

The tom Dieck splitting theorem in equivariant motivic homotopy theory

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Abstract

We establish, in the setting of equivariant motivic homotopy theory for a finite group, a version of tom Dieck’s splitting theorem for the fixed points of a suspension spectrum. Along the way we establish structural results and constructions for equivariant motivic homotopy theory of independent interest. This includes geometric fixed point functors and the motivic Adams isomorphism.

Key Words. Motivic homotopy theory, equivariant homotopy theory, Adams isomorphism, tom Dieck splitting.

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1 Introduction

In his 1970 ICM talk [Seg71], Segal sketched a computation of the endomorphism ring of the equivariant sphere spectrum for a finite group G , identifying this endomorphism ring with the Burnside ring of finite G -sets. Using other methods, this computation was recovered and massively generalized by tom Dieck’s splitting theorem [tD75]. These results form a crucial layer of the foundations on which the successes of equivariant homotopy in the ensuing decades were built, from early foundations [LMSM86], to Carlsson’s

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resolution of the Segal completion conjecture [Car84], to the Hill-Hopkins-Ravenel solution of the Kervaire invariant one problem [HHR16].

An equivariant version of motivic homotopy theory was introduced by Voevodsky [Del09] to study quotients of motives by group actions, which played a role in his work on the Bloch-Kato conjecture. A number of authors have subsequently further developed this theory, and variants, including Hu-Kriz-Ormsby [HKO11], Heller-Krishna-Østvær [HKØ15], Herrmann [Her13], and Carlsson-Joshua [CJ14]. The state-of-the-art is Hoyois’s [Hoy17], where he develops the formalism of Grothendieck’s six operations in this theory.

In this paper we establish an analogue of tom Dieck’s splitting in the context of stable motivic homotopy theory for finite group actions. Throughout, we assume that G is a finite group whose order is coprime to the characteristics of the residue fields of the base-scheme B ; in other words, the group scheme associated to G is linearly reductive over B . Our splitting theorem, proved in Theorem 7.4 below, computes the N -fixed points of suspension spectra (more generally of “split spectra”) as a motivic G/N -spectrum, where $N \trianglelefteq G$ is a normal subgroup. In case $N = G$, this takes the following form, where (H) denotes the conjugacy class of a subgroup.

Theorem 1.1 (Motivic tom Dieck splitting). *Let G be a finite group whose order is invertible on B . Let X be a based motivic G -space over B . There is an equivalence of motivic spectra*

$$\Theta_X : \bigoplus_{(H)} (\Sigma^\infty(X^H))_{\mathbf{h}WH} \xrightarrow{\sim} (\Sigma^\infty X)^G$$

Corollary 1.2. *For integers a and b , there is a canonical isomorphism*

$$\pi_{a,b}^G(\mathbb{1}_B) \cong \bigoplus_{(H)} \pi_{a,b}(\mathbf{B}WH_+).$$

The reader familiar with tom Dieck’s theorem [tD75] will recognize this result as taking a very similar form as the classical result. The key difference here is that the functor $(-)_\mathbf{h}G$, is an algebro-geometric, or motivic, version of the homotopy orbits functor rather than the familiar categorical construction. Recall that the ordinary homotopy orbits functor is defined as follows. A G -spectrum Y determines a diagram on the category $BG \simeq B_{\text{Nis}}G$ and the homotopy orbits is the colimit of this diagram: $Y_{hG} \simeq \text{colim}_{BG} Y$. The motivic version should then be thought of as a motivic, or parameterized, colimit of Y over the category $B_{\text{ét}}G$ of étale G -torsors. We don’t make the version of the definition, as just stated, precise here (this can be done using [BEH21]) instead we provide a direct construction of the functor $(-)_\mathbf{h}G$. First, recall that Morel-Voevodsky [MV99] introduce a geometric model for the classifying space of étale G -torsors. This construction is distinct from the usual simplicial construction of the classifying space; rather, the simplicial construction is a model for the classifying space of Nisnevich G -torsors. The equivariant manifestation of this fact is that the universal free motivic G -space $\mathbf{E}G$ is not equivalent to the usual simplicial construction $\mathbf{E}_\bullet G$. The motivic homotopy orbits of a G -spectrum Y is defined here by a variant on the standard formula $Y_{hG} \simeq (\mathbf{E}_\bullet G_+ \otimes Y)/G$, obtained by replacing the use of $\mathbf{E}_\bullet G$ by $\mathbf{E}G$. That is, we take $Y_{\mathbf{h}G} \simeq (Y \otimes \mathbf{E}G_+)/G$ as the definition of the motivic homotopy orbits of Y .

Before explaining the intermediate results leading to the splitting theorem, we pause to point out an obvious, but important, difference between ordinary equivariant and motivic equivariant homotopy theory. In the topological case, equivalences are detected by the fixed point functors for subgroups $H \leq G$. This corresponds to the fact that a set of generators is given by the orbits G/H , or that the homotopy theory of G -spaces can be presented as presheaves of spaces on the category of G -orbits. On the other hand, generators for equivariant motivic homotopy theory are smooth schemes over B with a G -action. Orbits G/H are examples of smooth G -schemes over B , but of course there are many more. Equivalences between motivic

G -spaces or G -spectra are not detected by fixed points,¹ because smooth G -schemes cannot in any meaningful way be decomposed into pieces of the form $G/H \times X$ (where X has trivial action). However, by analyzing filtrations of equivariant motivic homotopy theory arising from localizations and colocalizations determined by families of subgroups, as in § 3, one can see that equivalences can be detected using only (desuspensions of) smooth G -schemes of very special form, namely those of the form $G \times_K X$ such that there is a normal subgroup $N \trianglelefteq K$ which acts trivially on X and the quotient K/N acts freely on X . These are the G -schemes whose stabilizers are concentrated at a single conjugacy class.

In addition to the six functor formalism established in [Hoy17], the proof of Theorem 1.1 relies on several new results for equivariant motivic homotopy theory, which should be of independent interest. As with the splitting theorem, there are versions for all of these results established relative to a normal subgroup $N \trianglelefteq G$; for simplicity we discuss here in the introduction only the results for $N = G$.

The first key ingredient which we need is the geometric fixed points functor, constructed in § 4.3. The geometric fixed points $X^{\Phi G}$ of a motivic G -spectrum X may be obtained as the G -fixed points of a suitable localization of X , namely one determined by smooth G -schemes with trivial action. This functor satisfies analogues of the main features of the geometric fixed points functor from ordinary equivariant motivic homotopy theory, as follows.

1. It is a symmetric monoidal left adjoint and $(\Sigma^\infty Y_+)^{\Phi G} \simeq \Sigma^\infty (Y_+^G)$ for any $Y \in \text{Sm}_B$.
2. $X^{\Phi G} \simeq (X \otimes \tilde{\mathbf{E}}\mathcal{P})^G$ where $\tilde{\mathbf{E}}\mathcal{P}$ is the unreduced suspension of the universal motivic G -space associated to the family \mathcal{P} of proper subgroups of G .
3. $X^{\Phi G} \simeq (X[a^{-1}])^G$, where a is the euler class $a : S^0 \rightarrow T^{\bar{\rho}_G}$ and $\bar{\rho}_G$ is the reduced regular representation. Here, given a representation V , we write T^V for the associate motivic sphere (i.e, its Thom space). The connection between the second two items is provided by a geometric presentation for universal motivic G -spaces for families, established in § 3, analogous to Morel-Voevodsky’s geometric description of classifying spaces. In particular, for the family of proper subgroups, what we find is that $\tilde{\mathbf{E}}\mathcal{P}$ may be described by the formula $\tilde{\mathbf{E}}\mathcal{P} \simeq T^{\infty \bar{\rho}_G} = \text{colim}_n T^{n \bar{\rho}_G}$, analogous to the familiar formula from topology.

Also of interest is that the construction of the geometric fixed points functor here permits a motivic version of the Tate square for C_p -equivariant motivic spectra in § 4.4. This is a homotopy pushout square of motivic spectra

$$\begin{array}{ccc} X^{C_p} & \longrightarrow & X^{\Phi C_p} \\ \downarrow & & \downarrow \\ X^{\mathbf{h}C_p} & \longrightarrow & X^{\mathbf{t}C_p}, \end{array}$$

where $X^{\mathbf{h}C_p}$ is a motivic version of the homotopy fixed points functor, defined using $\mathbf{E}C_p$.

A second key ingredient entering into the splitting theorem is the motivic Adams isomorphism, proved in Theorem 6.36 below. This fundamental result identifies the quotient of a free G -spectrum with its fixed points. There is a natural transformation from the former to the latter and the bulk of the work in the section § 6 is devoted to verifying that this morphism is an equivalence. Our strategy is to first check that this transformation is an equivalence on the full subcategory of dualizable free G -spectra. Of course, unless the base is a field of characteristic zero, this doesn’t suffice to conclude the result in general. But since $\mathbf{E}G_+$ is a colimit of dualizable spectra, it does imply that the fixed points of $\mathbf{E}G_+$ coincides with $\mathbf{B}G_+$. Using that $\mathbf{B}G_+$ contains $\mathbb{1}_B$ as a summand, this lets us define an inverse to the Adams transformation to obtain the general result. It is worth pointing out that if $f : T \rightarrow B$ is an étale torsor, then the Adams isomorphism for $f_{\#}\mathbb{1}_T$ is a straightforward consequence of ambidexterity, proved in [Hoy17], for the finite étale map f . An obvious strategy presents itself. If $q : X \rightarrow B$ is a smooth G -scheme over B with free action, then $g : X \rightarrow X/G$ is an étale torsor and $p : X/G \rightarrow B$ is smooth. Since $(g_{\#}\mathbb{1}_X)^G \simeq (g_{\#}\mathbb{1}_X)/G$ and $p_{\#}$ commutes

¹Of course one can define a homotopy theory which has this property, but as pointed out by Herrmann [Her13], equivariant algebraic K -theory is not representable in the resulting homotopy category.

with the quotient functor, to verify the Adams isomorphism for $p_{\#}(g_{\#}\mathbb{1}_X)$, it would suffice to check that the fixed points functor commutes with $p_{\#}$. Establishing this change-of-base formula directly appears to be as difficult as the Adams isomorphism itself and we actually obtain this base change formula as a *consequence* of the Adams isomorphism. It is interesting to note that from the viewpoint of motivic homotopy theory of stacks, this is an instance of smooth-proper base change formula, along the non-representable map $BG \rightarrow B$.

Once all of the foundational results are in place, the proof of the splitting theorem is actually fairly straightforward. It is not hard to write down the map Θ_X of the statement of the theorem and to check it is an equivalence it suffices to check it is an equivalence when X is concentrated at a single conjugacy class, a case which follows from the analysis in § 3 of localizations and colocalizations of equivariant motivic homotopy.

Outline: We utilize the language of ∞ -categories throughout this paper. We begin in § 2 by recalling the construction of the ∞ -categories of motivic G -spaces and motivic G -spectra from [Hoy17], as well as a few extensions used in this paper. In § 3 we study the colocalizations and localizations of equivariant motivic homotopy theory which are determined by a family. In § 4 we define fixed point functors and the geometric fixed points. In § 5 we define the quotient functor on N -free spectra and in § 6 we prove the motivic Adams isomorphism. Finally in § 7, we prove the motivic tom Dieck splitting theorem.

