prod_{u \in U} F(\text{Spec } \mathcal{O}_{U,u})$$

then define $G^n F = G \circ \dots \circ GF$ to be the $(n+1)$ -fold composition of G . The inclusions and projections of the various factors gives rise to the cosimplicial spectrum $n \mapsto G^n F(U)$ and define $\mathcal{G}F(U) = \text{Tot } G^\bullet F(U)$.

In general for a presheaf of spectra, $F(-)$, define

$$\mathcal{G}F(U) = \text{Tot } G^\bullet QF(U)$$

where Q is a fixed functorial fibrant replacement for spectra. Then $QF(-)$ is a presheaf of fibrant spectra and the homotopy groups of $\mathcal{G}F(U)$ define the Zariski hypercohomology with coefficients in F ,

$$\pi_k \mathcal{G}F(U) = \mathbb{H}_{Zar}^{-k}(U; F).$$

Moreover $\pi_k \mathcal{G}F(U) = [S^k \wedge U_+, F]_{Zar}$, where $[-, -]_{Zar}$ denotes maps in the Zariski-local homotopy category of presheaves of spectra on Sch/\mathbb{R} ([29, 40]).

Now we consider the case when F' is a presheaf of naive $\mathbb{Z}/2$ -spectra on Sch/\mathbb{R} obtained by $F' = p_* p^* F$ for some presheaf F on Sch/\mathbb{R} , where $p : Sch/\mathbb{R} \rightarrow Sch/\mathbb{C}$ is the map of sites specified by $X \mapsto X_{\mathbb{C}}$. Observe that if F is additive, i.e. $F(U \amalg V) = F(U) \times F(V)$, then for a real variety U we have

$$GF'(U) = \prod_{u \in U} F((\text{Spec } \mathcal{O}_{U,u})_{\mathbb{C}}) = \prod_{v \in U_{\mathbb{C}}} F((\text{Spec } \mathcal{O}_{U_{\mathbb{C}},v})) = p_* p^* GF(U).$$

Indeed $(\mathrm{Spec} \mathcal{O}_{U,u})_{\mathbb{C}} = \mathrm{Spec} \mathcal{O}_{U_{\mathbb{C}}, \pi^{-1}\{u\}} = \coprod_{u'_i \in \pi^{-1}\{u\}} \mathrm{Spec} \mathcal{O}_{U_{\mathbb{C}}, u'_i}$, where $\pi : U_{\mathbb{C}} \rightarrow U$ is the projection. Take the fibrant replacement functor Q on spectra used above to be a $\mathbb{Z}/2$ -equivariant fibrant replacement functor. Further suppose that canonical map $F(U) \rightarrow F(U_{\mathbb{C}})^{\mathbb{Z}/2}$ is a weak equivalence. For a cosimplicial spectrum A^{\bullet} we have that $\mathrm{Tot}(A^{\bullet})_n = \mathrm{Tot}(A_n^{\bullet})$ and from the description of $\mathrm{Tot}(A_n^{\bullet})$ as the simplicial function complex $\mathrm{Tot}(A_n^{\bullet}) = \underline{\mathrm{HOM}}(\Delta, A_n^{\bullet})$ on cosimplicial simplicial sets, we see that $\mathrm{Tot}(-)$ commutes with fixed points. Because $(GF(U_{\mathbb{C}}))^{\mathbb{Z}/2} = G(F(U_{\mathbb{C}})^{\mathbb{Z}/2}) = GF(U)$ and $[\mathrm{Tot} G^{\bullet} F(U)]^{\mathbb{Z}/2} = \mathrm{Tot}(G^{\bullet} F(U)^{\mathbb{Z}/2})$ we see that

$$\mathcal{G}F(U) \rightarrow (\mathcal{G}F(U_{\mathbb{C}}))^{\mathbb{Z}/2}$$

is a weak equivalence as well.

The reason for using this specific fibrant replacement functor is the following proposition which is needed during our discussion of the topological spectral sequence in Section 2.3. We remind the reader that the presheaf $\mathcal{K}^{(n)}(-)$ is obtained by applying the Godement resolution to the presheaf $\mathcal{W}^{(n)}(-) = \mathcal{K}_{\mathrm{geom}}(-, \mathbb{P}^{\wedge n})$ and $\mathcal{K}_{\mathrm{geom}}(X, Y)$ is defined to be the spectrum associated to a certain special Γ -space (Definition 2.3). The presheaf $\mathcal{M}^{(n)}$ is obtained by applying the Godement resolution to the presheaf of spectra associated to the presheaf of simplicial abelian groups $z_{\mathrm{equi}}(\mathbb{P}^{\wedge n}, 0)(- \times \Delta^{\bullet})$. See Section 2.1 for more details on these constructions and Section 2.2 for a recollection of the meaning of the notation $F(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{R}} -)$ and $F(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{C}} -)$ for a presheaf $F(-)$ on Sch/\mathbb{R} , used below.

