EQUIVARIANT ALGEBRAIC COBORDISM

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Abstract. We construct an equivariant algebraic cobordism theory for schemes with an action by a linear algebraic group over a field of characteristic zero.

1. Introduction

In [13] Levine-Morel construct a theory Ω_*, called algebraic cobordism, for schemes over a field of characteristic zero. This theory is constructed so that its restriction to smooth schemes is the universal oriented cohomology theory. In this paper we extend their construction to the equivariant setting and define an equivariant algebraic cobordism Ω_*^G for schemes equipped with an action of a linear algebraic group G. Our construction is based on an idea used by Totaro [19] to define Chow groups of a classifying space and Edidin-Graham [3] to define equivariant Chow groups. Their construction is motivated by the construction of equivariant cohomology theories in algebraic topology using the Borel construction. The Borel construction of a manifold M with action by a Lie group G is the space $(M \times EG)/G$, where EG is a contractible G-space with free G-action. In the algebro-geometric setting, the classifying space of a linear algebraic group cannot exist as a scheme. However reasonable approximations of the classifying space do exist and the idea of the construction of the equivariant Chow groups is to approximate the Borel construction. Chow groups have the property that the groups $\mathrm{CH}_n(X)$ vanish when $n > \dim X$. Because of this, in a fixed degree, there is a finite dimensional approximation to the Borel construction which is a sufficiently good approximation. On the other hand algebraic cobordism $\Omega_n(X)$ can be nonzero for arbitrarily large values of n. This leads us to define equivariant algebraic cobordism $\Omega_*^G(X)$ in Section 3 as the limit

$$\Omega_*^G(X) = \varprojlim_i \Omega_* \left(X \times^G U_i\right)$$

over successively better approximations to the Borel construction. This involves making a choice of a sequence of approximations to the Borel construction and we show in Theorem 16 that our definition does not depend on the choice of sequence of approximations. The rest of the section is devoted to some basic computations of the resulting equivariant algebraic cobordism groups.

In Section 4 we establish the basic properties of equivariant algebraic cobordism. It has all of the equivariant analogues of the basic properties expected of an oriented Borel-Moore homology theory. These include pull-back maps for equivariant l.c.i.-morphisms, push-forwards for equivariant projective morphisms and appropriate compatibilities with pull-backs, homotopy invariance, projective bundle formula, and a localization exact sequence. Additionally, equivariant algebraic cobordism has properties that one would expect from an equivariant cohomology theory for G-schemes, for example it is equipped with natural restriction and induction maps.

Suppose that $X \to X/G$ is a geometric quotient of X by a proper action of a reductive group G. There is in this case a natural map $\Omega_*(X/G) \to \Omega^G_{*+\dim(G)}(X)$ relating the algebraic cobordism of X/G to the equivariant algebraic cobordism of X. In Theorem 28 this map is shown to be an isomorphism with rational coefficients. As a consequence we obtain a naturally defined ring structure on the rational algebraic cobordism of the quotient X/G of a smooth scheme.

If G is split reductive, with maximal torus T, Weyl group W, and X is a smooth G-scheme then we show in Theorem 33 that $\Omega^*_G(X)_{\mathbb{Q}} \cong \Omega^*_T(X)^W_{\mathbb{Q}}$. If G is special then this is an injection integrally, and under certain assumptions (see Remark 35) can be shown to be an isomorphism. In order to prove this result we require a cobordism version of the Leray-Hirsch type theorem for smooth projective fibrations $p : X \to Y$ with cellular fiber F. This is done in Proposition 7 where we show that $\Omega^*(X)_{\mathbb{Q}} = \Omega^*(Y)_{\mathbb{Q}} \otimes_{\Omega^*(k)_{\mathbb{Q}}} \Omega^*(F)_{\mathbb{Q}}$ and it is an integral isomorphism if the fibration is Zariski-locally trivial.

We finish in Section 5 with a brief discussion of oriented equivariant Borel-Moore homology theories. We focus on the examples of such theories that arise from Chow groups and algebraic K-theory, and its relations with equivariant Chow groups [3] and equivariant K-theory [18].

Independently, another definition of equivariant algebraic cobordism has been given in [12]. The version there is based on Desphande's definition of algebraic cobordism for classifying spaces [1], which involves taking an inverse limit over approximations to the Borel construction as well as quotients by a niveau filtration. That definition gives an equivariant algebraic cobordism theory isomorphic to the one we have defined here, see Remark 14.

Notation and Conventions. Throughout k will denote a field of characteristic zero. Let $\mathbf{Sch}_k$ denote the category whose objects are separated, quasi-projective schemes of finite type over k, and let $\mathbf{Sm}_k$ be the full-subcategory of smooth quasi-projective k-schemes. If $A = \lim_i A_i$ is an inverse limit of groups, we write $A_{\mathbb{Q}} = \lim_i (A_i \otimes \mathbb{Q})$.

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2. Preliminaries

2.1. Algebraic Cobordism.

In this section we recall definitions and terminology from [13].

2.1.1. Let $\mathbf{Ab}_*$ be the category of graded Abelian groups. Let $\mathbf{Sch}'_k$ be the subcategory of $\mathbf{Sch}_k$ which has the same objects and the morphisms are the projective morphisms. An *oriented Borel-Moore homology theory (OBM)* on $\mathbf{Sch}_k$ consists of an additive functor $A_* : \mathbf{Sch}'_k \to \mathbf{Ab}_*$, pull-backs maps $f^* : A_*(X) \to A_{*+d}(Y)$ for each l.c.i. morphism $f : Y \to X$ of relative dimension d, and an associative and commutative external product $A_*(X) \otimes A_*(Y) \to A_*(X \times Y)$, $u \otimes v \mapsto u \times v$, with unit $1 \in A_0(k)$.

These data satisfy certain axioms: functoriality of pull-back maps, compatibility of push-forwards maps and pull-backs maps in transverse Cartesian squares, compatibility of variances with product of schemes, and

(EH) *Extended homotopy.* Let E be a vector bundle of rank r over X in $\mathbf{Sch}_k$. Let $p : V \to X$ be a E-torsor. Then the induced morphism $p^* : A_*(X) \to A_{*+r}(V)$ is an isomorphism.

(PB) *Projective Bundle Formula.* Let E be a vector bundle of rank $r+1$ over X in $\mathbf{Sch}_k$. Let $q : \mathbb{P}(E) \to X$ be the associated projective space. Consider the canonical quotient line bundle $\mathcal{O}(1) \to \mathbb{P}(E)$ with zero section $s : \mathbb{P}(E) \to \mathcal{O}(1)$. Consider the operator $\xi : A_*(\mathbb{P}(E)) \to A_{*-1}(\mathbb{P}(E))$, $\eta \mapsto s^*(s_*(\eta))$. For $i = 0, \ldots, r$, let

$$\xi^{(i)} : A_{*+i-r}(X) \xrightarrow{q^*} A_{*+i}(\mathbb{P}(E)) \xrightarrow{\xi^i} A_*(\mathbb{P}(E)).$$

Then $\sum_{i=0}^r \xi^{(i)} : \oplus_{i=0}^r A_{*+i-r}(X) \to A_*(\mathbb{P}(E))$ is an isomorphism.

(CD) *Cellular Decomposition.* Let $r, N > 0$. Consider $W = \mathbb{P}^N \times \cdots \times \mathbb{P}^N$ with r factors, and let $p_i : W \to \mathbb{P}^N$ be the i-th projection. Let $X_0, \ldots, X_N$ be the standard homogeneous coordinates on $\mathbb{P}^N$. Let $n_1, \ldots, n_r$ be non-negative integers, and let $i : E \to W$ be the subscheme defined by $\prod_{i=1}^r p_i^*(X_N)^{n_i} = 0$. Then $i_* : A_*(E) \to A_*(W)$ is injective.

A *morphism* of OBMs is a natural transformation of functors. Given an OBM A_*, the *Chern class endomorphism* $\tilde{c}_1(L) : A_*(X) \to A_{*-1}(X)$ associated to a line bundle $L \to X$ is given by $\tilde{c}_1(L) = s^* s_*$ where $s : X \to L$ is the zero-section. There is an infinite series $F_A(u,v)$ with coefficients in $A_*(k)$, such that for any line bundles L, M over X, $\tilde{c}_1(L \otimes M) = F_A(\tilde{c}_1(L), \tilde{c}_1(M))$.

2.1.2. Let $\mathbf{CRng}^*$ be the category whose objects are commutative graded rings with unit and maps the ring morphisms. An *oriented cohomology theory (OCT)* on $\mathbf{Sm}_k$ consists of an additive functor $A^* : (\mathbf{Sm}_k)^{\mathrm{op}} \to \mathbf{CRng}^*$, endowed with morphism of graded $A^*(X)$-modules $f_* : A^*(Y) \longrightarrow A^{*+d}(X)$ for each projective morphism $f : Y \to X$ of relative codimension d satisfying certain axioms: functoriality of push-forwards maps, compatibility of variances in transverse Cartesian squares, as well as the analogues of the extended homotopy axiom (EH) and the projective bundle formula (PB).

A *morphism* of OCTs is a natural transformation of contravariant functors that also commutes with the push-forwards maps. Setting $\mathrm{c}_1(L) := \tilde{c}_1(L)(1)$ we obtain a *first Chern class* element. For every OCT A^* the pair $(A^*(k), F_A)$ defines a formal group law ([13, Lemma [1.1.3]). Thus, there is a unique ring morphism $\theta_A : \mathbb{L} \to A^*(k)$ classifying $(A^*(k), F_A)$.

2.1.3. Let A_* be an OBM and let X be in $\mathbf{Sm}_k$ be of pure dimension d. Set $A^n(X) := A_{d-n}(X)$ and extend this definition to any smooth scheme by additivity over the connected components. Let $\delta_X : X \to X \times X$ be the diagonal morphism. The product $a \cup_X b := \delta^*(a \times b)$ makes $A^*(X)$ a commutative ring with unit $1_X := p^*(1)$, where $p : X \to \mathrm{Spec}(k)$ is the structural morphism.

Proposition 1 (Levine-Morel)**.** *The correspondence $A_* \mapsto A^*$ gives an equivalence of the category of OBMs on $\mathbf{Sm}_k$ with the category of OCTs on $\mathbf{Sm}_k$.*

2.1.4. We summerize the main properties of algebraic cobordism in the following

Theorem 2 (Levine-Morel). *Let k be a field that admits resolution of singularities.*

(1) *There is a universal oriented Borel-Moore homology theory on $\mathbf{Sch}_k$, $X \mapsto \Omega_*(X)$, called* algebraic cobordism. *The restriction of Ω_* to $\mathbf{Sm}_k$ yields the universal oriented cohomology theory Ω^* on $\mathbf{Sm}_k$.*
(2) *Let F_Ω be the formal group law associated to Ω^*. Then the morphism $\theta_\Omega : \mathbb{L} \longrightarrow \Omega^*(k)$ classifying F_Ω is an isomorphism.*
(3) *(Localization Sequences) If $\imath : Z \to X$ is a closed immersion and $j : U := X - Z \to X$ is the open complement, then the following sequence is exact:*

$$\Omega_*(Z) \xrightarrow{\imath_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \longrightarrow 0. \tag{LS}$$

As an Abelian group, $\Omega_*(X)$ is generated by isomorphism classes $\mathrm{M}_n(X)$ of projective morphisms $Y \to X$, with Y in $\mathbf{Sm}_k$ and irreducible of dimension n.

2.1.5. Given any formal group law (R, F_R), the canonical morphism $\Omega^*(k) \to R$ induces an OCT $\Omega^* \otimes_{\mathbb{L}} R$, defined by $X \mapsto \Omega^*(X) \otimes_{\mathbb{L}} R$, which is universal for OCTs with formal group law (R, F_R).

Example 3. (1) Let $\Omega^*_+ := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}$ be the OCT classifying the additive formal group law $(\mathbb{Z}, u + v)$. Then we have a canonical morphism $\Omega^*_+ \to \mathrm{CH}^*$.
(2) Let $\Omega^*_\times := \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$ be the OCT classifying the multiplicative periodic formal group law $(\mathbb{Z}[\beta, \beta^{-1}], u+v-uv\beta)$. Then we have a canonical morphism $\Omega^*_\times \to \mathrm{K}^0[\beta, \beta^{-1}]$.