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Notation: Throughout, B is a quasi-compact, quasi-separated base scheme and G is a finite group whose order is invertible in \mathcal{O}_B . We write Sch_B^G for the category of G -schemes which are finitely presented and G -quasi-projective over B . For $S \in \text{Sch}_B^G$, write Sch_S^G for the slice category over S and $\text{Sm}_S^G \subseteq \text{Sch}_S^G$ for the full subcategory whose objects are smooth over S .

If \mathcal{E} is a locally free \mathcal{O}_S -module, we write

$$\mathbb{V}_S(\mathcal{E}) := \underline{\text{Spec}}(\text{Sym}(\mathcal{E}^\vee)) \quad \text{and} \quad \mathbb{P}_S(\mathcal{E}) := \underline{\text{Proj}}(\text{Sym}(\mathcal{E}^\vee))$$

respectively for the associated vector bundle scheme and the associated projective bundle on S . A representation of G over B will mean a locally free G -module on B , for a recollection of the definition, see [Tho87]. If M is a G -set we let

$$\rho_M = \mathcal{O}_B[M] := \mathcal{O}_B \otimes \mathbb{Z}[M]$$

denote the associated permutation representation. In particular, ρ_G is the regular representation.

We use the language of ∞ -categories in this paper and mostly follow the terminology in [Lur09, Lur17] with the exception that we write Cat_∞ for the ∞ -category of not necessarily small ∞ -categories. We write $\text{Map}_{\mathcal{C}}(x, y)$ for the space of maps in an ∞ -category \mathcal{C} and $\underline{\text{Map}}_{\mathcal{C}}(x, y)$ for the spectrum of maps in a stable ∞ -category. If \mathcal{C} is a closed symmetric monoidal ∞ -category, we write $F_{\mathcal{C}}(x, y)$ for the internal mapping object.

2 Equivariant motivic homotopy theory

We recall definitions and basic properties of equivariant motivic homotopy theory. We will use the ∞ -categorical approach to equivariant motivic spectra introduced by Hoyois [Hoy17]. Since we are working

with finite groups, the unstable homotopy category agrees with those constructed by Voevodsky [Del09] and Heller-Krishna-Østvær [HKØ15] and the stable homotopy category agrees with the one from [HVØ19, Appendix A.4].

2.1 Equivariant geometry

Let $S \in \text{Sch}_B^G$. If $\phi : G \rightarrow K$ is a group homomorphism, we write

$$\phi^{-1} : \text{Sch}_S^K \rightarrow \text{Sch}_{\phi^{-1}S}^G$$

for the restriction functor, which regards a K -scheme over S as a G -scheme over S via ϕ . When no confusion should arise, we write Sch_S^G instead of $\text{Sch}_{\phi^{-1}S}^G$.

Let $X \in \text{Sch}_B^G$. Say that G acts freely on X if the action of $G(T)$ on $X(T)$ is free for any $T \in \text{Sch}_B$. If G acts freely on X , then the fppf-quotient $(X/G)_{\text{fppf}}$ is representable by an object $X/G \in \text{Sch}_B$ (since all of our schemes are quasiprojective), see [Sta18, Tag 07S7]. Since G is smooth, the fppf-torsor $X \rightarrow X/G$ is an étale torsor (as it is a smooth map and so étale locally admits sections).

Definition 2.1. The stabilizer of a point $x \in X$ is the subgroup $\text{Stab}(x) \leq G$ defined by

$$\text{Stab}(x) = \{g \in G \mid g \cdot x = x \text{ and } g \text{ acts as id on } k(x)\}.$$

Then, G acts freely on X provided $\text{Stab}(x) = \{e\}$ for all $x \in X$. More generally, if $H \leq G$ is a subgroup which acts freely on X then the quotient X/H inherits an action of the Weyl group $W(H) = W_G(H) := N_G(H)/H$, so defines a functor $(-)/H : \text{Sch}_B^{G, H\text{-free}} \rightarrow \text{Sch}_B^{WH}$. This is the composite of the restriction functor $\text{Sch}_B^{G, H\text{-free}} \rightarrow \text{Sch}_B^{NH, H\text{-free}}$ and followed by the quotient $\text{Sch}_B^{NH, H\text{-free}} \rightarrow \text{Sch}_B^{WH}$.

Let $N \trianglelefteq G$ be a normal subgroup. Suppose that either

- (i) N acts trivially on S , or
- (ii) N acts freely on S .

In either case, the quotient functor yields a functor

$$\text{Sm}_S^{G, N\text{-free}} \rightarrow \text{Sm}_{S/N}^{G/N}.$$

Write $q : S \rightarrow S/N$ for the quotient map of schemes and $q^{-1} : \text{Sm}_{S/N}^{G/N} \rightarrow \text{Sm}_S^{G/N}$ for the functor defined by $q^{-1}(Y) = Y \times_{S/N} S$.

Proposition 2.2. *Suppose that N acts freely on S , then $(-)/N$ and $\pi^{-1}q^{-1}$ are inverse equivalences*

$$(-)/N : \text{Sm}_S^G \rightleftarrows \text{Sm}_{S/N}^{G/N} : \pi^{-1}q^{-1}.$$

Proof. Let $f : X \rightarrow S$ be in Sm_S^G . By descent we have a cartesian square in Sm_S (hence in Sm_S^G)

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow & & \downarrow \\ X/N & \longrightarrow & S/N. \end{array}$$

It follows that $(-)/N$ is fully faithful. It is also essentially surjective since if $Y \in \text{Sm}_{S/N}^{G/N}$ then $Y \cong (Y \times_{S/N} S)/N$. \square

Let $\iota : H \hookrightarrow G$ be a monomorphism of groups. The induction-restriction adjunction

$$\iota_! : \mathrm{Sch}_S^H \rightleftarrows \mathrm{Sch}_S^G : \iota^{-1}$$

restricts to an adjunction

$$\iota_! : \mathrm{Sm}_S^H \rightleftarrows \mathrm{Sm}_S^G : \iota^{-1}.$$

When $H \leq G$ is a subgroup and ι is the inclusion, we often write $\iota_!(X) = G \times_H X$ and this scheme is described concretely as follows. The scheme $G \times X$ becomes an H -scheme under the action $h(g, x) = (gh^{-1}, hx)$ and we define

$$G \times_H X = (G \times X)/H.$$

The scheme $G \times_H X$ has a left G -action through the action of G on itself. We can describe $G \times_H X$ in slightly more concrete terms as follows. Choose a complete set of left coset representatives g_i , then $G \times_H X = \coprod_{g_i} X_i$, each X_i is a copy of X and $g \in G$ acts as $k : X_i \rightarrow X_j$ where $k \in H$ satisfies $gg_i = g_jk$.

Let $X \in \mathrm{Sch}_B^G$. The presheaf of sets X^G is the presheaf of sets on Sch_B defined by

$$X^G(Y) = \{y \in X(Y) \mid y \text{ is fixed by } G\}.$$

If $X \rightarrow B$ is separated, the presheaf X^G is represented by a closed subscheme of X which is finitely presented over B , which is moreover smooth over B if X is, [DG71, Proposition XII.9.2, Corollaire XII.9.8] or [CGP10, Proposition A.8.10]. Note that if $H \leq G$ is a subgroup, the fixed point subscheme X^H comes equipped with an action of the Weyl group $W(H)$.

Now, suppose that $N \trianglelefteq G$ is a normal subgroup which acts trivially on S . Write $\pi : G \rightarrow G/N$ for the quotient map. Restricting action along π defines a functor $\pi^{-1} : \mathrm{Sch}_S^{G/N} \rightarrow \mathrm{Sch}_S^G$, which is left adjoint to fixed points. We will usually simply write again X instead of $\pi^{-1}X$, whenever context makes the meaning clear. Now, restricting attention to smooth S -schemes we obtain the adjunction

$$\pi^{-1} : \mathrm{Sm}_S^{G/N} \rightleftarrows \mathrm{Sm}_S^G : (-)^N.$$

2.2 Families of subgroups

Families of subgroups provide a convenient way to filter equivariant motivic homotopy theory.

Definition 2.3. A *family* \mathcal{F} of subgroups of G is a set of subgroups which is closed under taking subgroups and conjugation.

Example 2.4. The following families play an important role.

1. The trivial family $\mathcal{F}_{\mathrm{triv}} := \{e\}$.
2. The family of all subgroups $\mathcal{F}_{\mathrm{all}} := \{H \leq G \mid H \text{ is a subgroup}\}$.
3. The family of proper subgroups $\mathcal{P} := \{H \leq G \mid H \text{ is a proper subgroup}\}$.
4. For a normal subgroup $N \trianglelefteq G$, define $\mathcal{F}[N] := \{H \leq G \mid N \not\subseteq H\}$. Note that $\mathcal{P} = \mathcal{F}[G]$.
5. For a normal subgroup $N \trianglelefteq G$, define $\mathcal{F}(N) := \{H \leq G \mid H \cap N = \{e\}\}$. Note that $\mathcal{F}(G) = \mathcal{F}_{\mathrm{triv}}$.

If \mathcal{F} is a family, we write $\mathrm{co}(\mathcal{F}) := \mathcal{F}_{\mathrm{all}} - \mathcal{F}$ for its complement. Note that $\mathrm{co}(\mathcal{F})$ is not a family.²

Remark 2.5. A family of subgroups can be equivalently viewed as a sieve on the orbit category Orb^G . The sieve corresponding to \mathcal{F} is the full subcategory $\mathrm{Orb}^G[\mathcal{F}] \subseteq \mathrm{Orb}^G$ of orbits such that $G/H \in \mathrm{Orb}^G[\mathcal{F}]$ if and only if $H \in \mathcal{F}$.

²Rather, it is a *cofamily*, meaning it is closed under conjugation and $K \in \mathrm{co}(\mathcal{F})$ whenever K contains a subgroup in $\mathrm{co}(\mathcal{F})$

A family \mathcal{F} determines a sieve on Sm_S^G by letting $\mathrm{Sm}_S^G[\mathcal{F}] \subseteq \mathrm{Sm}_S^G$ be the full subcategory whose objects are smooth G -schemes over S such that all stabilizers are contained in \mathcal{F} . It is useful to make the following more general definition.

Definition 2.6. Let \mathcal{E} be a set of subgroups of G which is closed under conjugacy. Write $\mathrm{Sm}_S^G[\mathcal{E}] \subseteq \mathrm{Sm}_S^G$ for the full subcategory whose objects are those smooth G -schemes X over S such that $\mathrm{Stab}(x) \in \mathcal{E}$ for every point $x \in X$.

Notation 2.7. Let $X \in \mathrm{Sch}_S^G$. Write

$$X^{\mathcal{F}} := \bigcup_{H \in \mathrm{co}(\mathcal{F})} X^H$$

and

$$X(\mathcal{F}) := X - X^{\mathcal{F}}.$$

The subset $X(\mathcal{F}) \subseteq X$ is the set of points whose stabilizers are in \mathcal{F} . Observe that $X(\mathcal{F}) \subseteq X$ an open invariant subscheme and $X^{\mathcal{F}} \subseteq X$ is a closed invariant subscheme since $X^{\mathcal{F}}$ is a finite union of closed subschemes.