Proposition B.1. *Let X be a real variety. Then*

$$\mathcal{K}^{(n)}(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{R}} X) \rightarrow \mathcal{K}^{(n)}(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{C}} X)^{\mathbb{Z}/2}$$

and

$$\mathcal{M}^{(n)}(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{R}} X) \rightarrow \mathcal{M}^{(n)}(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{C}} X)^{\mathbb{Z}/2}$$

are weak equivalences.

Proof. The spectrum $\mathcal{K}_{\mathrm{geom}}(X, Y)$ is defined to be the spectrum associated to the special Γ -space $m \mapsto \mathrm{Hom}_{\mathbb{R}}(X \times \Delta_{\mathbb{R}}^{\bullet}, G_Y^{(m)})$. By Proposition A.1 we see that $\mathcal{K}_{\mathrm{geom}}(X_{\mathbb{C}}, Y)^G = \mathcal{K}_{\mathrm{geom}}(X, Y)$ because $\mathrm{Hom}_{\mathbb{R}}(X, G_Y^{(m)}) = \mathrm{Hom}_{\mathbb{R}}(X_{\mathbb{C}}, G_Y^{(m)})^G$. In Section 2.1 the spectrum $\mathcal{W}^{(n)}(X)$ is defined by a homotopy colimit of a certain diagram of spectra, $\mathcal{W}^{(n)}(X) = \mathrm{hocolim}_I \mathcal{K}_{\mathrm{geom}}(X, \mathbb{P}_I^{\wedge n})$. Taking fixed points commutes with homotopy colimits of naive G -spectra (see [12, Remark 5.6] for an argument in terms of G -spaces, note that the same argument applies to G -simplicial sets and homotopy colimits of spectra are computed degreewise) and so we have that $\mathcal{W}^{(n)}(X) \simeq \mathcal{W}^{(n)}(X_{\mathbb{C}})^G$. Similarly we see that $\mathcal{M}(z_{\mathrm{equi}}(\mathbb{P}^{\wedge n}, 0)(X \times_{\mathbb{R}} \Delta_{\mathbb{R}}^{\bullet})) \simeq \mathcal{M}(z_{\mathrm{equi}}(\mathbb{P}^{\wedge n}, 0)(X_{\mathbb{C}} \times_{\mathbb{R}} \Delta_{\mathbb{R}}^{\bullet}))^G$. These are additive presheaves and so from the discussion above on Godement resolutions we conclude that $\mathcal{K}^{(n)}(-) = \mathcal{G}\mathcal{W}^{(n)}(-)$ and $\mathcal{M}^{(n)}(-) = \mathcal{G}\mathcal{M}(z_{\mathrm{equi}}(\mathbb{P}^{\wedge n}, 0)(- \times_{\mathbb{R}} \Delta_{\mathbb{R}}^{\bullet}))$ also satisfy $\mathcal{K}^{(n)}(X) \simeq \mathcal{K}^{(n)}(X_{\mathbb{C}})^{\mathbb{Z}/2}$ and $\mathcal{M}^{(n)}(X) \simeq \mathcal{M}^{(n)}(X_{\mathbb{C}})^{\mathbb{Z}/2}$.

For a presheaf F , the colimit $F(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{C}} X) = \mathrm{colim}_{T \rightarrow U(\mathbb{R})} F(U_{\mathbb{C}} \times_{\mathbb{R}} X)$ is a filtered colimit and so fixed points commutes with this colimit,

$$F(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{C}} X)^{\mathbb{Z}/2} = \mathrm{colim}_{T \rightarrow U(\mathbb{R})} F(U_{\mathbb{C}} \times_{\mathbb{R}} X)^{\mathbb{Z}/2}.$$

We conclude that $\mathcal{K}^{(n)}(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{R}} X) \simeq \mathcal{K}^{(n)}(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{C}} X)^{\mathbb{Z}/2}$ and $\mathcal{M}^{(n)}(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{R}} X) \simeq \mathcal{M}^{(n)}(\Delta_{\mathrm{top}}^{\bullet} \times_{\mathbb{C}} X)^{\mathbb{Z}/2}$. \square

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