Theorem 4 (Levine-Morel). *Let k be a field of characteristic zero. Then, the canonical morphisms $\Omega^*_+ \to \mathrm{CH}^*$ and $\Omega^*_\times \to \mathrm{K}^0[\beta, \beta^{-1}]$ are isomorphisms of OCTs.*

2.1.6. We recall the technique of twisting an OBM [13, §7.4.1]. Fix an OBM A_*. Let $\tau = (\tau_i)_{i \geq 0}$ be a sequence where $\tau_i \in A^{-i}(k)$ for all i and τ_0 is a unit. The *τ-inverse Todd class operator* of a line bundle $L \to X$ is defined as $\widetilde{\mathrm{Td}}_\tau^{-1}(L) := \sum_{i \geq 0} \tau_i \, \tilde{\mathrm{c}}_1^i(L)$. This definition is extended to any vector bundle via the splitting.

The *twisted OBM* $A_*^{(\tau)}$ is the OBM given by the data: set $A_*^{(\tau)}(X) := A_*(X)$ for any X in $\mathbf{Sch}_k$, set $f_*^{(\tau)} := f_*$ for any projective morphism f, and for any l.c.i. morphism $f : Y \to X$, given $x \in A_*^{(\tau)}(X)$, set $f^*_{(\tau)}(x) := \widetilde{\mathrm{Td}}^{-1}(N_f) \cdot f^*(x)$, where N_f denotes the virtual normal bundle of f.

Example 5. Let $\mathbb{Z}[\mathbf{t}] := \mathbb{Z}[t_i \mid i \geq 1]$, where t_i is a variable of degree $-i$. Set $t_0 = 1$ and define $\mathbf{t} = (t_0, t_1, \ldots)$. By means of the "universal exponential" it can be shown that that there is an isomorphism $\log : \mathbb{Q}[\mathbf{t}] \to \mathbb{L}_{\mathbb{Q}}$ of $\mathbb{Q}$-algebras. Levine and Morel used this to show [13, Theorem 4.1.28, Theorem 4.5.1] that the canonical map $\Theta_{\exp} : (\Omega_*)_{\mathbb{Q}} \to \mathrm{CH}_* \otimes_{\mathbb{Z}} \mathbb{Q}[\mathbf{t}]^{(\mathbf{t})}$ is an isomorphism of Borel-Moore homology theories.

2.1.7. For later applications we need a cobordism version of the Leray-Hirsch theorem for smooth projective fibrations with cellular fiber, proved for Chow groups in [4, Lemma 2.8].

A morphism $X \to Y$ is said to be a *locally isotrivial fibration* with fiber F provided that every $y \in Y$ has an open neighborhood U admitting a finite étale morphism $f : V \to U$ such that $f^{-1}X = V \times F$, and the map $f^{-1}X \to V$ agrees with the projection $V \times F \to V$.

A smooth variety F is said to be *cellular* provided it has a filtration

$$\varnothing = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_{N-1} \subseteq F_N = F,$$

where $\alpha_i : F_i \hookrightarrow F_{i+1}$ is a closed embedding and $F_i - F_{i-1} \cong \mathbb{A}^{n_i}$ for all i. Since a cellular variety is built from affine spaces its cobordism is easy to describe as an $\Omega^*(k)$-module.

Proposition 6. *([9, Theorem 2.5]) Let F be a smooth, projective cellular variety with filtration $\{F_i\}$ as above. Then $\Omega_*(F)$ is a free module over $\Omega_*(k)$. A basis is given by $[\widetilde{F}_i \to F_i \subseteq F_m]$ where $\widetilde{F}_i \to F_i$ is a resolution of singularities for each $i \leq m$. If L/k be a field extension, then the the map induced by base change $\Omega_*(F) \to \Omega_*(F_L)$ is an isomorphism.*

Proposition 7. *Suppose that F is a smooth, projective cellular variety and $p : X \to Y$ is a projective, locally isotrivial fibration between smooth varieties with fiber F. Let $\iota_y : F_y \to X$ be the inclusion of a fiber over a point $y \in Y$.*

(1) *The morphism $\iota_y^* : \Omega^*(X)_{\mathbb{Q}} \to \Omega^*(F_y)_{\mathbb{Q}}$ is surjective for any point $y \in Y$. Moreover, under the isomorphism $\Omega^*(F_y) \cong \Omega^*(F)$ induced by base change along $k \to k(y)$, we have that $\iota_y^* = \iota_{y'}^*$ for any two points y, y' in the same connected component.*
(2) *Let $e_i \in \Omega^*(X)_{\mathbb{Q}}$ be elements such that $\eta_i = \iota_y^* e_i$ form a basis for $\Omega^*(F_y)_{\mathbb{Q}}$ as an $\Omega^*(k)_{\mathbb{Q}}$-module. Define the morphism*

$$\Psi : \Omega^*(Y)_{\mathbb{Q}} \otimes_{\Omega^*(k)_{\mathbb{Q}}} \Omega^*(F)_{\mathbb{Q}} \to \Omega^*(X)_{\mathbb{Q}},$$

by $\Psi(\sum \alpha_i \otimes \eta_i) = \sum p^*(\alpha_i) \cup e_i$. Then Ψ is an isomorphism of $\Omega^*(k)_{\mathbb{Q}}$-modules.
(3) *If $p : X \to Y$ is Zariski-locally trivial then statements (1) and (2) hold integrally.*

Proof. (1) We may assume Y is connected. Let $y \in Y$ be any point. We show that ι_y^* is surjective. Let $j : U \to Y$ be an open neighborhood of y which admits a finite étale morphism $f : V \to U$ over which the fibration trivializes. Write $f_V : V \times F \to U \times_Y X$ for the induced morphism. Let $\beta_i \in \Omega^*(F)_{\mathbb{Q}}$ be a basis (as an $\Omega^*(k)_{\mathbb{Q}}$-module) and consider $(f_V)_*(1_V \times \beta_i) \in \Omega^*(U \times_Y X)_{\mathbb{Q}}$. Now $j_U : U \times_Y X \to X$ is open and take $e_i \in \Omega^*(X)_{\mathbb{Q}}$ such that $j_U^*(e_i) = (f_V)_*(1_V \times \beta_i)$. Then we have that $\iota_y^*(e_i) = \deg(f)\beta_i \in \Omega^*(F_y)_{\mathbb{Q}}$ and thus forms a basis.

Now we show that $\iota_y^* = \iota_\eta^*$ where η is the generic point of Y, and so of U as well. Let η' be the generic point of V and $y' \in V$ a point such that $f(y') = y$. The statement follows from consideration of the following commutative diagram, where the bottom vertical arrow is induced by $V \times_U X \cong V \times F \to F$

$$\begin{tikzcd}
\Omega^*(F_y) \arrow[d, "\cong"'] & \Omega^*(X) \arrow[l] \arrow[r] \arrow[d] & \Omega^*(F_\eta) \arrow[d, "\cong"] \\
\Omega^*(F_{y'}) & \Omega^*(V \times_U X) \arrow[l] \arrow[r] & \Omega^*(F_{\eta'}). \\
& \Omega^*(F) \arrow[ul, "\cong"] \arrow[u] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

(2) We begin by showing that Ψ is surjective. Let $f : W \to Y$ be a morphism of k-schemes and write $X_W = W \times_Y X$. Let $p_W : X_W \to W$ and $f_W : X_W \to X$ be the morphism obtained by base-change. The morphisms p_W and f_W induce $\langle p_W, f_W\rangle : X_W \to W \times X$. This is an l.c.i.-morphism since both X_W and $W \times X$ are smooth over W. Define the morphism of $\Omega^*(k)$-modules

$$\Psi_f : \Omega_*(W) \otimes_{\Omega^*(k)} \Omega^*(F) \to \Omega_*(X_W), \quad \sum w_i \otimes \eta_i \mapsto \sum \langle p_W, f_W\rangle^*(w_i \times e_i).$$

If f is smooth then $\Psi_f\left(\sum w_i \otimes \eta_i\right) = \sum f_W^*(w_i) \cup e_i$, in particular Ψ_{id_Y} is the map of the proposition. We proceed by induction on the dimension of W to show that Ψ_f is surjective. The zero-dimensional case follows directly from the definition of Ψ_f. Suppose that $\Psi_{f'}$ is surjective where $f' : W' \to Y$ with $\dim W' < \dim W$. Let $j : U \to W$ be an open over which the fibration $X_W \to W$ becomes isotrivial and let $i : Z \to W$ be the closed complement. Consider the comparison of exact sequences (where the tensor product is over $\Omega^*(k)_{\mathbb{Q}}$)

$$\begin{tikzcd}
\Omega_*(Z)_{\mathbb{Q}} \otimes \Omega^*(F)_{\mathbb{Q}} \arrow[r, "i_*\otimes \mathrm{id}"] \arrow[d, "\Psi_{f_{|Z}}"] & \Omega_*(W)_{\mathbb{Q}} \otimes \Omega^*(F)_{\mathbb{Q}} \arrow[r, "j^*\otimes \mathrm{id}"] \arrow[d, "\Psi_f"] & \Omega_*(U)_{\mathbb{Q}} \otimes \Omega^*(F)_{\mathbb{Q}} \arrow[r] \arrow[d, "\Psi_U"] & 0 \\
\Omega_*(X_Z)_{\mathbb{Q}} \arrow[r, "i'_*"] & \Omega_*(X_W)_{\mathbb{Q}} \arrow[r, "(j')^*"] & \Omega_*(X_U)_{\mathbb{Q}} \arrow[r] & 0.
\end{tikzcd}$$

This diagram commutes and the left-hand vertical map is surjective by induction. It suffices to conclude that the right-hand vertical map is surjective. Let $g : V \to U$ be a finite, étale morphism over which $X_U \to U$ becomes trivial. We have the commutative square

$$\begin{tikzcd}
\Omega_*(V)_{\mathbb{Q}} \otimes \Omega^*(F)_{\mathbb{Q}} \arrow[r, "g_*\otimes \mathrm{id}"] \arrow[d, "\Psi_V"] & \Omega_*(U)_{\mathbb{Q}} \otimes \Omega^*(F)_{\mathbb{Q}} \arrow[d, "\Psi_U"] \\
\Omega_*(V \times F)_{\mathbb{Q}} \arrow[r, "g'_*"] & \Omega_*(X_U)_{\mathbb{Q}}.
\end{tikzcd}$$

The morphism g'_* is surjective since $g'_*(g')^*$ is multiplication by $\deg(g')$. It only remains to see that Ψ_V is surjective as well in order to conclude that Ψ is surjective. First note that Ψ_V is the morphism $\Omega_*(V) \otimes_{\Omega^*(k)} \Omega_*(F) \to \Omega_*(V \times F)$ induced by external product $\alpha \otimes \beta \mapsto \alpha \times \beta$. Let $\{F_i\}$ be a filtration of F as above and consider the commutative diagram with exact rows

$$\begin{tikzcd}
\Omega_*(V) \otimes \Omega_*(F_k) \arrow[r] \arrow[d] & \Omega_*(V) \otimes \Omega_*(F_{k+1}) \arrow[r] \arrow[d] & \Omega_*(V) \otimes \Omega_*(F_{k+1} - F_k) \arrow[r] \arrow[d] & 0 \\
\Omega_*(V \times F_k) \arrow[r] & \Omega_*(V \times F_{k+1}) \arrow[r] & \Omega_*(V \times (F_{k+1} - F_k)) \arrow[r] & 0.
\end{tikzcd}$$

The right hand map is always a surjection and the left hand map is a surjection by induction and therefore the middle map is also a surjection.