2.3 Motivic G -spaces

Recall that the Nisnevich topology can be defined via a cd -structure.

Definition 2.8 ([Voe10]). Let \mathcal{C} be a small category which has an initial object \emptyset .

1. A cd -structure on \mathcal{C} is a collection \mathcal{A} of commutative squares in \mathcal{C} such that if $Q \in \mathcal{A}$ and if Q' is isomorphic to Q , then $Q' \in \mathcal{A}$.

2. Given a cd -structure \mathcal{A} in \mathcal{C} , the Grothendieck topology $t_{\mathcal{A}}$ generated by \mathcal{A} is the smallest topology on \mathcal{C} such that

- (a) the empty sieve is a covering sieve of \emptyset , and
- (b) given any square in \mathcal{A}

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X, \end{array}$$

the sieve generated by $\{U \rightarrow X, Y \rightarrow X\}$ is a covering sieve.

Definition 2.9. An equivariant map $f : Y \rightarrow X$ is said to be *fixed point reflecting* at $y \in Y$ if f induces an isomorphism $\mathrm{Stab}(y) \cong \mathrm{Stab}(f(y))$. If this condition holds at every $y \in Y$, then f is simply said to be *fixed point reflecting*.

Let $\mathcal{C}_S \subseteq \mathrm{Sch}_S^G$ be a full subcategory containing \emptyset . We often require \mathcal{C}_S to satisfy one or both of the following properties:

- (P) If $Y \rightarrow X$ is fixed point reflecting and étale, then $Y \in \mathcal{C}_S$ whenever $X \in \mathcal{C}_S$.
- (H) If $X \in \mathcal{C}_S$ then so is $X \times_S \mathbb{A}^1$.

Our primary examples of interest are the categories $\mathrm{Sm}_S^G[\mathcal{E}]$, where \mathcal{E} is a set of subgroups closed under conjugacy. More generally, we could also consider the following property.

- (P') If $Y \rightarrow X$ is an equivariant étale map, then $Y \in \mathcal{C}_S$ whenever $X \in \mathcal{C}_S$.

The condition (P) on \mathcal{C}_S guarantees that the fixed point Nisnevich cd -structure (defined below) is complete while the condition (P') guarantees that the Nisnevich cd -structure is complete. Some categories of interest in this paper, e.g., $\mathrm{Sm}_S^G[\mathrm{co}(\mathcal{F})]$ for a family \mathcal{F} , do not satisfy (P') but do satisfy the weaker property (P). We will see below in [Proposition 2.13](#) that when \mathcal{C}_S satisfies (P') then the topology associated to the fixed point Nisnevich cd -structure coincides with the Nisnevich topology.

Definition 2.10. Let $\mathcal{C}_S \subseteq \mathrm{Sch}_S^G$ be a full subcategory containing \emptyset .

1. The *Nisnevich cd -structure* Nis on \mathcal{C}_S consists of cartesian squares

$$\begin{array}{ccc} V & \hookrightarrow & Y \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X, \end{array} \quad (2.11)$$

where j is open immersion, p is étale, and the map $(Y - V)_{\mathrm{red}} \rightarrow (X - U)_{\mathrm{red}}$ is an isomorphism in Sch_S .

2. The *fixed point Nisnevich cd -structure* fpNis on \mathcal{C}_S consists of cartesian squares as in the Nisnevich cd -structure, but with the added condition that p is a fixed point reflecting étale map.

Remark 2.12. In general, if G is a flat group scheme over B , one may choose a scheme structure on $Z := X \setminus U$ so that Z is invariant under the G -action [[Hoy17](#), Lemma 2.1]. Since G is a finite discrete group, Z_{red} is invariant and the map $p^{-1}(Z)_{\mathrm{red}} \rightarrow Z_{\mathrm{red}}$ is equivariant.

Proposition 2.13. *Suppose that \mathcal{C}_S satisfies (P'). The topology t_{fpNis} coincides with the Nisnevich topology on \mathcal{C}_S .*

Proof. Every t_{fpNis} -cover is a Nisnevich cover. We show that the reverse implication holds. It suffices to show that any Nisnevich square

$$\begin{array}{ccc} V & \hookrightarrow & Y \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X. \end{array}$$

admits a t_{fpNis} -refinement.

Write $\mathrm{fpr}(Y) \subseteq Y$ for the set of points where p is fixed point reflecting. Since $Y \rightarrow X$ is unramified and $\mathrm{Stab}(X) \rightarrow X$ is universally closed, [[Ryd13](#), Proposition 3.5] applies to show that the set $\mathrm{fpr}(Y) \subseteq Y$ is an invariant open subset. We have that $Y \setminus V \subseteq \mathrm{fpr}(Y)$. It follows that the outer square

$$\begin{array}{ccc} \mathrm{fpr}(V) & \hookrightarrow & \mathrm{fpr}(Y) \\ \downarrow & & \downarrow i \\ V & \hookrightarrow & Y \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array} \quad (2.14)$$

is a fixed point Nisnevich square (as is the top square). In particular, $\{j, pi\}$ is a t_{fpNis} -cover which refines $\{j, p\}$. \square

Write $\mathcal{P}(\mathcal{C}_S)$ for the ∞ -category of presheaves of spaces on \mathcal{C}_S . Note that \mathcal{C}_S does not necessarily contain a terminal object, in particular a terminal object $\mathrm{pt} \in \mathcal{P}(\mathcal{C}_S)$ is in general not representable. We write $\mathrm{pt} \in \mathcal{P}(\mathcal{C}_S)$ for this terminal object. Of course if $S \in \mathcal{C}_S$ then pt is representable; it is the presheaf represented by S .

Definition 2.15. Say that $F \in \mathcal{P}(\mathcal{C}_S)$ is *Nisnevich excisive* if

1. $F(\emptyset)$ is contractible, and
2. for any Nisnevich square (2.11) in \mathcal{C}_S , the square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U) \end{array}$$

is cartesian.

Write $\mathrm{Shv}_{\mathrm{Nis}}(\mathcal{C}_S) \subseteq \mathcal{P}(\mathcal{C}_S)$ for the subcategory of Nisnevich excisive presheaves of spaces on \mathcal{C}_S .

Temporarily say that F is “fixed point Nisnevich excisive” if $F(\emptyset) \simeq \mathrm{pt}$ and $F(\mathcal{Q})$ is cartesian for any fixed point Nisnevich square \mathcal{Q} . There is no real need for this extra terminology by the following proposition.

Proposition 2.16. *Let $F \in \mathcal{P}(\mathcal{C}_S)$ and suppose \mathcal{C}_S satisfies (P'). Then the following are equivalent.*

1. F is fixed point Nisnevich excisive.
2. F is Nisnevich excisive.
3. F is a sheaf for the Nisnevich topology.

Proof. If \mathcal{C}_S satisfies (P'), then the cd -structures fpNis and Nis both satisfy the conditions of [AHW17, Theorem 3.2.5], i.e., they are complete and regular, in the terminology of [Voe10]. Together with Proposition 2.13, it follows that (1), (2) and (3) are equivalent. □

Definition 2.17. Let $\mathcal{C}_S \subseteq \mathrm{Sch}_S$ be a full subcategory which contains \emptyset and satisfies properties (P) and (H). Say that $F \in \mathcal{P}(\mathcal{C}_S)$ is \mathbb{A}^1 -homotopy invariant if for any $Y \in \mathcal{C}_S$, the projection map $\pi : Y \times \mathbb{A}^1 \rightarrow Y$ induces an equivalence $F(Y) \simeq F(Y \times \mathbb{A}^1)$. We write $\mathcal{P}_{\mathbb{A}^1}(\mathcal{C}_S) \subseteq \mathcal{P}(\mathcal{C}_S)$ for the full subcategory consisting of the \mathbb{A}^1 -homotopy invariant presheaves.

The property that a presheaf is Nisnevich excisive is defined by a small set of conditions. It follows from [Lur09, Section 5.5.4] that the inclusion $\mathrm{Shv}_{\mathrm{Nis}}(\mathcal{C}_S) \subseteq \mathcal{P}(\mathcal{C}_S)$ is an accessible localization. Write L_{Nis} for the resulting localization endofunctor on $\mathcal{P}(\mathcal{C}_S)$. Moreover, this localization is *left-exact* in the sense of [Lur09, Section 6.2.2].

Similarly, the property that a presheaf is \mathbb{A}^1 -homotopy invariant is defined by a small set of conditions, so that the inclusion $\mathcal{P}_{\mathbb{A}^1}(\mathcal{C}_S) \subseteq \mathcal{P}(\mathcal{C}_S)$ is also an accessible localization. Write $L_{\mathbb{A}^1}$ for the resulting localization endofunctor. It can be described explicitly by the formula $L_{\mathbb{A}^1} \simeq \mathrm{Sing}_{\mathbb{A}^1}$, where

$$\mathrm{Sing}_{\mathbb{A}^1}(F)(U) := \mathrm{colim}_{\Delta} (n \mapsto F(U \times \Delta_S^n)); \quad (2.18)$$

see, for instance, [AE17, Theorem 4.25] for details. A map $f : F_1 \rightarrow F_2$ in $\mathrm{Shv}_{\mathrm{Nis}}(\mathcal{C}_S)$ is a *Nisnevich equivalence* provided $L_{\mathrm{Nis}}(f)$ is an equivalence and an \mathbb{A}^1 -equivalence provided $L_{\mathbb{A}^1}(f)$ is an equivalence.

Remark 2.19. Say that $F \in \mathcal{P}(\mathcal{C}_S)$ is *strongly \mathbb{A}^1 -homotopy invariant* if for any projection $E \rightarrow X$ of a G -affine bundle in \mathcal{C}_S , the induced map is an equivalence $F(X) \simeq F(E)$. Any $X \in \mathrm{Sm}_S^G$ is Nisnevich locally affine. This implies that if F is Nisnevich excisive, then F is strongly \mathbb{A}^1 -invariant if and only if it is \mathbb{A}^1 -homotopy invariant (as $E \rightarrow X$ always has local sections in this case), see [Hoy17, Remark 3.13].

In particular, the motivic localization considered here agrees with the one in [Hoy17].

Definition 2.20. 1. A *motivic G -space over S* is a Nisnevich excisive and \mathbb{A}^1 -homotopy invariant presheaf $F \in \mathcal{P}(\mathrm{Sm}_S^G)$.

2. Write $\mathrm{Spc}^G(S)$ for the ∞ -category of motivic G -spaces. The category of based motivic G -spaces is $\mathrm{Spc}_\bullet^G(S) = \mathrm{Spc}^G(S)_{\mathrm{pt}/}$.

3. More generally, write $\mathrm{Spc}(\mathcal{C}_S)$ for the ∞ -categories of Nisnevich excisive \mathbb{A}^1 -homotopy invariant presheaves on \mathcal{C}_S and $\mathrm{Spc}_\bullet(\mathcal{C}_S) = \mathrm{Spc}(\mathcal{C}_S)_{\mathrm{pt}/}$.

When \mathcal{E} is a set of subgroups, closed under conjugacy, we use the notation

$$\begin{aligned}\mathrm{Spc}^{G,\mathcal{E}}(S) &:= \mathrm{Spc}(\mathrm{Sm}_S^G[\mathcal{E}]) \\ \mathrm{Spc}_\bullet^{G,\mathcal{E}}(S) &:= \mathrm{Spc}_\bullet(\mathrm{Sm}_S^G[\mathcal{E}]).\end{aligned}$$

The inclusion $\mathrm{Spc}(\mathcal{C}_S) \subseteq \mathcal{P}(\mathcal{C}_S)$ is an accessible localization and we write

$$L_{\mathrm{mot}} : \mathcal{P}(\mathcal{C}_S) \rightarrow \mathrm{Spc}(\mathcal{C}_S)$$

for the corresponding localization endofunctor. The motivic localization functor may be computed by the formula [MV99, Lemma 3.2.6]

$$L_{\mathrm{mot}}(F) = L_{\mathrm{Nis}} \operatorname{colim}_{n \rightarrow \infty} (L_{\mathbb{A}^1} \circ L_{\mathrm{Nis}})^n(F). \quad (2.21)$$

Proposition 2.22. *The motivic localization functor $L_{\mathrm{mot}} : \mathcal{P}(\mathcal{C}_S) \rightarrow \mathrm{Spc}(\mathcal{C}_S)$ is locally cartesian³ and preserves finite products. In particular, colimits in $\mathrm{Spc}^G(\mathcal{C}_S)$ are universal.*

Proof. The localization functors $L_{\mathrm{Nis}}, L_{\mathbb{A}^1}$ satisfy these properties and therefore so does L_{mot} using (2.21). \square

We will make use of the notion of the tensor product of presentable ∞ -categories as defined and studied in [Lur17, Section 4.8.1], see especially [Lur17, Proposition 4.8.1.17]. The category $\mathrm{Spc}(\mathcal{C}_S)$ is cartesian monoidal (i.e., it is symmetric monoidal with respect to the cartesian product). The symmetric monoidal product on $\mathrm{Spc}(\mathcal{C}_S)$ extends to one on

$$\mathrm{Spc}_\bullet(\mathcal{C}_S) \simeq \mathrm{Spc}(\mathcal{C}_S) \otimes \mathcal{S}_\bullet,$$

where $\mathcal{S}_\bullet := \mathcal{S}_{\mathrm{pt}/}$ denotes the ∞ -category of pointed spaces, see [GGN15, Lemma 3.6] and [Lur17, Proposition 4.8.2.11]. We write \wedge for the symmetric monoidal product on $\mathrm{Spc}_\bullet(\mathcal{C}_S)$. Sometimes we will need to use the symmetric monoidal product on $\mathcal{P}_\bullet(\mathcal{C}_S) := \mathcal{P}(\mathcal{C}_S)_{\mathrm{pt}/}$, which we denote $\wedge^{\mathcal{P}}$ to avoid confusion. The symmetric monoidal localization functor $L_{\mathrm{mot}} : \mathcal{P}(\mathcal{C}_S) \rightarrow \mathrm{Spc}(\mathcal{C}_S)$ induces a unique symmetric monoidal localization functor $L_{\mathrm{mot}} : \mathcal{P}_\bullet(\mathcal{C}_S) \rightarrow \mathrm{Spc}_\bullet(\mathcal{C}_S)$. In particular, given $X, Y \in \mathcal{P}_\bullet(\mathcal{C}_S)$, then

$$L_{\mathrm{mot}}(X) \wedge L_{\mathrm{mot}}(Y) \simeq L_{\mathrm{mot}}(X \wedge^{\mathcal{P}} Y).$$

Given a functor $u : \mathcal{C} \rightarrow \mathcal{D}$, the functor $u^* : \mathcal{P}(\mathcal{D}) \rightarrow \mathcal{P}(\mathcal{C})$, defined by precomposition with u , preserves small limits and colimits, as these are computed objectwise in presheaf ∞ -categories [Lur09, Corollary 5.1.2.3], which are presentable ∞ -categories [Lur09, Theorem 5.5.1.1]. It follows from [Lur09, Corollary 5.5.2.9] that u^* admits both a left adjoint $u_!$ as well as a right adjoint u_* ,

$$\begin{array}{ccc} & u_! & \\ \mathcal{P}(\mathcal{C}) & \xleftarrow{u^*} & \mathcal{P}(\mathcal{D}) \\ & u_* & \end{array}$$

³A localization endofunctor $L : \mathcal{D} \rightarrow \mathcal{D}$ is *locally cartesian* if $L(A \times_B X) \rightarrow A \times_B L(X)$ is an equivalence for any maps $A \rightarrow B, X \rightarrow B$ in \mathcal{D} where $A, B \in L(\mathcal{D})$ [GK17, §1].

A functor $u : \mathcal{C} \rightarrow \mathcal{D}$ between small ∞ -categories equipped with Grothendieck topologies is called *topologically cocontinuous*⁴ if for every $Y \in \mathcal{C}$ and every covering sieve R on $u(Y)$, the sieve $u^*R \times_{u^*u_1Y} Y \hookrightarrow Y$, consisting of arrows $Z \rightarrow Y$ such that $uZ \rightarrow uY$ factors through R , is a covering sieve on Y .

Lemma 2.23. *Let $u : \mathcal{C} \rightarrow \mathcal{D}$ be a topologically cocontinuous functor between small ∞ -categories equipped with Grothendieck topologies $\tau_{\mathcal{C}}$ and $\tau_{\mathcal{D}}$ respectively. Then $u^* : \mathcal{P}(\mathcal{D}) \rightarrow \mathcal{P}(\mathcal{C})$ preserves all $\tau_{\mathcal{D}}$ -local equivalences.*

Proof. The set of $\tau_{\mathcal{D}}$ -local equivalences in $\mathcal{P}(\mathcal{D})$ is the closure under pushouts, small colimits, and 2-out-of-3, of the set of maps $R \hookrightarrow X$, where R is a covering sieve on $X \in \mathcal{D}$. Since u^* preserves colimits, it suffices to show that $u^*R \rightarrow u^*X$ is a $\tau_{\mathcal{C}}$ -local equivalence. Since colimits are universal in $\mathcal{P}(\mathcal{C})$, it suffices to show that $u^*R \times_{u^*X} Y \rightarrow Y$ is an equivalence for any map $Y \rightarrow u^*X$ where $Y \in \mathcal{C}$. To see this, note that it is in fact a covering sieve. Indeed, $R \times_X uY \rightarrow uY$ is a covering sieve and therefore $u^*(R \times_X uY) \times_{u^*u_1Y} Y \simeq u^*R \times_{u^*X} Y \rightarrow Y$ is covering since u is cocontinuous. \square

For the remainder of this section

$$u : \mathcal{C}_S \subseteq \mathcal{D}_S \tag{2.24}$$

is the inclusion between full subcategories of Sch_S^G , both containing \emptyset and both satisfying properties (P) and (H). Main examples of interest to keep in mind are the inclusions $\text{Sm}_S^G[\mathcal{E}] \subseteq \text{Sm}_S^G[\mathcal{E}']$.

Lemma 2.25. *Let $u : \mathcal{C}_S \subseteq \mathcal{D}_S$ be as in (2.24). Then u is topologically cocontinuous.*

Proof. Let $X \in \mathcal{C}_S$. Since the cd -structure fpNis on each of these categories is complete, the covering sieves on X and on $u(X)$, in both cases, are exactly those which contain the sieve generated by a simple covering, see [Voe10, Section 2]. Property (P) implies that any simple covering of X in \mathcal{D}_S is a simple covering in \mathcal{C}_S . It follows that the pullback u^*R of any covering sieve R is again a covering sieve. \square

Proposition 2.26. *Let $u : \mathcal{C}_S \subseteq \mathcal{D}_S$ be as in (2.24).*

1. *The functor $u_! : \mathcal{P}(\mathcal{C}_S) \rightarrow \mathcal{P}(\mathcal{D}_S)$ preserves all Nisnevich and all motivic equivalences.*
2. *The functor $u^* : \mathcal{P}(\mathcal{D}_S) \rightarrow \mathcal{P}(\mathcal{C}_S)$ preserves all Nisnevich and all motivic equivalences.*

Proof. Recall that if \mathcal{A} is a presentable ∞ -category and S is a set of morphisms in \mathcal{A} , then the class of S -local equivalences is the closure of S under pushouts, small colimits, and 2-out-of-3, see e.g., [Lur09, Proposition 5.5.4.15].

The first statement then follows from the fact that $u_!$ preserves colimits, fixed point Nisnevich squares, and that $u_!(X \times \mathbb{A}^1) \simeq u_!(X) \times \mathbb{A}^1$. The second statement follows from Lemma 2.23, Lemma 2.25, and that $u^*(X \times \mathbb{A}^1) \simeq u^*(X) \times \mathbb{A}^1$. \square

Corollary 2.27. *Let $u : \mathcal{C}_S \subseteq \mathcal{D}_S$ be as above. There are natural equivalences $u^* \circ L_{\text{Nis}} \simeq L_{\text{Nis}} \circ u^*$ and $u^* \circ L_{\text{mot}} \simeq L_{\text{mot}} \circ u^*$ of functors $\mathcal{P}(\mathcal{D}_S) \rightarrow \mathcal{P}(\mathcal{C}_S)$.*

\square

The adjoint pairs $(u_!, u^*)$ and (u^*, u_*) extend to adjoint pairs on \mathbb{A}^1 -invariant Nisnevich sheaves. Overloading notation, we continue to write $u_!, u^*, u_*$ for the induced functors on categories of \mathbb{A}^1 -invariant Nisnevich sheaves.

Proposition 2.28. *Let $u : \mathcal{C}_S \subseteq \mathcal{D}_S$ be as in (2.24).*

1. *The restriction functor $u^* : \mathcal{P}(\mathcal{D}_S) \rightarrow \mathcal{P}(\mathcal{C}_S)$ preserves \mathbb{A}^1 -invariant Nisnevich sheaves.*
2. *The induced functor $u^* : \text{Spc}(\mathcal{D}_S) \rightarrow \text{Spc}(\mathcal{C}_S)$ is symmetric monoidal and has a left adjoint $u_!$ and a right adjoint u_* .*

⁴This is called ‘‘cocontinuous’’ in [AGV71, III.2.1]. We follow the terminology in [Kha16] to avoid confusion with the category theorist’s terminology ‘‘cocontinuous functor’’.

3. Similarly $u^* : \mathrm{Spc}_\bullet(\mathcal{D}_S) \rightarrow \mathrm{Spc}_\bullet(\mathcal{C}_S)$ is symmetric monoidal and has a left and a right adjoint, which we again denote respectively by $u_!$ and u_* .