To show injectivity we first observe that the surjectivity result in the previous paragraph implies a decomposition of the class of the diagonal $\Delta_*(1_X) \in \Omega^*(X \times_Y X)_{\mathbb{Q}}$. Write $\pi_k : X \times_Y X \to X$ for the projection to the k-th factor. The map $\pi_2 : X \times_Y X \to X$ is a locally isotrivial fibration with fiber F and the elements $\pi_1^*(e_i)$ restrict to a basis of the fiber. Therefore $\Omega^*(X\times_Y X)_{\mathbb{Q}}$ is generated by the elements $\pi_1^*(e_i)$ as an $\Omega^*(X)_{\mathbb{Q}}$-module, where $\Omega^*(X\times_Y X)_{\mathbb{Q}}$ is viewed as an $\Omega^*(X)_{\mathbb{Q}}$-module via π_2^*.

This means that there are elements $\alpha_i\in\Omega^*(X)_{\mathbb{Q}}$ so that $\Delta_*(1_X)=\sum\pi_1^*(e_i)\cup\pi_2^*(\alpha_i)$ where $\pi_i : X\times_Y X\to X$ are the projections. Define a morphism of $\Omega^*(k)_{\mathbb{Q}}$-modules $\rho:\Omega^*(X)_{\mathbb{Q}}\to\Omega^*(Y)_{\mathbb{Q}}\otimes_{\Omega^*(k)_{\mathbb{Q}}}\Omega^*(F)_{\mathbb{Q}}$ by $\rho(x)=\sum p_*(x\cup\alpha_i)\otimes\eta_i$. Then injectivity of Ψ follows from the equalities

$$\begin{aligned}\Psi(\rho(x)) &= \sum p^*p_*(x\cup\alpha_i)\cup e_i=\sum(\pi_1)_*\pi_2^*(x\cup\alpha_i)\cup e_i\\ &=(\pi_1)_*\left(\sum\pi_2^*(x)\cup\pi_2^*(\alpha_i)\cup\pi_1^*(e_i)\right)=(\pi_1)_*\left(\pi_2^*(x)\cup\Delta_*(1_X)\right)\\ &=(\pi_1)_*\Delta_*\left(\Delta^*\pi_2^*(x)\right)=(\mathrm{id}_X)_*(\mathrm{id}_X)^*(x)=x.\end{aligned}$$

(3) If $p:X\to Y$ is Zariski-locally trivial then $V=U$ in the proof of (1), so the argument given works integrally. Similarly for (2).

□

2.2. **Algebraic Groups and Algebraic Quotients.**

2.2.1. Given a linear algebraic group G over k and a G-scheme X with action σ, we have the action map $\Psi:=(\sigma,\mathrm{pr_X}):G\times X\to X\times X$. We say that the action $\sigma:G\times X\to X$ is *proper* if Ψ is proper and *free* if Ψ is a closed embedding.

Let X be a scheme with G-action σ. Say that a morphism $\pi:X\to Q$ of k-schemes is a *geometric quotient* of X by G if: $\pi\circ\sigma=\pi\circ\mathrm{pr}_X$, π is surjective, the image of Ψ is $X\times_Q X$, $U\subset Q$ is open if and only if $\pi^{-1}(U)$ is open, and the structure sheaf $\mathcal{O}_Y$ is the subsheaf of $\pi_*(\mathcal{O}_X)$ consisting of invariant functions. We write $X\to X/G$ for the geometric quotient when it exists.

If the geometric quotient X/G exists, X is called a *principal G-bundle* over X/G if $\pi:X\to X/G$ is faithfully flat and $\Psi:G\times X\to X\times_{X/G}X$ is an isomorphism. By [16, Lemme XIV 1.4] this is equivalent to the condition that π is a locally isotrivial fibration with fiber G. If G acts freely on X and the geometric quotient X/G exists then by [15, Proposition 0.9] it is a principle G-bundle.

2.2.2. We frequently use faithfully flat descent for certain properties of morphisms. We briefly summarize the main results we use, see [8, §2] for details. Let **P** be a property of morphisms of schemes that is stable under flat base change. Say that **P** satisfies *faithfully flat descent* if, given $f:X\to Y$ and a faithfully flat morphism $Y'\to Y$, such that $f':X\times_Y Y'\to X$ satisfies **P**, then f satisfies **P**. Among the properties which satisfy descent are: separated, finite type, proper, open immersion, closed immersion, finite, reduced, normal, quasi-compact, regular, flat, étale and smooth.

2.2.3. Let X be a G-scheme with action σ. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is called a *G-module* (see [18, §1.2]) if there is an isomorphism $\phi:\sigma^*\mathcal{F}\to\mathrm{pr}_X^*\mathcal{F}$ of quasi-coherent $\mathcal{O}_{G\times X}$-modules such that the co-cycle condition $\mathrm{p}_{23}^*(\phi)\circ(\mathrm{Id}_G\times\sigma)^*(\phi)=(\mu\times\mathrm{Id}_X)^*(\phi)$ holds on $G\times G\times X$, where $p_{23}:G\times G\times X\to G\times X$ is the projection onto the second and third factors, and $\mu:G\times G\to G$ is the multiplication on G. A morphism $f:M\to N$ between G-modules is called a *G-morphism* if $\phi_N\circ\sigma^*(f)=\mathrm{pr}_X^*(f)\circ\phi_M$.

Say that $\mathcal{L}$ is *G-linearizable* if it can be given a G-module structure. A choice of such a structure is a *G-linearization* of $\mathcal{L}$. The set of G-linearized line bundles over X is a group under tensor product and is written $\mathrm{Pic}^G(X)$. An equivariant morphism $f : X \to Y$ induces the morphism $f^* : \mathrm{Pic}^G(Y) \longrightarrow \mathrm{Pic}^G(X)$. If $\pi : X \to Y$ is a principal G-bundle, then $\pi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}^G(X)$ is an isomorphism (see [15, Ch. 1, § 3] for details).

2.2.4. Projective and quasi-projective morphisms are not stable under descent. To resolve the resulting difficulties in our situation we consider the following category of G-schemes. Let $G - \mathbf{Sch}_k$ be the category whose objects are schemes X with a G-action and which possess a G-linearizable ample line bundle. Morphisms are equivariant maps. Similarly $G - \mathbf{Sm}_k$ consists of smooth G-schemes which possess a G-linearizable ample line bundle.

Remark 8. If G is connected then every normal, quasi-projective G-scheme is in $G - \mathbf{Sch}_k$ by [17, Theorem 1.6].

A map $f : Y \to X$ in $G - \mathbf{Sch}_k$ is said to be an *equivariant l.c.i.-morphism* provided that we can write $f = g \circ i$ where both i and g are in $G - \mathbf{Sch}_k$, $i : Y \to W$ is a regular closed embedding, and $g : W \to X$ is smooth map.

Lemma 9. (1) *Let A be in $G - \mathbf{Sch}_k$ such that a principal G-bundle $A \to A/G$ exists with A/G in $\mathbf{Sch}_k$. Then a principal G-bundle $X \times A \to (X \times A)/G$ exists with $(X \times A)/G$ in $\mathbf{Sch}_k$ for any X in $G - \mathbf{Sch}_k$.*

(2) *Suppose that principal G-bundles $X \to X/G$ and $Y \to Y/G$ exist with X/G and Y/G in $\mathbf{Sch}_k$. Let $f : Y \to X$ be an equivariant projective (resp. an equivariant l.c.i.-morphism). Then the induced morphism $\phi : Y/G \to X/G$ is projective (resp. ϕ is an l.c.i.-morphism).*

Proof. (1) Let $p_2 : X \times A \to A$ be the projection. Then $X \times A$ has a G-linearizable p_2-ample line bundle because of the assumption on X. The statement follows from [15, Proposition 7.1].

(2) If f is projective then it is proper and by descent ϕ is proper as well, but a proper, quasi-projective morphism between k-schemes is projective. If f is an equivariant l.c.i.-morphism we may factor it in $G - \mathbf{Sch}_k$ as a regular immersion $i : Y \to W$ followed by a smooth morphism $g : W \to X$. If $\mathcal{L}$ is a linearizable ample bundle on W then it is g-ample [7, Proposition 4.6.13(v)] and so by [15, Proposition 7.1] the principle G-bundle quotient $W \to W/G$ exists and $W/G \to X/G$ is smooth. By descent the closed embedding $Y/G \to W/G$ is regular and therefore $\phi : Y/G \to X/G$ is an l.c.i.-morphism.

□

3. Equivariant Algebraic Cobordism

Fix a linear algebraic group G. From now on, all the G-schemes to be considered are in the category $G - \mathbf{Sch}_k$ introduced in the previous section. In this section we define the equivariant algebraic cobordism of a G-scheme.

3.1. Construction.

Definition 10. Say that $\{(V_i, U_i)\}$ is a *good system of representations* for a linear algebraic group G if each V_i a G-representation, $U_i \subseteq V_i$ is a G-invariant open satisfying the following conditions:

(1) G acts freely on U_i and U_i/G exists in $\mathbf{Sch}_k$.
(2) For each i there is a G-representation W_i so that $V_{i+1} = V_i \oplus W_i$.
(3) $U_i \subseteq U_{i+1}$ and the inclusion factors as $U_i = U_i \oplus \{0\} \subseteq U_i \oplus W_i \subseteq U_{i+1}$.
(4) $\lim_{i\to\infty} \dim V_i = \infty$.
(5) $\operatorname{codim}_{V_i}(V_i - U_i) < \operatorname{codim}_{V_j}(V_j - U_j)$, for $i < j$.

Remark 11. The existence of one such system follows from [19, Remark 1.4].

Let X be in $G-\mathbf{Sch}_k$ and consider a good system of representations $\{(V_i, U_i)\}$. For convenience we write $X \times^G U_i = (X \times U_i)/G$. By Lemma 9 this quotient exists, $X \times^G U_i$ is quasi-projective and the morphisms $\phi_{ij} : X \times^G U_i \to X \times^G U_j$ are l.c.i.-morphisms. If X is smooth then by descent we have that $X \times^G U_i$ is in $\mathbf{Sm}_k$.

Definition 12. Let G be a linear algebraic group. Let $\{(V_i, U_i)\}$ be a good system of representations. For X in $G - \mathbf{Sch}_k$ we define the *equivariant algebraic cobordism group* as

$$\Omega_*^G(X) = \varprojlim_i \Omega_* \left(X \times^G U_i\right).$$

The *n-th equivariant algebraic cobordism group* of X is defined as

$$\Omega_n^G(X) = \varprojlim_i \Omega_{n-\dim G+\dim U_i} \left(X \times^G U_i\right).$$

If X is an equidimensional and smooth G-scheme, define $\Omega^*_G(X) = \varprojlim_i \Omega^* \left(X \times^G U_i\right)$ and $\Omega^n_G(X) = \varprojlim_i \Omega^n \left(X \times^G U_i\right)$. Equivariant algebraic cobordism with rational coefficients is defined by the completed tensor product and we write $\Omega^*_G(X)_{\mathbb{Q}} = \Omega^*_G(X)\widehat{\otimes}\mathbb{Q}$.

Remark 13. In the rest of the paper we consider the ungraded group $\Omega_*^G(X)$ for simplicity. The reader interested in the graded situation may replace this by $\oplus\Omega_n^G(X)$, all results hold in this case with a few appropriate changes.

Remark 14. In [12] another version of equivariant algebraic cobordism was defined. The definition there yields a theory isomorphic to the one we have defined, which we briefly explain. We restrict our discussion to smooth schemes so that we may index by codimension but the discussion applies more generally provided one indexes by dimension. Define the coniveau filtration

$$F^r\Omega^n(X) = \{x \in \Omega^n(X) \,|\, j^*(x) = 0 \text{ for some } j : X - S \to X,\ S\,\text{closed}, \operatorname{codim}(S) \geq r\}.$$

Then with our notational conventions for the meaning of the pairs $\{(V_j, U_j)\}$, the definition of equivariant algebraic cobordism given in [12] is $\lim_i \Omega^n(X \times^G U_i)/F^{c(i)}\Omega^n(X \times^G U_i)$ where $c(i) = \operatorname{codim}_{V_i}(V_i - U_i)$. For a morphism $f : X \to Y$ we have $f^*(F^r\Omega^n(Y)) \subseteq F^r\Omega^n(X)$. Also $F^r\Omega^n(X) = 0$ for $r > \dim X$. This means that whenever j is large enough so that $c(j) > \dim(X \times^G U_i)$ we have that $\phi_{ij}^*(F^{c(j)}\Omega^n(X \times^G U_j)) = 0$.

This implies that

$$\Omega^n(X \times^G U_j)/F^{c(j)}\Omega^n(X \times^G U_j) \to \Omega^n(X \times^G U_i)/F^{c(i)}\Omega^n(X \times^G U_i)$$

factors through $\Omega^n(X \times^G U_i) \to \Omega^n(X \times^G U_i)/F^{c(i)}\Omega^n(X \times^G U_i)$ and so the map of towers $\{\Omega^n(X \times^G U_i)\}_i \to \{\Omega^n(X \times^G U_i)/F^{c(i)}\Omega^n(X \times^G U_i)\}_i$ induces an isomorphism on inverse limits.

3.1.1. To see that our theory Ω_*^G is well-defined we will require the following.

Proposition 15. *Let $\pi : E \to X$ a vector bundle over a scheme X of rank r. Let $U \subseteq E$ be an open subscheme with closed complement $S = E - U$.*

(1) *If X is affine and $\mathrm{codim}_E S > \dim X$ then $\pi^*_{|U} : \Omega_k(X) \to \Omega_{k+r}(U)$ is an isomorphism for all k.*
(2) *For a non affine scheme X, there is an integer $n(X)$ depending only on X, such that $\pi^*_{|U} : \Omega_k(X) \to \Omega_{k+r}(U)$ is an isomorphism for all k whenever $\mathrm{codim}_E S > n(X)$.*

Proof. (1) We always have the commutative diagram

$$\begin{tikzcd} \Omega_{k+r}(E) \arrow[r, "j^*"] & \Omega_{k+r}(U) \arrow[r] & 0 \\ \Omega_k(X) \arrow[u, "\pi^*"] \arrow[ur, "\pi^*_{|U}"'] & & \end{tikzcd}$$

where the top row is exact and π^* is an isomorphism. In particular

$$\pi^*_{|U} : \Omega_k(X) \to \Omega_{k+r}(U)$$

is surjective for all k. To show injectivity we proceed in cases. First suppose that $E = \mathbb{A}^r \times X$ is trivial. It suffices to find a section $s : X \to U$ of $\pi_{|U}$.

For each rational point $\xi \in \mathbb{A}^r$ define $Z(\xi) = (\{\xi\} \times X) \cap S$. This is a closed subscheme $Z(\xi) \subseteq S$. If we can find a rational point $\xi \in \mathbb{A}^r$ such that $Z(\xi) = \varnothing$ then the inclusion of $\{\xi\} \times X$ in U defines a section of $\pi_{|U} : U \to X$ and we are done. Note that the condition $\mathrm{codim}_E S > \dim X$ is equivalent to the condition $r > \dim S$.

Now consider the projection $\pi_{|S} : S \subseteq \mathbb{A}^r \times X \to \mathbb{A}^r$ and the closure of the image $\overline{\pi(S)} \subseteq \mathbb{A}^r$. Since $S \to \overline{\pi(S)}$ is a dominant morphism between k-schemes we have $r > \dim S \geq \dim \overline{q(S)}$. In particular the complement $\mathbb{A}^r - \overline{\pi(S)} \subseteq \mathbb{A}^r$ is a dense open subset. We conclude that $\mathbb{A}^r - \overline{\pi(S)}$ has a rational point (k is infinite). Let α be a rational point in $\mathbb{A}^r - \overline{\pi(S)}$. Since $Z(\xi) \neq \varnothing$ whenever $\pi Z(\xi) = \xi$ and $\alpha \notin \overline{\pi(S)}$ we must have that $Z(\alpha) = \varnothing$.

More generally, since X is affine every vector bundle admits a surjection from a trivial bundle. Let $p : \mathbb{A}^N \times X \to E$ be such a surjection of vector bundles on X. Let $W = p^{-1}U$ and set $\tilde{S} = (\mathbb{A}^N \times X) - W$. Then $\dim \tilde{S} = \dim S + N - r < N$. By the proof of the first case above we have a section $X \to W$ and therefore we obtain a section $X \to W \to U$ of $\pi_{|U} : U \to X$.

(2) First assume that X is affine. If $\operatorname{codim}_E S > \dim X$, then by the first part of the proposition we have $\pi^*_{|U} : \Omega_k(X) \to \Omega_{k+\operatorname{rank}E}(U)$. Thus in this case we may take $n(X) = \dim X$.

For the general case we employ Jouanolou's trick to find an affine torsor $p : \tilde{X} \to X$ with $\tilde{X}$ affine. In this case we claim that we may take $n(X) = n(\tilde{X})$. Consider the vector bundle $\tilde{E} := p^*E \to \tilde{X}$ and the open subscheme $\tilde{U} = p^*U \subseteq p^{-1}E$. Set $\tilde{S} = \tilde{E} - \tilde{U}$. From the first part of the proposition we see that since $\tilde{X}$ is affine we have that $\Omega_*(\tilde{X}) \to \Omega_*(\tilde{U})$ is an isomorphism for $\operatorname{codim}_E S = \operatorname{codim}_{\tilde{E}} \tilde{S} > \dim \tilde{X}$. Since $\tilde{X} \to X$ and $\tilde{U} \to U$ are torsors for some vector bundle we have the following chain of isomorphisms

$$\Omega_k(X) \xrightarrow{p^*} \Omega_k(\tilde{X}) \xrightarrow{\tilde{\pi}^*_{|U}} \Omega_{k+r}(\tilde{U}) \xleftarrow{p^*_{|U}} \Omega_{k+r}(U).$$

□

Theorem 16. *For any $X \in G - \mathbf{Sch}_k$, $\Omega^G_*(X)$ is well defined.*

Proof. To see that our definition does not depend on the choice of the sequence $\{(V_i, U_i)\}$ we proceed as in [19] by using Bologomolov's double fibration argument. Let $\{(V'_i, U'_i)\}$ be some other good system of representations. Consider a fixed U_i. Since G acts freely on U_i it acts freely on $U_i \oplus V'_j$ too. Thus $X \times^G (U_i \oplus V'_j) \to X \times^G U_i$ is a vector bundle. The second part of Proposition 15 says that there is an integer $N_i = N(X \times^G U_i)$ such that $\Omega_*(X \times^G U_i) \cong \Omega_*(X \times^G (U_i \oplus U'_j))$ for $j > N_i$. Thus $\varprojlim_i \Omega_*(X \times^G U_i) \cong \varprojlim_i \varprojlim_j \Omega_*(X \times^G (U_i \oplus U'_j))$. A similar argument for $X \times^G U'_i$ shows that $\varprojlim_i \Omega_*(X \times^G U'_i) \cong \varprojlim_i \varprojlim_j \Omega_*(X \times^G (U_i \oplus U'_j))$. □

Example 17. If $G \cong \langle e \rangle$ is the trivial linear algebraic group, then the projections $X \times U_i \to U_i$ induce an isomorphism $\Omega_*(X) \xrightarrow{\cong} \Omega^{\langle e \rangle}_*(X)$. Indeed, we have that $X \times^G U_i = X \times U_i$ for any U_i in the system. The statement follows from Proposition 15.

3.1.2. We verify that the Mittag-Leffler condition holds on the system defining Ω^*_G.

Lemma 18. *Let $\{(V_i, U_i)\}$ be a good system of representations. For every $i < j$ in the system, let $\phi_{ij} : X \times^G U_i \to X \times^G U_j$ be the induced morphism of schemes. Then*

$$\phi^*_{ij} : \Omega_* \left(X \times^G U_j\right) \longrightarrow \Omega_* \left(X \times^G U_i\right)$$

is a surjection.

Proof. The morphism $U_i \to U_j$ factors as $U_i \to U_i \oplus W \subseteq U_j$, where W is a representation (depending on i and j). Now, $X \times^G U_i \to X \times^G (U_i \oplus W)$ is the inclusion of the zero section of a vector bundle and $X \times^G (U_i \oplus W) \subseteq X \times^G U_j$ is an open inclusion. Both maps induce surjections on algebraic cobordism. □

3.1.3. Let $\{(V_i, U_i)\}$ be a good system of representations. Let X be in $G-\mathbf{Sch}_k$. For $U_i \subseteq U_j$, write $\phi_{ij} : X \times^G U_i \to X \times^G U_j$ for the induced morphism. By definition, a G-equivariant cobordism class on X is a sequence of cobordism classes $\{\alpha_i \in \Omega_*(X \times^G U_i) \mid \phi_{ij}^*([\alpha_j]) = [\alpha_i]\}$.

An equivariant projective morphism $f : Y \to X$ with Y in $G-\mathbf{Sm}_k$ defines an equivariant cobordism class as follows. The assumption on Y implies that $f_i : Y \times^G U_i \to X \times^G U_i$ is projective and $Y \times^G U_i$ is smooth. Therefore for each i we have the induced cobordism class $[f_i : Y \times^G U_i \to X \times^G U_i]$ in $\Omega_*(X \times^G U_i)$. By descent we have the transverse Cartesian square

$$\begin{tikzcd} Y \times^G U_i \arrow[r] \arrow[d] & Y \times^G U_j \arrow[d] \\ X \times^G U_i \arrow[r, "\phi_{ij}"] & X \times^G U_j \end{tikzcd}$$

which shows that $\phi_{ij}^*[f_i : Y \times^G U_i \to X \times^G U_i] = [f_j : Y \times^G U_j \to X \times^G U_j]$. Therefore $f : Y \to X$ induces the class $([Y \times^G U_i \to X \times^G U_i])$ in $\Omega_*^G(X)$. Note however that $Y \to X$ is not a unique representation for this class.

3.2. **Computations.**

3.2.1. *Coefficient Ring in the Case of a Torus Action.* Let $T = (\mathbb{G}_m)^r$. Consider the good system of representations $\{(V_i, U_i)\}$, where $V_i = \left(\mathbb{A}^i\right)^r$ and $U_i = \left(\mathbb{A}^i - \{0\}\right)^r$. The action of T on V_i is given by letting the k-th factor of $\mathbb{G}_m$ act on the k-th factor of $\mathbb{A}^i$ via the formula

$$\mathbb{G}_m \times \mathbb{A}^i \to \mathbb{A}^i, \qquad (g, a_1, \dots, a_i) \mapsto (g \cdot a_1, \dots, g \cdot a_i).$$

Then T acts freely on U_i and $U_i/T = \left(\mathbb{P}^{i-1}\right)^r$. A direct computation shows

$$\Omega_T^*(k) = \varprojlim_i \Omega^*\left(\left(\mathbb{P}^{i-1}\right)^r\right) = \varprojlim_i \frac{\Omega^*(k)[t_1, \dots, t_r]}{\left(t_1^{i-1}, \dots, t_r^{i-1}\right)} = \Omega^*(k)[[t_1, \dots, t_r]],$$

where each t_i is a variable of degree 1.