Moreover, these functors fit into commutative diagrams

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{C}_S) & \xrightleftharpoons[u^*]{u_!} & \mathcal{P}(\mathcal{D}_S) & & \mathcal{P}(\mathcal{D}_S) & \xrightleftharpoons[u_*]{u^*} & \mathcal{P}(\mathcal{C}_S) \\
\mathrm{L}_{\mathrm{mot}} \downarrow \uparrow & & \mathrm{L}_{\mathrm{mot}} \downarrow \uparrow & & \mathrm{L}_{\mathrm{mot}} \downarrow \uparrow & & \mathrm{L}_{\mathrm{mot}} \downarrow \uparrow \\
\mathrm{Spc}(\mathcal{C}_S) & \xrightleftharpoons[u^*]{u_!} & \mathrm{Spc}(\mathcal{D}_S) & \text{and} & \mathrm{Spc}(\mathcal{D}_S) & \xrightleftharpoons[u_*]{u^*} & \mathrm{Spc}(\mathcal{C}_S) \\
(-)_+ \downarrow \uparrow & & (-)_+ \downarrow \uparrow & & (-)_+ \downarrow \uparrow & & (-)_+ \downarrow \uparrow \\
\mathrm{Spc}_\bullet(\mathcal{C}_S) & \xrightleftharpoons[u^*]{u_!} & \mathrm{Spc}_\bullet(\mathcal{D}_S) & & \mathrm{Spc}_\bullet(\mathcal{D}_S) & \xrightleftharpoons[u_*]{u^*} & \mathrm{Spc}_\bullet(\mathcal{C}_S).
\end{array}$$

Proof. Since $u_!$ preserves fixed point Nisnevich squares and $u_!(X \times \mathbb{A}^1) \simeq u_!(X) \times \mathbb{A}^1$, it follows that u^* preserves \mathbb{A}^1 -invariant Nisnevich sheaves on \mathcal{C}_S and thus restricts to a limit preserving functor $u^* : \mathrm{Spc}(\mathcal{D}_S) \rightarrow \mathrm{Spc}(\mathcal{C}_S)$. As these categories are presentable, u^* has a left adjoint $u_!$. It follows from [Proposition 2.26](#) that $u^* : \mathrm{Spc}(\mathcal{D}_S) \rightarrow \mathrm{Spc}(\mathcal{C}_S)$ preserves colimits. In particular it has a right adjoint u_* . Note that u^* is symmetric monoidal since it preserves limits and $\mathrm{Spc}(\mathcal{D}_S)$ is cartesian monoidal.

The functors u^* and u_* preserve final objects so induce adjoint pairs on based spaces. Since u^* preserves limits, it has a left adjoint $u_!$. It is straightforward to verify that these fit into the displayed commutative diagrams. □

Remark 2.29. Suppose that \mathcal{C}_S and \mathcal{D}_S are also closed under binary products. Then $u_! : \mathrm{Spc}(\mathcal{C}_S) \rightarrow \mathrm{Spc}(\mathcal{D}_S)$ preserves binary products (the monoidal product on these categories). However, $u_!$ is not in general a symmetric monoidal functor since it does not always preserve the unit object pt (because it is not in general representable) but there is always a canonical map $u_!(\mathrm{pt}) \rightarrow \mathrm{pt}$ adjoint to the equivalence $\mathrm{pt} \simeq u^*(\mathrm{pt})$. Similarly, $u_! : \mathrm{Spc}_\bullet(\mathcal{C}_S) \rightarrow \mathrm{Spc}_\bullet(\mathcal{D}_S)$ preserves the smash product, but need not preserve the unit object $S^0 \simeq \mathrm{pt} \amalg \mathrm{pt}$, though again there is a canonical map $u_!(S^0) \rightarrow S^0$.

Proposition 2.30. *Let $u : \mathcal{C}_S \subseteq \mathcal{D}_S$ be as above. The functors*

$$\begin{aligned}
u_!, u_* &: \mathrm{Spc}(\mathcal{C}_S) \rightarrow \mathrm{Spc}(\mathcal{D}_S) \\
u_!, u_* &: \mathrm{Spc}_\bullet(\mathcal{C}_S) \rightarrow \mathrm{Spc}_\bullet(\mathcal{D}_S)
\end{aligned}$$

are all full and faithful. In particular if \mathcal{E} is a family of subgroups closed under conjugacy and $u : \mathrm{Sm}_S^G[\mathcal{E}] \subseteq \mathrm{Sm}_S^G$ is the inclusion, $u_!, u_ : \mathrm{Spc}^{G, \mathcal{E}}(S) \rightarrow \mathrm{Spc}^G(S)$ and $u_!, u_* : \mathrm{Spc}_\bullet^{G, \mathcal{E}}(S) \rightarrow \mathrm{Spc}_\bullet^G(S)$ are full and faithful.*

Proof. First we note that the unit of the adjunction $u_!^{\mathcal{P}} : \mathcal{P}(\mathcal{C}_S) \rightleftarrows \mathcal{P}(\mathcal{D}_S) : u^*$ is an equivalence $\eta : \mathrm{id} \simeq u^* u_!^{\mathcal{P}}$ and the counit of $u^* : \mathcal{P}(\mathcal{D}_S) \rightleftarrows \mathcal{P}(\mathcal{C}_S) : u_*$ is an equivalence $\epsilon : u^* u_* \simeq \mathrm{id}$. Indeed $u_!^{\mathcal{P}} F$ and $u_* F$ are respectively computed by the left and right Kan extensions of F along u and so $u_!^{\mathcal{P}} F(W) \simeq \mathrm{colim}_{W \rightarrow X} F(X)$ and $u_* F(W) \simeq \mathrm{lim}_{X \rightarrow W} F(X)$. Here the indexing categories are respectively the categories of morphisms $W \rightarrow X$ and $X \rightarrow W$ where $X \in \mathrm{Sm}_S^G[\mathcal{E}]$. In particular, if $W \in \mathrm{Sm}_S^G[\mathcal{E}]$ these have an initial, respectively terminal, object and so $\mathrm{id} \simeq u^* u_!^{\mathcal{P}}$ and $u^* u_* \simeq \mathrm{id}$, as claimed.

The counit of $u^* : \mathrm{Spc}(\mathcal{D}_S) \rightleftarrows \mathrm{Spc}(\mathcal{C}_S) : u_*$ is thus an equivalence as $u^* u_* \simeq \mathrm{id}$. To show that the unit of $u_! : \mathrm{Spc}(\mathcal{C}_S) \rightleftarrows \mathrm{Spc}(\mathcal{D}_S) : u^*$ is an equivalence $\mathrm{id} \simeq u^* u_!$, we note that by [Corollary 2.27](#) (and writing $\iota : \mathrm{Spc}^G(\mathcal{C}_S) \subseteq \mathcal{P}(\mathcal{C}_S)$ for the inclusion) we have $u^* u_! \simeq u^* \mathrm{L}_{\mathrm{mot}} u_!^{\mathcal{P}} \iota \simeq \mathrm{L}_{\mathrm{mot}} u^* u_!^{\mathcal{P}} \iota \simeq \mathrm{L}_{\mathrm{mot}} \iota \simeq \mathrm{id}$. Thus both of the functors $u_!, u_* : \mathrm{Spc}(\mathcal{C}_S) \rightarrow \mathrm{Spc}(\mathcal{D}_S)$ are full and faithful.

The pointed cases follow immediately from the above considerations. □

Let $\iota : H \hookrightarrow G$ be a group monomorphism. Let $\mathcal{C}_S^H \subseteq \text{Sch}_S^H$ and $\mathcal{C}_S^G \subseteq \text{Sch}_S^G$ be full subcategories satisfying (P) and (H), as above. Suppose further that the induction-restriction adjunction $\iota_! : \text{Sch}_S^H \rightleftarrows \text{Sch}_S^G : \iota^*$ restricts to the adjunction

$$\iota_! : \mathcal{C}_S^H \rightleftarrows \mathcal{C}_S^G : \iota^*.$$

Lemma 2.31. *Let $\iota, \mathcal{C}_S^G, \mathcal{C}_S^H$ be as above.*

1. *The restriction functor $\iota^* : \mathcal{P}(\mathcal{C}_S^G) \rightarrow \mathcal{P}(\mathcal{C}_S^H)$ preserves \mathbb{A}^1 -invariant Nisnevich sheaves.*
2. *The induced functor $\iota^* : \text{Spc}(\mathcal{C}_S^G) \rightarrow \text{Spc}(\mathcal{C}_S^H)$ is symmetric monoidal and has a left adjoint $\iota_!$ and a right adjoint ι_* .*
3. *Similarly $\iota^* : \text{Spc}_\bullet(\mathcal{C}_S^G) \rightarrow \text{Spc}_\bullet(\mathcal{C}_S^H)$ is symmetric monoidal and has a left and a right adjoint, which we again denote respectively by $\iota_!$ and ι_* .*

Moreover, these functors fit into commutative diagrams

$$\begin{array}{ccc} \mathcal{C}_S^H & \xrightarrow{\iota_!} & \mathcal{C}_S^G \\ \downarrow & & \downarrow \\ \text{Spc}(\mathcal{C}_S^H) & \xleftarrow[\iota^*]{\iota_!} & \text{Spc}(\mathcal{C}_S^G) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{C}_S^G & \xrightarrow{\iota^*} & \mathcal{C}_S^H \\ \downarrow & & \downarrow \\ \text{Spc}(\mathcal{C}_S^G) & \xleftarrow[\iota_*]{\iota^*} & \text{Spc}(\mathcal{C}_S^H) \end{array}$$

$$\begin{array}{ccc} \text{Spc}_\bullet(\mathcal{C}_S^H) & \xleftarrow[\iota_*]{\iota_!} & \text{Spc}_\bullet(\mathcal{C}_S^G) \\ \downarrow & & \downarrow \\ \text{Spc}_\bullet(\mathcal{C}_S^H) & \xleftarrow[\iota_*]{\iota_!} & \text{Spc}_\bullet(\mathcal{C}_S^G) \end{array}$$

Proof. The functors $\iota_! : \mathcal{C}_S^H \rightarrow \mathcal{C}_S^G$, $\iota^* : \mathcal{C}_S^G \rightarrow \mathcal{C}_S^H$ both send distinguished squares to distinguished squares and $\iota_!(X) \times \mathbb{A}^1 \cong \iota_!(X \times \mathbb{A}^1)$ and $\iota^*(X) \times \mathbb{A}^1 \cong \iota^*(X \times \mathbb{A}^1)$. It follows that $\iota_! : \mathcal{P}(\mathcal{C}_S^H) \rightarrow \mathcal{P}(\mathcal{C}_S^G)$ and $\iota^* : \mathcal{P}(\mathcal{C}_S^G) \rightarrow \mathcal{P}(\mathcal{C}_S^H)$ preserve all motivic equivalences. It follows that these induce functors on the category of motivic spaces as displayed above. Since $\iota^* : \text{Spc}(\mathcal{C}_S^G) \rightarrow \text{Spc}(\mathcal{C}_S^H)$ preserves limits and these categories are cartesian monoidal, it follows that ι^* is symmetric monoidal.