3.2.2. $\mathbb{P}^n$ *with a Weighted* $\mathbb{G}_m$*-Action.* Let $\mathbb{G}_m$ act on $\mathbb{P}^n$ with the action

$$\mathbb{G}_m \times \mathbb{P}^n \to \mathbb{P}^n, \qquad (g, (a_0 : \dots : a_n)) \mapsto (g^{m_0} \cdot a_0 : \dots : g^{m_n} \cdot a_n).$$

Let $\{(V_i, U_i)\}$ be the good system of representations considered in §3.2.1. For each U_i in the system we have the $\mathbb{P}^n$-bundle $\mathbb{P}^n \times^{\mathbb{G}_m} U_i \to \mathbb{P}^{i-1}$ is a $\mathbb{P}^n$-bundle. As a $\mathbb{P}^{i-1}$-scheme, we have that $\mathbb{P}^n \times^{\mathbb{G}_m} U_i \cong \mathbb{P}\left(\mathcal{O}(m_0) \oplus \dots \oplus \mathcal{O}(m_n)\right)$. By [13, Lemma 4.1.4] we get

$$\Omega^*\left(\mathbb{P}^n \times^{\mathbb{G}_m} U_i\right) = \frac{\Omega^*\left(\mathbb{P}^{i-1}\right)[\xi]}{(\xi - \mathrm{c}_1(\mathcal{O}(m_0))) \cdots (\xi - \mathrm{c}_1(\mathcal{O}(m_n)))},$$

where ξ is a variable of order one. If we let $t = \mathrm{c}_1(\mathcal{O}(1)) \in \Omega^*(\mathbb{P}^{i-1})$ then $\mathrm{c}_1(\mathcal{O}(a)) = [a]_\Omega(t)$ where $[a]_\Omega(t)$ is defined by $F_\Omega(t, [a-1]_\Omega t)$. Taking the limit we obtain

$$\Omega_{\mathbb{G}_m}^*(\mathbb{P}^n) = \frac{\Omega^*(k)[[t]][\xi]}{(\xi - [m_0]_\Omega(t)) \cdots (\xi - [m_n]_\Omega(t))}.$$

3.2.3. *A Torus Acting Trivially.* Let $T = \mathbb{G}_m^{\times r}$ and let X be a scheme considered in $T - \mathbf{Sm}_k$ with the trivial action. Then $\Omega^*_T(X) \cong \Omega^*(X)\widehat{\otimes}_{\Omega^*(k)}\Omega^*_T(k)$ of $\Omega^*(k)$-modules. This follows from

$$\Omega^*_T(X) = \varprojlim_i \Omega^*\left(X \times \left(\mathbb{P}^{i-1}\right)^r\right) \cong \varprojlim_i \frac{\Omega^*(X)[t_1, \ldots, t_r]}{\left(t_1^i, \ldots, t_r^i\right)} \cong \Omega^*(X)[[t_1, \ldots, t_r]],$$

where the first isomorphism is the statement of [13, Lemma 4.1.4].

3.2.4. *Coefficient Ring in the Case of a GL_n-Action.* Let $M_{n\times(n+i)}$ be the space of $n \times (n+i)$ matrices. Consider the good system of representations $\{(V_i, U_i)\}$, where $V_i = M_{n\times(n+i)}$ with GL_n acting by multiplication on the left and U_i is the subset of matrices of maximal rank. We have that $U_i/\mathrm{GL}_n \cong \mathrm{Gr}(n, n+i)$, where $\mathrm{Gr}(n, n+i)$ is the Grassmannian of n-planes in k^{n+i}. Let $\mathbb{F}_{n+i}$ denote the variety of complete flags in k^{n+i}. Let $\phi : \mathbb{F}_{n+i} \to \mathrm{Gr}(n, n+i)$ be the map which sends the flag $Fl_{n+i} = \{0 \subseteq F^1 \subseteq \cdots \subseteq F^{n+i} = k^{n+i}\}$ to the n-plane $F^n \subseteq k^{n+i}$. The induced map ϕ^* is injective on cobordism with rational coefficients by Proposition 7 and because the Grassmannian is cellular we conclude that $\phi^* : \Omega^*(\mathrm{Gr}(n, n+i)) \to \Omega^*(\mathbb{F}_{n+i})$ is injective integrally. Let $\mathcal{W}_k$ be the tautological k-plane bundle on $\mathbb{F}_{n+i}$ (i.e. the fiber of $\mathcal{W}_k$ on the flag Fl_{n+i} is F^k) and let $\mathcal{L}_k = \mathcal{W}_k/\mathcal{W}_{k-1}$. In [9, Theorem 2.6] the cobordism of the complete flag variety is shown to be

$$\Omega^*(\mathbb{F}_{n+i}) \cong \Omega^*(k)[x_1, \ldots x_{n+i}]/IS_{n+i},$$

where $x_j = c_1(\mathcal{L}_j)$ and S_{n+i} is the graded ring of symmetric polynomials in the x_j with coefficients in $\Omega^*(k)$ and IS_{n+i} is the ideal generated by the symmetric polynomials of strictly positive polynomial degree.

The cobordism of $\mathrm{Gr}(n, n+i)$ is generated by the Chern classes $c_r(\mathcal{E}_n)$, where $\mathcal{E}_n$ is the tautological n-plane bundle on $\mathrm{Gr}(n, n+i)$ (so $\phi^*\mathcal{E}_n = \mathcal{W}_n$). The total Chern class of $\mathcal{W}_n$ is $c(\mathcal{W}_n) = \prod_{k=1}^n c(\mathcal{L}_k) = \prod_{k=1}^n (1 + x_k)$ from which we see that $\phi^* c_k(\mathcal{E}_n)$ is the k-th elementary symmetric polynomial in the $x_1, \ldots, x_n$. Thus ϕ^* gives us the identification

$$\Omega^*(\mathrm{Gr}(n, n+i)) = S_n/IS_{n+i} \subseteq \Omega^*(k)[x_1, \ldots x_{n+i}]/IS_{n+i},$$

and therefore we see that $\Omega^*_{\mathrm{GL}_n}(k) \cong \Omega^*(k)[[\eta_1, \ldots, \eta_n]]$, where η_j is of degree j.

3.2.5. *Roots of unity.* Let μ_n be the algebraic group of roots of unity. Let X be in $\mu_n - \mathbf{Sm}_k$. We show that $\Omega^*_{\mu_n}(k) \cong \Omega^*(k)[[\xi]]/[n]_\Omega \cdot \xi$.

Consider the Kummer sequence $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \to 1$, which is exact in the étale topology. We obtain an étale $\mathbb{G}_m$-torsor

$$\mathbb{A}^i - \{0\}/\mu_n \longrightarrow \mathbb{A}^i - \{0\}/\mathbb{G}_m = \mathbb{P}^{i-1}.$$

Since $\mathrm{Pic}(X) = \mathrm{H}^1_{Zar}(X; \mathcal{O}_X^*) = \mathrm{H}^1_{et}(X; \mathbb{G}_m)$, we know that $\mathbb{G}_m$-torsors correspond to line bundles on X. The line bundle associated to this $\mathbb{G}_m$-torsor is $\mathcal{O}_{\mathbb{P}^{i-1}}(-n) = L$. Let $\pi : L \to \mathbb{P}^{i-1}$ denote the structural morphism, and write L_0 for the complement of the zero section s. There is an embedding $j : \mathbb{A}^i - \{0\}/\mu_n \to L$ such that $j\left(\mathbb{A}^i - \{0\}/\mu_n\right) = L_0$. Now from the localization sequence

$$\Omega^*(\mathbb{P}^{i-1}) \xrightarrow{s_*} \Omega^*(L) \to \Omega^*(L_0) \to 0$$

we see that $\Omega^*(\mathbb{P}^{i-1}) \xrightarrow{\pi^*} \Omega^*(L) \to \Omega^*(L_0)$ is surjective.

Since $s^*: \Omega^*(L) \to \Omega^*(\mathbb{P}^{i-1})$ is an isomorphism and $s^*s_*(L) = \mathrm{c}_1(L)$ [13, Proposition 4.1.15] it follows that $\Omega^*(\mathbb{A}^i - \{0\}/\mu_n) = \Omega^*(\mathbb{P}^{i-1})/\mathrm{c}_1(\mathcal{O}(-n))$. The result follows from the equalities $\mathcal{O}(-n) = \mathcal{O}(-1)^{\otimes n}$ and $\xi = c_1\mathcal{O}(-1)$.

4. Formal Properties of Equivariant Algebraic Cobordism

In this section we establish some properties of Ω_*^G. Mainly the properties are of two types, one the equivariant analogues of the formal properties of an oriented Borel-Moore theory, and the other are expected from an equivariant cohomology theory. Proceeding as in the end of § 3.1.1, shows that all of the following properties are independent of the choice of a good system of representations.

From now on, let $\{(V_i, U_i)\}$ be a fixed good system of representations for a linear algebraic group G. All the G-schemes are in the category $G - \mathbf{Sch}_k$ introduced in § 2.2.4.

4.1. **Variances.** Since any l.c.i. morphism $f : Y \to X$ of relative dimension d in $G - \mathbf{Sch}_k$ induces for any U_i in the system a l.c.i. morphism $f_i : X \times^G U_i \to Y \times^G U_i$ in $\mathbf{Sch}_k$, we obtain a sequence of pull-backs maps $f_i^* : \Omega_*(X \times^G U_i) \to \Omega_*(Y \times^G U_i)$. By naturality of Ω_* we have a functorial *pull-back map*

$$f_G^* := \varprojlim_i f_i^* : \Omega_*^G(X) \to \Omega_{*+d}^G(Y),$$

which is a morphism of Abelian groups.

If $f : Y \to X$ is a projective morphism in $G-\mathbf{Sch}_k$, by Lemma 9 we have a sequence of projective morphisms $f_i : Y \times^G U_i \to X \times^G U_i$ in $\mathbf{Sch}_k$. We have a transverse Cartesian diagram

$$\begin{tikzcd} Y \times^G U_i \arrow[r, "f_i"] \arrow[d] & X \times^G U_i \arrow[d] \\ Y \times^G U_j \arrow[r, "f_j"'] & X \times^G U_j \end{tikzcd}$$

for any $i < j$. Therefore the f_i are compatible with the transition maps in the system and we obtain an induced functorial *push-forward map* $f_*^G : \Omega_*^G(Y) \to \Omega_*^G(X)$.

Proposition 19. (1) *Let $f : X' \to X$ and $g : Y' \to Y$ be l.c.i. morphisms in $G - \mathbf{Sch}_k$. If $u \in \Omega_*^G(X)$ and $v \in \Omega_*^G(Y)$ then $(f \times g)_G^*(u \times v) = f_G^*(u) \times g_G^*(v)$.*

(2) *Let $f : X' \to X$ and $g : Y' \to Y$ be projective morphisms in $G - \mathbf{Sch}_k$. If $u' \in \Omega_*^G(X')$ and $v' \in \Omega_*^G(Y')$ then $(f \times g)_*^G(u' \times v') = f_*^G(u') \times g_*^G(v')$.*

(3) *Let $f : X \to Z$ be a projective morphism in $G-\mathbf{Sch}_k$, and $g : Y \to Z$ is a l.c.i. morphism in $G - \mathbf{Sch}_k$. If we have a transverse Cartesian diagram*

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

then $g_G^* \circ f_*^G = f'^G_* \circ g'^*_G$.

Proof. (1) and (2) follows from taking the limit of the corresponding identities. Now we proceed to show (3). By descent and flat base change for Tor_n we obtain the transverse Cartesian diagram

$$\begin{tikzcd} W \times^G U_i \arrow[r, "g_i'"] \arrow[d, "f_i'"] & Y \times^G U_i \arrow[d, "f_i"'] \\ Z \times^G U_i \arrow[r, "g_i"'] & X \times^G U_i \end{tikzcd}$$

in $\mathbf{Sch}_k$ for each i. Since $g_i^* \circ f_{i\,*} = f'_{i\,*} \circ g_i'^{\,*}$ and this is compatible with the transition maps in the inverse system, we obtain $g_G^* \circ f_*^G = f_*'^G \circ g_G'^{\,*}$. □

4.2. **Localization Sequence.**

Theorem 20. *Let X in $G - \mathbf{Sch}_k$. Let $\imath : Z \to X$ be an invariant closed subscheme and let $j : U \to X$ be the open complement. Then we have the exact sequence*

$$(\mathbf{LS}) \qquad \Omega_*^G(Z) \xrightarrow{\imath_*^G} \Omega_*^G(X) \xrightarrow{j_G^*} \Omega_*^G(U) \longrightarrow 0\,.$$

Proof. The proof follows directly from (2.1.4), Lemma 18 and naturality of the push-forwards and pull-backs. □

4.3. **Projective Bundle Axiom.** Let $E \to X$ be a G-equivariant vector bundle of rank $r+1$. We know that $E \times^G U_i$ is a vector bundle of rank $r+1$ over $X \times^G U_i$ for each U_i in the system. Let $q_i : \mathbb{P}_i := \mathbb{P}\left(E \times^G U_i\right) \to X \times^G U_i$ be the associated projective bundle. As a projective bundles over $X \times^G U_i$ we have that $\mathbb{P}(E \times^G U_i)$ is isomorphic to $\mathbb{P}(E) \times^G U_i$. For convenience we will work with $\mathbb{P}(E \times^G U_i)$. By (PB) for Ω_* we have an isomorphism

$$\Phi_i := \textstyle\prod_{n=0}^r \ \xi_i^n \cdot q_i^* : \displaystyle\prod_{n=0}^{r} \Omega_{*-r+n}\left(X \times^G U_i\right) \longrightarrow \Omega_*\left(\mathbb{P}_i\right),$$

where $\xi_i = \mathrm{c}_1\left(\mathcal{O}_{\mathbb{P}_i}(1)\right)$. Consider the morphisms $\phi_{ij} : E \times^G U_i \to E \times^G U_j$ for $i < j$. Notice that $\mathcal{O}_{\mathbb{P}_i}(1) = \mathbb{P}(\phi_{ij})^*\left(\mathcal{O}_{\mathbb{P}_j}(1)\right)$, so $\mathbb{P}(\phi_{ij})^*\left(\xi_j\right) = \xi_i$. Thus, the isomorphisms Φ_i induce the isomorphism

$$\varprojlim_i \left(\textstyle\prod_{n=0}^r \ \xi_i^n \cdot q_i^*\right) : \varprojlim_i \left(\displaystyle\prod_{n=0}^{r} \Omega_{*-r+n}\left(X \times^G U_i\right)\right) \longrightarrow \varprojlim_i \Omega_*\left(\mathbb{P}(E \times^G U_j)\right).$$

We have proved the following.

Proposition 21. *Let $E \to X$ be a G-equivariant vector bundle of rank $r+1$ in $G - \mathbf{Sch}_k$. With the notation above, set $\xi_*^G := \varprojlim_i \xi_i$, and $\Omega_G^*\left(\mathbb{P}(E)\right) := \varprojlim_i \Omega_*\left(\mathbb{P}(E \times^G U_i)\right)$. Then*

$$(\mathbf{PB}) \qquad \Phi_{X,E}^G := \prod_{n=0}^{r} \left(\xi_*^G\right)^n \cdot q_G^* : \prod_{n=0}^{r} \Omega_{*-r+n}^G(X) \longrightarrow \Omega_*^G\left(\mathbb{P}(E)\right)$$

is an isomorphism.

4.4. **Extended Homotopy Axiom.**

Proposition 22. *Let $E \to X$ be a G-vector bundle of rank n and $p : Y \to X$ be a E-torsor in $G - \mathbf{Sch}_k$, then*

$$(\mathbf{EH}) \qquad p_G^* : \Omega_*^G(X) \longrightarrow \Omega_{*+n}^G(Y)$$

is an isomorphism.

Proof. We have that $E \times^G U_i \times_{X \times^G U_i} Y \times^G U_i$ is isomorphic to $(E \times U_i \times_{X \times U_i} Y \times U_i)/G$ for each i. Therefore the action map $\Psi : E \times_X Y \to Y \times_X Y$ induces an action map

$$\Psi_i : E \times^G U_i \times_{X \times^G U_i} Y \times^G U_i \longrightarrow Y \times^G U_i \times_{X \times^G U_i} Y \times^G U_i$$

and Ψ_i is an isomorphism for all i. This makes $Y \times^G U_i$ an $E \times^G U_i$-torsor over $X \times^G U_i$. The result then follows from the extended homotopy property for algebraic cobordism. □

Corollary 23 (Homotopy Invariance)**.** *Let X be in $G - \mathbf{Sch}_k$. If G acts linearly on $\mathbb{A}^n$ then $p_G^* : \Omega_*^G(X) \to \Omega_{*+n}^G(\mathbb{A}^n \times X)$ is an isomorphism.*

4.5. **Chern Classes of G-Equivariant Vector Bundles.** Let $E \to X$ be an G-equivariant vector bundle in $G - \mathbf{Sch}_k$. We define the n-th *G-equivariant Chern operator* $\tilde{\mathrm{c}}_n^G(E)$ as

$$\tilde{\mathrm{c}}_n^G(E) := \varprojlim_i \tilde{\mathrm{c}}_{n,i}(E),$$

where $\tilde{\mathrm{c}}_{n,i}(E)$ is the n-th Chern operator of $E \times^G U_i \to X \times^G U_i$ induced by (PB). If X is smooth, given a G-equivariant vector bundle E over X, define the n-th *G-equivariant Chern class* $\mathrm{c}_n^G(E)$ in $\Omega_{-n}^G(X)$ as $\mathrm{c}_n^G(E) := \tilde{\mathrm{c}}_n^G(E)(1_X)$.

Remark 24. Restrict to smooth schemes.

(1) We could have defined $\mathrm{c}_n^G(E)$ by means of (**PB**) following Grothendieck's method [6], so that $\sum_{n=0}^r (-1)^n \mathrm{c}_n^G(E)\, \xi_G^{r-n} = 0$ holds, where $r = \mathrm{rank}(E)$.
(2) If $E = \oplus_{i=1}^r L_i$, with $L_i \to X$ a G-linear bundle, then $\mathrm{c}_n^G(E)$ is the n-th elementary symmetric polynomial at the $\mathrm{c}_1^G(L_i)$.

Our equivariant Chern operators have the expected properties.

Lemma 25. *Let X and Y be in $G - \mathbf{Sch}_k$.*

(1) (Commutativity) *Let E and F be G-vector bundles over X. For any i and j we have that $\tilde{\mathrm{c}}_i^G(E) \circ \tilde{\mathrm{c}}_j^G(F) = \tilde{\mathrm{c}}_j^G(F) \circ \tilde{\mathrm{c}}_i^G(E)$.*
(2) (Naturality) *For any l.c.i. morphism $f : Y \to X$ in $G - \mathbf{Sch}_k$ and any G-equivariant vector bundle $E \to X$, we have that $\tilde{\mathrm{c}}_n^G \circ f^*E = f_G^* \circ \tilde{\mathrm{c}}_n^G(E)$, for all $n \geq 0$.*
(3) (Whitney Formula) *If $0 \to E' \to E \to E'' \to 0$ is an G-equivariant exact sequence of G-vector bundles over X, then $\tilde{\mathrm{c}}_n^G(E) = \sum_{l=0}^n \tilde{\mathrm{c}}_l^G(E')\, \tilde{\mathrm{c}}_{n-l}^G(E'')$, for all $n \geq 0$.*

4.6. **Restriction Maps.** In this section we relate Ω_*^G and Ω_* via restriction maps, which can be defined via $\langle e \rangle \to G$ or by restricting to the fiber. We show that both agree up to isomorphism on Ω_*.

4.6.1. *Restricting the Action.* Let $H \subseteq G$ be a closed normal subgroup of G. Since the induced action of H on each U_i is free and the quotient $U_i/H = G/H \times^G U_i$ exists we see that $\{(V_i, U_i)\}$ is also a good system for H. Moreover, we have smooth morphisms $X \times^H U_i \to X \times^G U_i$ that induces

$$\mathrm{res}_{G,H} : \Omega^G_*(X) \longrightarrow \Omega^H_*(X).$$

When $H = \langle e \rangle$, we will use res_G to denote $\mathrm{res}_{G,\langle e \rangle}$. From Example 17 we have the natural isomorphism $\Omega_*(X) \cong \Omega^{\langle e \rangle}_*(X)$ and so we obtain $\mathrm{res}_G : \Omega^G_*(X) \longrightarrow \Omega_*(X)$.

4.6.2. *Restriction to the Fiber.* Assume $X \in G - \mathbf{Sch}_k$ to be irreducible. Let $\eta \in U/G$ be a rational point, where U is the initial G-invariant open in the system being considered. For each i, the projection $X \times U_i \to U_i$ induces a flat morphism $X \times^G U_i \to U_i/G$ whose fiber over a rational point η_i of U_i/G equals X, where η_i is the image of η under the canonical morphism $U/G \to U_i/G$. We have an induced morphism $\mathrm{res}^G_i(\eta) : \Omega_*(X \times^G U_i) \to \Omega_*(X)$. For any $i < j$ in the system we have an induced commutative diagram

$$\begin{tikzcd}
\Omega^G_*(X) \arrow[d] \arrow[dr] & \\
\Omega_*(X \times^G U_j) \arrow[r] \arrow[dr] & \Omega_*(X \times^G U_i) \arrow[d] \\
& \Omega_*(X).
\end{tikzcd}$$

Thus, given a rational point η in U/G and for any i we have the *restriction map*

$$\mathrm{res}^G_\Omega(\eta) : \Omega^G_*(X) \longrightarrow \Omega_*\left(X \times^G U_i\right) \xrightarrow{\mathrm{res}^G_i(\eta)} \Omega_*(X).$$

If the group G is clear from the context, we will use the notation $\mathrm{res}_\Omega(\eta)$. When $G = \langle e \rangle$, the restriction $\mathrm{res}^{\langle e \rangle}_\Omega(\eta)$ is precisely the isomorphism of Example 17.

4.6.3. *Comparison of Restrictions.* Fix a rational point $\eta \in U/G$ as in the previous section. Let H be a normal closed subgroup of G. For every U_i in the system, let $P_H(i)$ and $P_G(i)$ be points in U_i/H and U_i/G respectively, so that $P_H(i) \mapsto P_G(i)$ under the canonical map $U_i/H \to U_i/G$, where $\eta \mapsto P_G(i)$ under $U/G \to U_i/G$. We have the commutative diagram

$$\begin{tikzcd}
& & X \arrow[dll, equal] \arrow[r] \arrow[d] & X \times^H U_i \arrow[dll] \arrow[d] \\
X \arrow[r] \arrow[d] & X \times^G U_i \arrow[d] & P_H(i) \arrow[r] \arrow[dll] & U_i/H \arrow[dll] \\
P_G(i) \arrow[r] & U_i/G & &
\end{tikzcd}$$

with Cartesian faces induced by the fiber squares.

Hence we have a commutative diagram

$$\begin{tikzcd} \Omega_*^G(X) \arrow[r, "\mathrm{res}_{G,H}"] \arrow[d, "\mathrm{res}^G_\Omega(\eta)"'] & \Omega_*^H(X) \arrow[d, "\mathrm{res}^H_\Omega(\eta)"] \\ \Omega_*(X) \arrow[r, "\mathrm{Id}"'] & \Omega_*(X). \end{tikzcd}$$

For $H = \langle e\rangle$, we have seen that $\mathrm{res}^H_\Omega(\eta)$ is an isomorphism, so $\mathrm{res}^G_\Omega(\eta)$ and res_G are equal up to isomorphism on $\Omega_*(X)$. In particular, $\mathrm{res}^G_\Omega(\eta)$ is independent of η. From now on, we will denote by res^G_Ω the restriction to the fiber map. With this notation, we have proved $\mathrm{res}^G_\Omega = \mathrm{res}^{\langle e\rangle}_\Omega \circ \mathrm{res}_G$.

4.6.4. We have the following.

Theorem 26 (Induction)**.** *Let H be a closed normal subgroup of G. Consider G as an H-scheme with the action $(h, g) \mapsto gh^{-1}$. Let X be in $G - \mathbf{Sch}_k$. Then*

$$\Omega^H_*(X) \cong \Omega^G_*(X \times^H G),$$

where $X \times^H G$ is given a G-action via its action on G. Moreover, if X is obtained by restriction of a G-action then

$$\Omega^H_*(X) \cong \Omega^G_*(X \times G/H).$$

These isomorphisms are natural with respect to the variances.

Proof. Let $\{V_i, U_i\}$ be a good system of representations for G. By restricting the action this provides a good system of representations for H as well. The first statement follows from the isomorphism $(X \times^H G) \times^G U_i \to X \times^H U_i$, given by $([x, g], u) \mapsto (x, g^{-1}u)$.

If X is an H-scheme obtained by restricting a G-action then $X \times^H G \to X \times G/H$, given by $[x, a] \mapsto (ax, aH)$, is an isomorphism of G-schemes. The second statement now follows from the first. □

4.7. **Geometric Quotients.** In this section we assume the geometric quotient $p : X \to X/G$ exists. We compare the ordinary cobordism of X/G and the equivariant algebraic cobordism X. As a consequence we see that the ordinary cobordism with rational coefficients of X/G, for smooth X, is equipped with a natural ring structure.