Since ι^*, ι_* preserve final objects they induce an adjoint pair on based spaces. Since ι^* preserves limits, it has a left adjoint $\iota_!$. It is straightforward to verify that these fit into commutative diagrams as displayed. \square

Let $f : T \rightarrow S$ be a map in Sch_B^G . Let $\mathcal{C}_T^G \subseteq \text{Sch}_T^G$ and $\mathcal{C}_S^G \subseteq \text{Sch}_S^G$ be full subcategories satisfying (P) and (H), as above. Suppose that the base change $f^{-1} : \text{Sch}_S^G \rightarrow \text{Sch}_T^G$ restricts to a functor $f^{-1} : \mathcal{C}_S^G \rightarrow \mathcal{C}_T^G$. We write $f_* : \mathcal{P}(\mathcal{C}_T^G) \rightarrow \mathcal{P}(\mathcal{C}_S^G)$ for precomposition with f^{-1} , i.e. $f_* = (f^{-1})^*$, in the notation above. The functor f_* preserves Nisnevich excisive presheaves and homotopy invariant presheaves, thus restricts to a functor $f_* : \text{Spc}(\mathcal{C}_T^G) \rightarrow \text{Spc}(\mathcal{C}_S^G)$. This functor preserves limits and thus has a left adjoint, which we write as $f^* : \text{Spc}(\mathcal{C}_S^G) \rightarrow \text{Spc}(\mathcal{C}_T^G)$. Suppose further that the adjunction $u : \text{Sch}_T^G \rightleftarrows \text{Sch}_S^G : f^{-1}$ restricts to an adjunction

$$u : \mathcal{C}_T^G \rightleftarrows \mathcal{C}_S^G : f^{-1}.$$

Then $f^* : \mathcal{P}(\mathcal{C}_S^G) \rightarrow \mathcal{P}(\mathcal{C}_T^G)$ is precomposition with the forgetful functor u and it preserves Nisnevich excisive presheaves and homotopy invariant presheaves. Thus f^* restricts to a functor $f^* : \text{Spc}(\mathcal{C}_S^G) \rightarrow \text{Spc}(\mathcal{C}_T^G)$, which is left adjoint to f_* . Also in this case f^* preserves limits and so has a left adjoint $f_\# : \text{Spc}(\mathcal{C}_T^G) \rightarrow \text{Spc}(\mathcal{C}_S^G)$.

The functors f^*, f_* preserve final objects and so induce an adjunction $f^* : \text{Spc}_\bullet(\mathcal{C}_S^G) \rightleftarrows \text{Spc}_\bullet(\mathcal{C}_T^G) : f_*$. In case the forgetful functor restricts to $u : \mathcal{C}_T^G \rightarrow \mathcal{C}_S^G$, then there is an adjunction $f_\# : \text{Spc}(\mathcal{C}_T^G) \rightleftarrows \text{Spc}(\mathcal{C}_S^G) : f^*$.

Remark 2.32. The adjunctions of Lemma 2.31 admit the following alternate description in terms change of base functors. There is an equivalence of categories

$$\text{Sch}_{G \times_H S}^G \xrightarrow{\sim} \text{Sch}_S^H \quad (2.33)$$

induced by taking the fiber over $\{e\} \times S \subseteq G \times_H S$. Write $\mathcal{C}_{G \times_H S} \subseteq \text{Sch}_{G \times_H S}^G$ for the full subcategory corresponding to \mathcal{C}_S^H under (2.33). The restriction functor ι^* corresponds to pullback along $f : G \times_H S \rightarrow S$. Moreover, the equivalence (2.33) induces an equivalence of motivic spaces

$$\text{Spc}(\mathcal{C}_{G \times_H S}) \simeq \text{Spc}(\mathcal{C}_S^H)$$

and under this equivalence the functors $i_!, i^*, i_*$ are respectively identified with the functors $f_#, f^*, f_*$.

2.4 Motivic G -spectra

We recall the construction of categories of motivic G -spectra from [Hoy17] and some variants.

Let \mathcal{C}^\otimes be a presentably symmetric monoidal ∞ -category and X a set of objects in \mathcal{C} . If $I = \{x_1, \dots, x_n\}$ is a finite subset of X , write $\bigotimes I = x_1 \otimes \dots \otimes x_n$. Write

$$\mathcal{C}[X^{-1}] := \text{colim}_{\substack{I \subseteq X \\ I \text{ finite}}} \mathcal{C}[(\bigotimes I)^{-1}],$$

where $\mathcal{C}[x^{-1}]$ denotes the symmetric monoidal inversion of an object $x \in \mathcal{C}$ in a presentable symmetric monoidal ∞ -category \mathcal{C} , see [Rob15, Section 2.1].

Alternatively, one may consider the stabilization in $\text{Mod}_{\mathcal{C}}$, the ∞ -category of \mathcal{C} -modules in $\text{Pr}^{\text{L}, \otimes}$. Recall that if $\mathcal{M} \in \text{Mod}_{\mathcal{C}}$ and $x \in \mathcal{C}$ then $\text{Stab}_x(\mathcal{M})$ is the colimit, in $\text{Mod}_{\mathcal{C}}$, of the sequence

$$\mathcal{M} \xrightarrow{-\otimes x} \mathcal{M} \xrightarrow{-\otimes x} \mathcal{M} \rightarrow \dots$$

More generally, for a set of objects X in \mathcal{C} , define

$$\text{Stab}_X(\mathcal{M}) := \text{colim}_{\substack{I \subseteq X \\ I \text{ finite}}} \text{Stab}_{\bigotimes I}(\mathcal{M}).$$

An object $x \in \mathcal{C}$ is n -symmetric if the cyclic permutation on $x^{\otimes n}$ is homotopic to the identity. If each $x \in X$ is n -symmetric for some $n \geq 2$, then the canonical map of $\mathcal{C}[X^{-1}]$ -modules is an equivalence

$$\mathcal{M} \otimes_{\mathcal{C}} \mathcal{C}[X^{-1}] \xrightarrow{\sim} \text{Stab}_X(\mathcal{M}),$$

see [Rob15, Corollary 2.22], [Hoy17, Section 6.1].

Let \mathcal{E} be a finite rank locally free G -module on S . Write $T^\mathcal{E} \in \text{Spc}_\bullet(S)$ for the associated motivic sphere, defined as the Thom space

$$T^\mathcal{E} = \mathbb{V}(\mathcal{E})/\mathbb{V}(\mathcal{E}) - z(S),$$

where $z : S \rightarrow \mathbb{V}(\mathcal{E})$ is the zero section. We will also write $\Sigma^\mathcal{E}$ for the associated endofunctor

$$\Sigma^\mathcal{E} \simeq T^\mathcal{E} \wedge -.$$

We will also be interested in stabilizing the categories $\text{Spc}_\bullet^{G, \mathcal{F}}(S)$, for a family \mathcal{F} . Observe that $\text{Spc}_\bullet^{G, \mathcal{F}}(S)$ is an $\text{Spc}_\bullet^G(S)$ -module and $u_! : \text{Spc}_\bullet^{G, \mathcal{F}}(S) \rightarrow \text{Spc}_\bullet^G(S)$ is a map of $\text{Spc}_\bullet^G(S)$ -modules since if $X \in \text{Sm}_S^G[\mathcal{F}]$ and $Y \in \text{Sm}_S^G$ then $X \times Y \in \text{Sm}_S^G[\mathcal{F}]$. In particular, even though spheres $T^\mathcal{E} \in \text{Spc}_\bullet^G(S)$ are generally not objects of $\text{Spc}_\bullet^{G, \mathcal{F}}(S)$, they still determine endofunctors $\Sigma^\mathcal{E} : \text{Spc}_\bullet^{G, \mathcal{F}}(S) \rightarrow \text{Spc}_\bullet^{G, \mathcal{F}}(S)$.

Write $\text{Sph}_B^G := \{T^\mathcal{E} \mid \mathcal{E} \in \text{Rep}_B^G\}$ where Rep_B^G is the set of finite rank G -vector bundles over B .

Definition 2.34. A subset $\mathcal{T} \subseteq \text{Sph}_B^G$ is *stabilizing* if there is some $T^\mathcal{E} \in \mathcal{T}$ such that $T^\mathcal{E} \simeq T \wedge T^{\mathcal{E}'}$, for some locally free G -module \mathcal{E}' .

Definition 2.35.

1. Let $p : S \rightarrow B$ be a G -scheme over B and $\mathcal{T} \subseteq \mathrm{Sph}_B^G$ a stabilizing subset. Write

$$\mathrm{Spt}_{\mathcal{T}}^G(S) := \mathrm{Spc}_{\bullet}^G(S)[(p^*\mathcal{T})^{-1}].$$

If $\mathcal{T} = \{T^\varepsilon\}$ consists of a single sphere, we write $\mathrm{Spt}_{T^\varepsilon}^G(S)$ in place of $\mathrm{Spt}_{\mathcal{T}}^G(S)$. When $\mathcal{T} = \mathrm{Sph}_B^G$, we simply write

$$\mathrm{Spt}^G(S) := \mathrm{Spt}_{\mathrm{Sph}_B^G}^G(S).$$

2. Let \mathcal{F} be a family of subgroups. Define

$$\mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S) := \mathrm{Spc}_{\bullet}^{G,\mathcal{F}}(S) \bigotimes_{\mathrm{Spc}_{\bullet}^G(S)} \mathrm{Spt}_{\mathcal{T}}^G(S).$$

Write

$$\Sigma_{\mathcal{T}}^{\infty} : \mathrm{Spc}_{\bullet}^{G,\mathcal{F}}(S) \rightarrow \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)$$

for the stabilization functor. In case $\mathcal{T} = \mathrm{Sph}_B^G$, we simply write Σ^{∞} . When no confusion should arise, given $X \in \mathrm{Spc}_{\bullet}^{G,\mathcal{F}}(S)$ we will write again X for its image in $\mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)$ instead of $\Sigma_{\mathcal{T}}^{\infty} X$. Given $T^{\vee} \in \mathcal{T}$, $p : S \rightarrow B$, $E \in \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)$, and $k \in \mathbb{Z}$, we typically write $\Sigma^{k\vee} E$ rather than $\Sigma^{kp^*\vee} E$ when no confusion should arise.

Proposition 2.36. *Let $S \in \mathrm{Sch}_B^G$. Then $\mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)$ is a symmetric monoidal stable ∞ -category satisfying the following properties.*

1. *There is a canonical equivalence of $\mathrm{Spc}_{\bullet}^G(S)$ -modules*

$$\mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S) \simeq \mathrm{Stab}_{\mathcal{T}}(\mathrm{Spc}_{\bullet}^{G,\mathcal{F}}(S)).$$

2. *Given a morphism $f : T \rightarrow S$ in Sch_B^G , there is an induced adjunction*

$$f^* : \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S) \rightleftarrows \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(T) : f_*$$

such that $f^\Sigma_{\mathcal{T}}^{\infty} X \simeq \Sigma_{\mathcal{T}}^{\infty} f^* X$. Similarly, if f is smooth, then there is an induced adjunction*

$$f_{\#} : \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(T) \rightleftarrows \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S) : f^*$$

such that $f_{\#}\Sigma_{\mathcal{T}}^{\infty} X \simeq \Sigma_{\mathcal{T}}^{\infty} f_{\#} X$.