The fiber of $\pi_i : X \times^G U_i \to X/G$ over $x \in X/G$ is given by $\pi_i^{-1}(x) = U_i/G_x$. Since U_i/G_x is smooth, by [14, Corollary 6.3.24] we have that each π_i is an l.c.i. morphism. The morphisms $\pi_i^* : \Omega_k(X/G) \to \Omega_{k+\dim U_i}(X \times^G U_i)$ induce

$$\pi^* : \Omega_*(X/G) \longrightarrow \Omega^G_{*+\dim G}(X).$$

Proposition 27. *Let X be in $G-\mathbf{Sch}_k$. Assume that $X \to X/G$ is a principal G-bundle. Then $\pi^* : \Omega_*(X/G) \to \Omega^G_{*+\dim G}(X)$ is an isomorphism.*

Proof. Since the stabilizers are trivial, $X \times^G V_i \to X/G$ is a vector bundle for every representation V_i. By Proposition 15 there is an integer N such that $\pi^*_{|U_j} : \Omega_*(X/G) \to \Omega_{*+\dim U_j}(X \times^G U_j)$ is an isomorphism for all $j > N$. □

Theorem 28. *Let X be in $G-\mathbf{Sch}_k$ with proper G-action. Then $\pi^*:\Omega_*(X/G)_{\mathbb{Q}}\to\Omega^G_{*+\dim G}(X)_{\mathbb{Q}}$ is an isomorphism.*

This theorem implies that we have a ring structure on the cobordism of a large class of interesting singular schemes.

Corollary 29. *Let X be in $G-\mathbf{Sm}_k$. Under the assumptions of the theorem, $\Omega^*(X/G)_{\mathbb{Q}}$ has a ring structure.*

Proof of Theorem 28. Write $g=\dim G$ and write $Y=X/G$ for the quotient. We proceed in a similar fashion as in [3, Theorem 3(a)]. By [3, Proposition 10], we have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & X \arrow[d, "p"'] \\ Y' \arrow[r, "f"] & Y, \end{tikzcd}$$

where $X'\to Y'$ is a principal G-bundle and both $X'\to X$ and $Y'\to Y$ are finite and surjective.

We obtain from the exact sequences from Proposition 30 proved below,

$$\Omega^G_*(X'\times_X X')_{\mathbb{Q}}\xrightarrow{pr_{1G}-pr_{2G}}\Omega^G_*(X')_{\mathbb{Q}}\xrightarrow{\tilde{f}_G}\Omega^G_*(X)_{\mathbb{Q}}\to 0,$$
$$\Omega_*(Y'\times_Y Y')_{\mathbb{Q}}\xrightarrow{pr_{1*}-pr_{2*}}\Omega_*(Y')_{\mathbb{Q}}\xrightarrow{f_*}\Omega_*(Y)_{\mathbb{Q}}\to 0.$$

Write $Y''=(X'\times_X X')/G$, we have then that $\Omega_*(Y'')\to\Omega_*(Y'\times_Y Y')$ is a surjection since $Y''\to Y'\times_Y Y'$ is a finite and surjective morphism. We obtain the comparison of exact sequences,

$$\begin{tikzcd} \Omega_*(Y'')_{\mathbb{Q}} \arrow[r, "pr_{1*}-pr_{2*}"] \arrow[d, "\pi''^*"] & \Omega_*(Y')_{\mathbb{Q}} \arrow[r, "f_*"] \arrow[d, "\pi'^*"] & \Omega_*(Y)_{\mathbb{Q}} \arrow[r] \arrow[d, "\pi^*"] & 0 \\ \Omega^G_{*+g}(X'\times_X X')_{\mathbb{Q}} \arrow[r, "pr_{1G}-pr_{2G}"] & \Omega^G_{*+g}(X')_{\mathbb{Q}} \arrow[r, "\tilde{f}_G"] & \Omega^G_{*+g}(X)_{\mathbb{Q}} \arrow[r] & 0. \end{tikzcd}$$

By Proposition 27 the two maps on the left are isomorphisms, so the third map too is. ☐

Proposition 30. *Let $\pi:X'\to X$ be a finite surjective map.*

(a) *The sequence* $\Omega_*(X'\times_X X')_{\mathbb{Q}}\xrightarrow{p_{1*}-p_{2*}}\Omega_*(X')_{\mathbb{Q}}\xrightarrow{\pi_*}\Omega_*(X)_{\mathbb{Q}}\to 0$ *is exact.*
(b) *Suppose that both X' and X are G-schemes and π is G-equivariant. Then the sequence* $\Omega^G_*(X'\times_X X')_{\mathbb{Q}}\xrightarrow{p^G_{1*}-p^G_{2*}}\Omega^G_*(X')_{\mathbb{Q}}\xrightarrow{\pi^G_*}\Omega^G_*(X)_{\mathbb{Q}}\to 0$ *is exact.*

Proof. (a) By [11, Proposition 1.8] the sequence

$$\mathrm{CH}_*(X'\times_X X')_{\mathbb{Q}}\xrightarrow{p_{1*}-p_{2*}}\mathrm{CH}_*(X')_{\mathbb{Q}}\xrightarrow{\pi_*}\mathrm{CH}_*(X)_{\mathbb{Q}}\to 0,$$

is exact. It follows that the sequence

$$\mathrm{CH}_*(X'\times_X X')[\mathbf{t}]^{(\mathbf{t})}_{\mathbb{Q}}\xrightarrow{p_{1*}-p_{2*}}\mathrm{CH}_*(X')[\mathbf{t}]^{(\mathbf{t})}_{\mathbb{Q}}\xrightarrow{\pi_*}\mathrm{CH}_*(X)[\mathbf{t}]^{(\mathbf{t})}_{\mathbb{Q}}\to 0$$

is also exact. The statement follows from Example 5.

(b) The morphism $X' \times^G U_i \to X \times^G U_i$ is finite and surjective by faithfully flat descent, and $\left(X' \times^G U_i\right) \times_{X \times^G U_i} \left(X' \times^G U_i\right) = (X' \times_X X') \times^G U_i$. Therefore

$$\Omega_* \left(\left(X' \times_X X'\right) \times^G U_i \right)_{\mathbb{Q}} \xrightarrow{p_{1*} - p_{2*}} \Omega_* \left(X' \times^G U_i\right)_{\mathbb{Q}} \xrightarrow{\pi_*} \Omega_* \left(X \times^G U_i\right)_{\mathbb{Q}} \to 0$$

is an exact sequence for each i. By Lemma 18 the system $\Omega_* \left(\_ \times^G U_i\right)_{\mathbb{Q}}$ satisfies the Mittag-Leffler condition. Hence the sequence remains exact upon taking inverse limits.

□

4.8. **Reduction to a Torus Action.** For the rest of the section G will be a connected reductive linear algebraic group containing a split maximal torus T and B be a Borel subgroup containing T. Let N be the normalizer of T in G and $W = N/T$ be the Weyl group. The Weyl group W acts on $\Omega^*_T(X)$ and the image of the restriction map is W-invariant and so we have the morphism $\mathrm{res}_{G,T} : \Omega^*_G(X) \to \Omega^*_T(X)^W$. In [3] it is shown that the analogous morphism in equivariant Chow groups is an isomorphism rationally. We show now that the analogous statement holds for equivariant algebraic cobordism.

Lemma 31. *Let X be in $B-$* **Sch**. *Suppose that the principle B-bundle $X \to X/B$ exists. Then $\Omega^*(X/B) \to \Omega^*(X/T)$ is an isomorphism. In particular for any X in $B-$* **Sch**, *the restriction map $\mathrm{res}_{B,T} : \Omega^B_*(X) \to \Omega^T_*(X)$ is an isomorphism.*

Proof. Since B/T is an affine space, the map $X/T \to X/B$ is an affine space bundle. The result follows from the extended homotopy axiom for cobordism. □

Using the isomorphism in the previous lemma we transport the W-action on $\Omega^*_T(X)$ to an action on $\Omega^*_B(X)$.

Lemma 32. *Let G be a connected, reductive linear algebraic group. Then $\Omega^*(G/B)^W_{\mathbb{Q}} \cong \Omega^*(k)_{\mathbb{Q}}$.*

Proof. Base change induces a W-equivariant isomorphism $\Omega^*(G/B) \cong \Omega^*((G/B)_L)$ for any field extension L/k by Proposition 6 because G/B is cellular. Because G and T are determined by root datum we may assume that $k = \mathbb{C}$. Specifically, if $k \subseteq \mathbb{C}$ then we have $\Omega^*(G/B) \cong \Omega^*((G/B)_{\mathbb{C}})$. If $\mathbb{C} \subseteq k$ then there is a split, connected reductive group G' and Borel subgroup B' which are defined over $\mathbb{C}$ and are such that $(G'/B')_k = G/B$ and so $\Omega^*(G/B) = \Omega^*((G'/B')_k)$.

Consider the principal W-bundle $\pi : G/T \to G/N$. We can regard π^* as the morphism $\Omega^*(G/N)_{\mathbb{Q}} \to \Omega^*(G/T)^W_{\mathbb{Q}}$. Write π'_* for the restriction of π_* to $\Omega^*(G/T)^W_{\mathbb{Q}}$. Since $\pi'_*\pi^*$ and $\pi^*\pi'_*$ coincide with multiplication by $|W|$ we have that π^* is an isomorphism. Similarly we have the isomorphisms $\pi^* : \mathrm{CH}^*(G/N)_{\mathbb{Q}} \to \mathrm{CH}^*(G/T)^W_{\mathbb{Q}} = \mathrm{CH}^*(G/B)^W_{\mathbb{Q}}$.

The cycle map $\mathrm{CH}^*(G/B) \to \mathrm{H}^*(G/B)$ is a W-equivariant isomorphism [5, Example 19.1.11]. By [10, Chapter III, §1 (B)] we have $\mathrm{H}^*(G/B; \mathbb{Q})^W = \mathbb{Q}$ and so $\mathrm{CH}^*(G/B)^W_{\mathbb{C}}$ from which it follows that $\mathrm{CH}^*(G/N)_{\mathbb{Q}} = \mathbb{Q}$. Now from the isomorphism in Example 5 we obtain the isomorphism $\Omega^*(\mathbb{C})_{\mathbb{Q}} = \Omega^*(G/N)_{\mathbb{Q}}$ and so $\Omega^*(\mathbb{C})_{\mathbb{Q}} = \Omega^*(G/T)^W_{\mathbb{Q}} = \Omega^*(G/B)^W_{\mathbb{Q}}$. □

Theorem 33. *Assume G is a connected, reductive linear algebraic group. Let X be in $G-\mathbf{Sm}_k$. Then there is an isomorphism*

$$\mathrm{res}_{G,T} : \Omega^*_G(X)_{\mathbb{Q}} \to \Omega^*_T(X)^W_{\mathbb{Q}}$$

of $\Omega^*(k)_{\mathbb{Q}}$-modules. If G is special then $\mathrm{res}_{G,T} : \Omega^*_G(X) \to \Omega^*_T(X)^W$ is injective.

Proof. By Lemma 31 it suffices to show that $\Omega^*_G(X)_{\mathbb{Q}} \to \Omega^*_B(X)^W_{\mathbb{Q}}$ is an isomorphism. By the following proposition we have $\Omega^*(X \times^G U_i)_{\mathbb{Q}} \to \Omega^*(X \times^B U_i)^W_{\mathbb{Q}}$ is an isomorphism for all i. Fixed points and inverse limits commute we are done. □

Proposition 34. *Let $Y \to Q$ be a principle G-bundle, with Y smooth. Write $p : Y/B \to Q$ for the resulting principal G/B-bundle. Then*

$$p^* : \Omega^*(Q)_{\mathbb{Q}} \to \Omega^*(Y/B)^W_{\mathbb{Q}}$$

is an isomorphism. If G is special then $p^* : \Omega^*(Q) \to \Omega^*(Y/B)^W$ is injective.

Proof. By Proposition 7, $\Omega^*(Y/B)_{\mathbb{Q}}$ is a free $\Omega^*(Q)_{\mathbb{Q}}$-module. We may take $1 \in \Omega^*(Y/B)_{\mathbb{Q}}$ as one of the basis elements. It follows that $p^* : \Omega^*(Q) \to \Omega^*(Y/B)^W$ is injective.

Let $p : Y/B \to Q$ be obtained from a principal G-bundle $Y \to Q$ with Y not necessarily smooth. We proceed by induction to show that p^* is surjective, the zero-dimensional case being done. We may assume that Q is irreducible. Let $U \subseteq Q$ be an open subscheme over which the bundle is isotrivial and let Z be its closed complement. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
\Omega_*(Z)_{\mathbb{Q}} \arrow[r] \arrow[d] & \Omega_*(Q)_{\mathbb{Q}} \arrow[r] \arrow[d] & \Omega_*(U)_{\mathbb{Q}} \arrow[r] \arrow[d] & 0 \\
\Omega_*(Y_Z)^W_{\mathbb{Q}} \arrow[r] & \Omega_*(Y)^W_{\mathbb{Q}} \arrow[r] & \Omega_*(Y_U)^W_{\mathbb{Q}} \arrow[r] & 0,
\end{tikzcd}$$

where the bottom row is exact because taking fixed points is an exact functor when $|W|$ is invertible. We see that it suffices to show that $\Omega_*(U)_{\mathbb{Q}} \to \Omega_*(Y_U)^W_{\mathbb{Q}}$ is surjective. Let $g : V \to U$ be a finite étale morphism over which the bundle becomes trivial. Consider the commutative square

$$\begin{tikzcd}
\Omega_*(V)_{\mathbb{Q}} \arrow[r, "g_*"] \arrow[d] & \Omega_*(U)_{\mathbb{Q}} \arrow[d] \\
\Omega_*(V \times G/B)^W_{\mathbb{Q}} \arrow[r] & \Omega_*(Y_U)^W_{\mathbb{Q}}.
\end{tikzcd}$$

The horizontal arrows are surjective and so we are reduced to the case of the trivial bundle. In the proof of Proposition 7 it is shown that the map $\Omega^*(V)_{\mathbb{Q}} \otimes_{\Omega^*(k)_{\mathbb{Q}}} \Omega^*(G/B)_{\mathbb{Q}} \to \Omega^*(V \times G/B)_{\mathbb{Q}}$ induced by external product is a surjection. This is an equivariant map and so surjectivity follows from the isomorphism

$$(\Omega^*(V)_{\mathbb{Q}} \otimes_{\Omega^*(k)_{\mathbb{Q}}} \Omega^*(G/B)_{\mathbb{Q}})^W = \Omega^*(V)_{\mathbb{Q}} \otimes_{\Omega^*(k)_{\mathbb{Q}}} \Omega^*(G/B)^W_{\mathbb{Q}} = \Omega^*(V)_{\mathbb{Q}}.$$

If G is special then p^* is injective by Proposition 7 and its image is contained in $\Omega^*(Y/B)^W$. □

Remark 35. If G is special, $T \subseteq G$ a maximal torus and we additionally assume that the restriction map $\Omega^*_T(X) \to \Omega^*_T(X^T)$ is injective and $\Omega^*(X^T)$ is torsion-free then we can see that $\mathrm{res}_{G,T} : \Omega^*_G(X) \to \Omega^*_T(X)^W$ is an isomorphism, which we explain below. In particular the case $X = \mathrm{spec}(k)$ shows that for G special we have the equality

$$\Omega^*_G(k) = \Omega^*_T(k)^W,$$

giving the cobordism generalization of the result of Edidin-Graham for integral Chow groups of classifying spaces of special groups [2].

Fix a good system $\{(V_i, U_i)\}$ of G-representations. Let $(x_i) \in \Omega^*_T(X)^W$ be a fixed class. Write $p_i : X \times^T U_i \to X \times^G U_i$ for the projection. By Proposition 7 and Lemma 31 we have that $\Omega^*(X \times^T U_i)$ is a free $\Omega^*(X \times^G U_i)$-module. A basis is given by choosing elements which restrict to a basis of $\Omega^*(G/T)$. We may choose a basis $e_{r,i} \in \Omega^*(X \times^G U_i)$ inductively so that $e_{r,j}$ is mapped to $e_{r,i}$ under the transition maps in the inverse system. Moreover we may choose each $e_{1,i} = 1$. Thus, we can write $x_i = \sum_k p_i^*(y_{k,i}) \cup e_{k,i}$ with $y_{k,i} \in \Omega^*(X \times^G U_i)$.

By the previous proposition, for each i there is an integer M_i and a $y_i \in \Omega^*(X \times^G U_i)$ so that $M_i x_i = p_i^*(y_i)$. Comparing the two expressions for $M_i x_i$ we see that $M_i x_i = M_i p_i^*(y_{1,i})$. Write $a_i = y_{1,i}$. Then for all i the element $x_i - p_i^*(a_i) \in \Omega^*(X \times^T U_i)$ is torsion. Note that the sequence (a_i) defines an element in $\Omega^*_G(X)$. Hence we have a sequence $(x_i - p_i^*(a_i))$ of torsion elements representing a class in $\Omega^*_T(X)$.

Write $c_i = x_i - p_i^*(a_i)$. Since $\Omega^*_T(X) \subseteq \Omega^*_T(X^T)$ we also write $(c_i) \in \Omega^*_T(X^T)$. Let $\{V'_i, U'_i\}$ be a good system of T-representations with the property that U_i/T is a product of projective spaces as in Example 3.2.1. Consider the products $U_i \times U'_j$ and projections $f_{ij} : U_i \times U'_j \to U_i$ and $g_{ij} : U_i \times U'_j \to U'_j$. The maps of systems (indexed by pairs of integers)

$$\{\Omega^*(X^T \times U_i/T)\} \xrightarrow{f^*_{ij}} \{\Omega^*(X^T \times (U_i \times U'_j)/T)\} \xleftarrow{g^*_{ij}} \{\Omega^*(X^T \times U'_j/T)\}$$

induces an isomorphism on limits since each system computes $\Omega^*_T(X^T)$. Therefore there are $b_j \in \Omega^*(X^T \times U'_j/T)$ such that $(f^*_{ij}(c_i)) = (g^*_{ij}(b_j))$. The element $f^*_{ij}(c_i)$ is torsion for all i, j. By Proposition 15, there are integers $C(i)$ and $D(j)$ such that f^*_{ij} is an isomorphism for $j > C(i)$ and g^*_{ij} is an isomorphism for $i > D(j)$. Since there is no torsion in $\Omega^*(X^T \times U'_j/T)$ we must have that $f^*_{ik}(c_i) = 0$ for $i > D(k)$ but then $f^*_{ij}(c_i) = 0$ for $j > k$ as well. By taking $j > C(i)$ we conclude that $c_i = 0$.

5. Oriented Equivariant Borel-Moore Homology Theories

Our construction of equivariant algebraic cobordism relies only on the general properties of an OBM theory and the localization sequence. Thus, if A_* is an OBM theory which has localization sequences and $\{(V_i, U_i)\}$ is a good system of representations, we can define the theory

$$A^G_*(X) := \varprojlim_i A_* \left(X \times^G U_i\right)$$

for any X in $G - \mathbf{Sch}_k$. Similarly we define $A^G_n(X) = \varprojlim_i A_{n+\dim U_i - \dim G}(X \times^G U_i)$. Because A_* has localization sequences the analogue of Proposition 15 is valid for A_*. This theory is then seen to be well-defined by reasoning as in Theorem 16.

When the OCT A^* classifies its formal group law, so that $\Omega^*(X) \otimes_{\Omega^*(k)} A^*(k) = A^*(X)$, define
$$A^*_G(X) = \varprojlim_i A^*\left(X \times^G U_i\right) \cong \Omega^*_G(X) \widehat{\otimes}_{\Omega^*(k)} A^*(k).$$
Set $A^n_G(X) = \varprojlim_i A^n\left(X \times^G U_i\right)$. Analogoues of all the computations and the results for equivariant algebraic cobordism can be carried out for any such A^G_*.

Define an *oriented equivariant Borel-Moore theory* as a functor $A^G_* : G - \mathbf{Sch}'_k \to \mathbf{Ab}$ endowed with pull-backs maps for every G-equivariant l.c.i. morphism satisfying the analogues of the properties listed in Sections 4.1, 4.2, 4.3 and 4.4.

Similarly, an *oriented equivariant cohomology theory* is a functor $A^*_G : (G - \mathbf{Sm}_k)^{op} \to \mathbf{Rng}$ endowed with morphisms $f_G : A^*_G(Y) \to A^{*+d}_G(X)$ of $A^*_G(k)$-modules for every projective morphism $f : Y \to X$ of relative dimension d in $G - \mathbf{Sm}_k$, satisfying the analogues of the axioms listed in Section 2.1.2.

Remark 36. If A^G_* is an oriented equivariant Borel-Moore theory arising from an OBM theory A_*, by construction we have a canonical natural transformation $\Omega^G_* \to A^G_*$.

Example 37. The universal additive OCT Ω^*_+ induces the oriented equivariant cohomology theory
$$X \mapsto \Omega^*_{+\,G}(X) = \varprojlim_i \Omega^*_+\left(X \times^G U_i\right).$$
Let CH^n_G be the n-th equivariant Chow group as defined by Edidin and Graham [3]. The Chow group is the universal additive OCT theory (see Section 2.1.5) and so there is M_n such that
$$\mathrm{CH}^n_G(X) := \mathrm{CH}^n\left(X \times^G U_i\right) \cong \Omega^n_+\left(X \times^G U_i\right)$$
for any $i > M_n$. Since $\mathrm{CH}^n(X \times^G U_j) \cong \mathrm{CH}^n(X \times^G U_i)$ for $i, j > M_n$, by taking limits we get a natural isomorphism $\Omega^n_{+\,G}(X) \cong \mathrm{CH}^n_G(X)$. This give us
$$\bigoplus_n \mathrm{CH}^n_G(X) = \left(\bigoplus_n \Omega^n_G(X)\right) \widehat{\otimes}_{\Omega^*(k)} \mathbb{Z}.$$
Also, from the remark above we obtain the commutative diagram
$$\begin{tikzcd} \Omega^n_G(X) \arrow[r] \arrow[d, "\mathrm{res}^G_\Omega"'] & \mathrm{CH}^n_G(X) \arrow[d, "\mathrm{res}"] \\ \Omega^n(X) \arrow[r] & \mathrm{CH}^n(X), \end{tikzcd}$$
where the restriction map $\mathrm{res} : \mathrm{CH}^n_G(X) \to \mathrm{CH}^n(X)$ is obtained by restricting to the fiber, as we did for res^G_Ω in Section 4.6.2.

Example 38. The universal multiplicative periodic OCT $\Omega^\times_*$ induces the oriented equivariant cohomology theory
$$X \mapsto \Omega^*_{\times\,G}(X) = \varprojlim_i \Omega^*_\times\left(X \times^G U_i\right).$$
Let K^0_G be the equivariant algebraic K-theory defined by Thomason [18] as the Grothendieck group of the category G-vector bundles (Section 2.2.3). By descent the category of G-vector bundles on $X \times U_i$ is equivalent to the category of vector bundles on $X \times^G U_i$.

We thus have the induced natural isomorphism $\mathrm{K}^0_G(X \times U_i) \cong \mathrm{K}^0(X \times^G U_i)$. Since $\mathrm{K}[\beta, \beta^{-1}]$ is the universal multiplicative periodic OCT (see Section 2.1.5), we can consider the composition

$$\mathrm{K}^0_G(X) \to \mathrm{K}^0(X \times^G U_i) \to \mathrm{K}^0(X \times^G U_i)[\beta, \beta^{-1}] \cong \Omega^*_\times(X \times^G U_i).$$

Taking limits defines the natural transformation

$$\mathrm{K}^0_G(X) \longrightarrow \Omega^*_{\times\, G}(X).$$

Moreover, by forgetting the action we have a restriction map res : $\mathrm{K}^0_G \to \mathrm{K}^0$ which fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{K}^0_G(X) \arrow[r] \arrow[d, "\mathrm{res}"'] & \Omega^*_{\times\, G}(X) \arrow[d, "\mathrm{res}_G"] \\
\mathrm{K}^0(X) \arrow[r, "id\otimes 1"'] & \mathrm{K}^0(X)[\beta, \beta^{-1}].
\end{tikzcd}$$

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