3. *The ∞ -category $\mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)$ is generated under sifted colimits by the compact objects*

$$\Sigma^{-k\vee} \Sigma_{\mathcal{T}}^{\infty} X_+,$$

where $k \geq 0$, $T^{\vee} \in \mathcal{T}$, $p : X \rightarrow S$ is in $\mathrm{Sm}_S^G[\mathcal{F}]$, and X is affine.

4. *The family of functors*

$$\{p^* : \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S) \rightarrow \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(X) \mid p : X \rightarrow S \text{ is in } \mathrm{Sm}_S^G[\mathcal{F}], X \text{ affine}\}$$

is jointly conservative.

Proof. That $\mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)$ is stable is a consequence of the fact that \mathcal{T} is stabilizing and that $T \simeq S^1 \wedge \mathbb{G}_m$. It is symmetric monoidal by construction. The arguments for the remaining points of (1) - (3) are similar to [Hoy17, Section 6]. Since $\mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)$ is stable, in order to establish (4), it suffices to show that $E \simeq 0$

whenever $p^*E \simeq 0$ for all $p : X \rightarrow S$ in $\mathrm{Sm}_S^G[\mathcal{F}]$ with X affine. If $T^\mathcal{V} \in \mathcal{T}$, then since $\mathcal{V} \in \mathrm{Rep}_B^G$, we have $\Sigma^{-k\mathcal{V}}\Sigma_{\mathcal{T}}^\infty X_+ \simeq p_{\#}(\Sigma^{-k\mathcal{V}}\mathbb{1}_X)$. Now, if $p^*E \simeq 0$, we have

$$\underline{\mathrm{Map}}_{\mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)}(\Sigma^{-k\mathcal{V}}\Sigma_{\mathcal{T}}^\infty X_+, E) \simeq \underline{\mathrm{Map}}_{\mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(X)}(\Sigma^{-k\mathcal{V}}\mathbb{1}_X, p^*E) \simeq 0.$$

Given E such that $p^*E \simeq 0$ for all $p : X \rightarrow S$ in $\mathrm{Sm}_S^G[\mathcal{F}]$ with X affine, let $\mathcal{C}_E \subseteq \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)$ be the full subcategory of whose objects are W such that $\underline{\mathrm{Map}}_{\mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)}(W, E) \simeq 0$, i.e., the E -acyclic objects. Then \mathcal{C}_E is closed under colimits and it contains $\Sigma^{-k\mathcal{V}}\Sigma_{\mathcal{T}}^\infty X_+$ for any $k \geq 0$ and $T^\mathcal{V} \in \mathcal{T}$. Thus by (3), $\mathcal{C}_E = \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)$ which implies that $E \simeq 0$ as required. \square

Remark 2.37. Over an affine base, every representation is the quotient of a finite sum of copies of the regular representation ρ_G . This implies that for any S ,

$$\mathrm{Spt}^G(S) \simeq \mathrm{Spt}_{T\rho_G}^G(S).$$

Let $N \trianglelefteq G$ be a normal subgroup and $\pi : G \rightarrow G/N$ the quotient homomorphism. This induces a function $\pi^* : \mathrm{Rep}_B^{G/N} \rightarrow \mathrm{Rep}_B^G$, and we write

$$N\text{-triv} = \{T^\mathcal{E} \mid \mathcal{E} \in \pi^*(\mathrm{Rep}_B^{G/N})\} \subseteq \mathrm{Sph}_B^G$$

for the associated set of “ N -trivial G -spheres”. This stabilizing set of spheres plays an important role in later sections.

Lemma 2.38. *Let \mathcal{F} be a family. The adjunction $u_! : \mathrm{Spc}_{\bullet}^{G,\mathcal{F}}(S) \rightleftarrows \mathrm{Spc}_{\bullet}^G(S) : u^*$ of $\mathrm{Spc}_{\bullet}^G(S)$ -modules induces an adjoint pair*

$$u_! : \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S) \rightleftarrows \mathrm{Spt}_{\mathcal{T}}^G(S) : u^*.$$

Moreover, u^* is symmetric monoidal and

$$u_! : \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S) \hookrightarrow \mathrm{Spt}_{\mathcal{T}}^G(S)$$

is full and faithful with essential image the localizing tensor ideal generated by $\Sigma^{-n\mathcal{V}}X_+$, where $T^\mathcal{V} \in p^*\mathcal{T}$ and $X \in \mathrm{Sm}_S^G[\mathcal{F}]$.

Proof. That the adjunction $(u_!, u^*)$ of $\mathrm{Spc}_{\bullet}^G(S)$ -modules induces an adjoint pair on categories of motivic spectra follows from the description of $\mathrm{Spt}_{\mathcal{T}}^G(S)$ and $\mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S)$ respectively as $\mathrm{Stab}_{\mathcal{T}}(\mathrm{Spc}_{\bullet}^G(S))$ and $\mathrm{Stab}_{\mathcal{T}}(\mathrm{Spc}_{\bullet}^{G,\mathcal{F}}(S))$, see the discussion preceding Definition 6.1 in [Hoy17]. This also implies that $u_!$ is full and faithful, since $u_! : \mathrm{Spc}_{\bullet}^{G,\mathcal{F}}(S) \hookrightarrow \mathrm{Spc}_{\bullet}^G(S)$ is, by Proposition 2.30. \square

Let \mathcal{C}^\otimes be a presentably symmetric monoidal ∞ -category and $E \in \mathcal{C}$ is an idempotent object. Recall [Lur17, Definition 4.8.2.1] that this means there is a map $e : \mathbb{1} \rightarrow E$ such that $\mathrm{id} \otimes e : E \simeq E \otimes \mathbb{1} \rightarrow E \otimes E$ and $e \otimes \mathrm{id} : E \simeq \mathbb{1} \otimes E \rightarrow E \otimes E$ are equivalences. Tensoring with E is a localization functor $L = E \otimes - : \mathcal{M} \rightarrow \mathcal{M}$ on any $\mathcal{M} \in \mathrm{Mod}_e$, see [Lur17, Proposition 4.8.2.4].

Lemma 2.39. *With notation as above,*

$$LM \simeq LC \otimes_e \mathcal{M}.$$

Proof. An argument similar to [Lur17, Proposition 4.8.2.10] shows that the forgetful functor $\mathrm{Mod}_E(\mathcal{M}) \rightarrow \mathcal{M}$ determines an equivalence of \mathcal{C} -modules $\mathrm{Mod}_E(\mathcal{M}) \simeq LM$. Since $\mathrm{Mod}_E(\mathcal{M}) \simeq \mathrm{Mod}_E(\mathcal{C}) \otimes_e \mathcal{M}$ by [Lur17, Theorem 4.8.4.6], the lemma follows. \square

We record the following result which we'll use a few times. A similar statement can be found in [Bac18, Lemma 26].

Lemma 2.40. *Let \mathcal{C}^\otimes be a presentably symmetric monoidal ∞ -category, X a set of objects. Suppose that $E \in \mathcal{C}$ is an idempotent object. Write $\mathbb{L} = E \otimes -$ for the associated symmetric monoidal localization endofunctor. Then there is an equivalence in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}, \otimes})$*

$$\mathbb{L}(\mathcal{C}[X^{-1}]) \simeq (\mathbb{L}\mathcal{C})[X^{-1}].$$

Proof. Both of these categories can be identified with $\mathbb{L}\mathcal{C} \otimes_{\mathcal{C}} \mathcal{C}[X^{-1}]$. □

Next we record the motivic version of the Wirthmueller isomorphism, which is a special case of the ambidexterity equivalence proved in [Hoy17]. If $H \leq G$ is a subgroup and ι is the inclusion we sometimes write

$$G_+ \times_H X := \iota_! X.$$

Proposition 2.41 (Wirthmueller isomorphism, [Hoy17]). *Let $\iota : H \hookrightarrow G$ be a group monomorphism. Let $\mathcal{F}, \mathcal{F}'$ be families of subgroups respectively of H and of G such that the induction-restriction adjunction restricts to $\iota_! : \mathrm{Sm}_S^H[\mathcal{F}] \rightleftarrows \mathrm{Sm}_S^G[\mathcal{F}'] : \iota^{-1}$. Then there is an induced adjunction*

$$\iota_! : \mathrm{Spt}^{H, \mathcal{F}}(S) \rightleftarrows \mathrm{Spt}^{G, \mathcal{F}'}(S) : \iota^*,$$

such that $\iota_!(\Sigma^\infty X) \simeq \Sigma^\infty(\iota_! X)$ and $\iota^(\Sigma^\infty X) \simeq \Sigma^\infty(\iota^{-1} X)$. Moreover ι^* admits a right adjoint ι_* and there is an equivalence*

$$\iota_! \xrightarrow{\sim} \iota_*.$$

Proof. The first statements are straightforward. We explain the last statement. Consider the G -equivariant map $f : G \times_H S \rightarrow S$. Then $\iota_!, \iota^*, \iota_*$ are identified with $f_\#, f^*, f_*$ via the equivalence $\mathrm{Spt}^{G, \overline{\mathcal{F}}}(G \times_H S) \simeq \mathrm{Spt}^{H, \mathcal{F}}(S)$, see Remark 2.32. Here $\overline{\mathcal{F}}$ is the family of subgroups of G generated by \mathcal{F} .

We have that $u^* \iota'_! v_! \simeq \iota_!$ and $u^* \iota'_* v_! \simeq \iota_*$, where we write $u : \mathrm{Sm}_S^{G, \mathcal{F}} \subseteq \mathrm{Sm}_S^G$ and $v : \mathrm{Sm}_S^{H, \mathcal{F}} \subseteq \mathrm{Sm}_S^H$ for the inclusions and $\iota'_!, \iota'_* : \mathrm{Spt}^H(S) \rightarrow \mathrm{Spt}^G(S)$ are the corresponding functors for the family $\mathcal{F}_{\mathrm{all}} = \mathcal{F}'$. In particular, the general case follows from the case $\mathcal{F} = \mathcal{F}_{\mathrm{all}}$. But in this case, the Wirthmueller isomorphism is the ambidexterity equivalence [Hoy17, Theorem 1.5] for the finite étale morphism f . □

2.5 Functoriality

We will make use of the functoriality of the ∞ -categories of equivariant motivic spaces and spectra with respect to the group G , the family of subgroups \mathcal{F} , and the base scheme S . To establish this, it will be convenient to introduce the following notation.

Definition 2.42. 1. The category Sch_B° of *equivariant B -schemes* has objects pairs (G, S) consisting of a finite group G (whose order is invertible in \mathcal{O}_B) and $S \in \mathrm{Sch}_B^G$. A morphism $(G', S') \rightarrow (G, S)$ is a pair (ϕ, f) where $\phi : G' \rightarrow G$ is a homomorphism of groups and $f : S' \rightarrow S$ is a ϕ -equivariant map of B -schemes.

2. The category $\mathrm{Sch}_B^\circ[\cdot]$ of *equivariant B -schemes and families* has objects consisting of triples (G, \mathcal{F}, S) where $(G, S) \in \mathrm{Sch}_B^\circ$ and \mathcal{F} is a family of subgroups of G . A morphism $(G', \mathcal{F}', S') \rightarrow (G, \mathcal{F}, S)$ is a triple (ϕ, i, f) where (ϕ, f) is a morphism in Sch_B° and $i : \phi^{-1}\mathcal{F} \subset \mathcal{F}'$ is an inclusion of posets.

The inclusion $i : \phi^{-1}\mathcal{F} \subset \mathcal{F}'$ is unique if it exists. When no confusion should arise, we write (ϕ, f) instead of (ϕ, i, f) for a morphism in $\mathrm{Sch}_B^\circ[\cdot]$.

We identify Sch_B° with the full subcategory of $\text{Sch}_B^\circ[\cdot]$ whose objects are triples of the form $(G, \mathcal{F}_{\text{all}}, S)$. The inclusion $\text{Sch}_B^\circ \subseteq \text{Sch}_B^\circ[\cdot]$ is left adjoint to the forgetful functor.

In this subsection we extend constructions in the previous sections to *functors*

$$\text{Spc}^\times, \text{Spc}_\bullet^\wedge, \text{Spt}^\otimes : (\text{Sch}_B^\circ[\cdot])^{\text{op}} \rightarrow \text{CAlg}(\mathcal{P}\text{r}^{\text{L}})$$

whose respective values on (G, \mathcal{F}, S) are the symmetric monoidal ∞ -categories $\text{Spc}^{G, \mathcal{F}}(S)$, $\text{Spc}_\bullet^{G, \mathcal{F}}(S)$, and $\text{Spt}^{G, \mathcal{F}}(S)$, and on morphisms, $(\phi, i, f)^* \simeq i_! \phi^* f^*$.

Lemma 2.43. *The categories Sch_B° and $\text{Sch}_B^\circ[\cdot]$ admit finite products. In particular, they are cartesian symmetric monoidal.*

Proof. It is straightforward to check that $(G, S) \times (G', S') \simeq (G \times G', S \times_B S')$ and $(G, \mathcal{F}, S) \times (G', \mathcal{F}', S') \simeq (G \times G', \mathcal{F} \times \mathcal{F}', S \times_B S')$, where $\mathcal{F} \times \mathcal{F}'$ denotes the family of subgroups of $G \times G'$ consisting of those subgroups of the form $H \times H'$ for $H \in \mathcal{F}$ and $H' \in \mathcal{F}'$. □

Corollary 2.44. *The assignment $(G, \mathcal{F}, S) \mapsto \text{Sm}_S^G[\mathcal{F}]$, $(\phi, i, f) \mapsto i\phi^{-1}f^{-1}$ extends to a functor*

$$\text{Sch}_B^\circ[\cdot]^{\text{op}} \rightarrow \text{CAlg}(\text{Cat}),$$

the category of commutative algebra objects of Cat with respect to the cartesian symmetric monoidal structure (equivalently, the category of symmetric monoidal categories). □

Composing with the symmetric monoidal presheaves functor $\text{Cat} \rightarrow \text{Cat}_\infty \rightarrow \mathcal{P}\text{r}^{\text{L}}$, we obtain a functor

$$(\text{Sch}_B^\circ[\cdot])^{\text{op}} \rightarrow \text{CAlg}(\mathcal{P}\text{r}^{\text{L}}),$$

which sends the object (G, \mathcal{F}, S) to $\mathcal{P}(\text{Sm}_S^G[\mathcal{F}])$ and the morphism (ϕ, i, f) to $i_! \phi^* f^*$.

To obtain the necessary functoriality of equivariant motivic spaces and spectra, we follow the techniques of [BH17, Section 6.1]. Recall from loc. cit. that we have a commutative diagram of ∞ -categories

$$\begin{array}{ccccc} \mathcal{M}\text{Cat}_\infty & \longrightarrow & \mathcal{O}\text{Cat}_\infty & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Cat}_\infty & \xrightarrow{\text{Fun}(\Delta^1, -)} & \text{Cat}_\infty & \longrightarrow & \text{Pos}, \end{array}$$

in which all squares are cartesian. Here Pos denotes the ∞ -category of (not necessarily small) posets, the lower right hand horizontal arrow sends an ∞ -category to the poset of subsets of the set of equivalence classes of objects, and $\mathcal{E} \rightarrow \text{Pos}$ is the universal cocartesian fibration, restricted to posets. The ∞ -categories $\mathcal{O}\text{Cat}_\infty$ and $\mathcal{M}\text{Cat}_\infty$ are respectively the ∞ -categories of ∞ -categories equipped with a collection of equivalence classes of objects respectively a collection of equivalence classes of arrows.

Lemma 2.45. *Let $(\mathcal{C}, W) \in \mathcal{M}\text{Cat}_\infty$ such that \mathcal{C} is presentable and W is of small generation.*

1. *The partial adjoint to*

$$\text{Cat}_\infty \rightarrow \mathcal{M}\text{Cat}_\infty, \quad \mathcal{C} \mapsto (\mathcal{C}, \text{equivalences})$$

is defined at (\mathcal{C}, W) and the localization $\mathcal{C}[W^{-1}]$ is again presentable.

2. Suppose that \mathcal{C} admits a symmetric monoidal structure $\mathcal{C}^\otimes \in \text{CAlg}(\text{Cat}_\infty)$ and W is stable under the monoidal product. Then (\mathcal{C}, W) lifts to $(\mathcal{C}, W)^\otimes \in \text{CAlg}(\mathcal{M}\text{Cat}_\infty)$ and the partial left adjoint to

$$\text{CAlg}(\text{Cat}_\infty) \rightarrow \text{CAlg}(\mathcal{M}\text{Cat}_\infty), \quad \mathcal{C} \mapsto (\mathcal{C}, \text{equivalences})$$

is defined at $(\mathcal{C}, W)^\otimes$.

Proof. The first item follows from [Lur09, Proposition 5.5.4.15, Proposition 5.5.4.20]. It follows from [Lur17, Proposition 2.2.1.9] that $\mathcal{C}[W^{-1}]$ inherits a monoidal structure such that $\mathcal{C} \rightarrow \mathcal{C}[W^{-1}]$ is monoidal. This implies the second item. \square

Consider the subfunctor of the composition

$$(\text{Sch}_B^\circlearrowleft[\cdot])^{\text{op}} \rightarrow \text{Cat} \rightarrow \text{Cat}_\infty \xrightarrow{\mathcal{P}} \mathcal{P}\text{r}^{\text{L}} \rightarrow \text{Fun}(\Delta^1, \text{Cat}_\infty)$$

whose value on the object (G, \mathcal{F}, S) is the full subcategory

$$W_{(G, \mathcal{F}, S)} \subset \text{Fun}(\Delta^1, \mathcal{P}(\text{Sm}_S^G[\mathcal{F}]))$$

consisting of the motivic equivalences. Since motivic equivalences are stable under smash product and are preserved by the functors f^* , ϕ^* , and $i_!$, the assignments

$$(G, \mathcal{F}, S) \mapsto (\mathcal{P}(\text{Sm}_S^G[\mathcal{F}]), W_{(G, \mathcal{F}, S)}), (\mathcal{P}_\bullet(\text{Sm}_S^G[\mathcal{F}]), W_{(G, \mathcal{F}, S)})$$

induce functors

$$(\text{Sch}_B^\circlearrowleft[\cdot])^{\text{op}} \rightarrow \mathcal{M}\text{Cat}_\infty.$$

The images of $(\mathcal{P}(\text{Sm}_S^G[\mathcal{F}]), W_{(G, \mathcal{F}, S)})$ and $(\mathcal{P}_\bullet(\text{Sm}_S^G[\mathcal{F}]), W_{(G, \mathcal{F}, S)})$ under the partial left adjoint to

$$\text{CAlg}(\text{Cat}_\infty) \rightarrow \text{CAlg}(\mathcal{M}\text{Cat}_\infty)$$

are respectively $\text{Spc}^{G, \mathcal{F}}(S)$ and $\text{Spc}_\bullet^{G, \mathcal{F}}(S)$.

Write Sph_S^G for the set of spheres $\{T^\mathcal{E}\}$ where \mathcal{E} is an equivariant vector bundle over S . The assignment $(G, \mathcal{F}, S) \mapsto \text{Sph}_S^G$ is a presheaf of sets on $\text{Sch}_B^\circlearrowleft[\cdot]^{\text{op}}$, which we write as Sph . Let

$$\mathcal{T} : \text{Sch}_B^\circlearrowleft[\cdot]^{\text{op}} \rightarrow \text{Set}$$

be a subpresheaf of Sph , which is closed under smash product and takes values in stabilizing sets of spheres, i.e., there is some $T^\mathcal{E} \in \mathcal{T}_{(G, \mathcal{F}, S)}$ such that $T^\mathcal{E} \simeq T \wedge T^{\mathcal{E}'}$. We obtain a functor

$$\text{Sch}_B^\circlearrowleft[\cdot]^{\text{op}} \rightarrow \text{CAlg}(\mathcal{O}\text{Cat}_\infty)$$

which on objects is the assignment $(G, \mathcal{F}, S) \mapsto (\text{Spc}_\bullet^{G, \mathcal{F}}(S), \mathcal{T}_{(G, \mathcal{F}, S)})$. By the following lemma, we obtain $\text{Spt}_{\mathcal{T}_{(G, \mathcal{F}, S)}}^{G, \mathcal{F}}(S)$ as the image of $(\text{Spc}_\bullet^{G, \mathcal{F}}(S), \mathcal{T}_{(G, \mathcal{F}, S)})$ under the partial left adjoint of $\text{CAlg}(\text{Cat}_\infty) \rightarrow \text{CAlg}(\mathcal{O}\text{Cat}_\infty)$. $\mathcal{C} \mapsto (\mathcal{C}, \pi_0\text{Pic}(\mathcal{C}))$.

Lemma 2.46. *Let (\mathcal{C}^\otimes, U) be an object of $\text{CAlg}(\mathcal{O}\text{Cat}_\infty)$ such that \mathcal{C} is presentable symmetric monoidal and U is small. Then the partial adjoint of*

$$\text{CAlg}(\text{Cat}_\infty) \rightarrow \text{CAlg}(\mathcal{O}\text{Cat}_\infty), \quad \mathcal{C} \mapsto (\mathcal{C}, \pi_0\text{Pic}(\mathcal{C}))$$

is defined at (\mathcal{C}^\otimes, U) .

Proof. This follows from [Hoy17, Section 6.1]. \square

Corollary 2.47. *The assignments $(G, \mathcal{F}, S) \mapsto \text{Spc}^{G, \mathcal{F}}(S)$ and $(G, \mathcal{F}, S) \mapsto \text{Spt}^{G, \mathcal{F}}(S)$ extend to functors $\text{Spc} : \text{Sch}_B^\circlearrowleft[\cdot]^{\text{op}} \rightarrow \text{CAlg}(\mathcal{P}\text{r}^{\text{L}})$ and $\text{Spt} : \text{Sch}_B^\circlearrowleft[\cdot]^{\text{op}} \rightarrow \text{CAlg}(\mathcal{P}\text{r}^{\text{L}})$, respectively.*

3 Filtering by isotropy

In this section we develop techniques to define and analyze filtrations of motivic G -spaces and spectra by families of isotropy.

3.1 Universal Motivic \mathcal{F} -spaces

Let \mathcal{F} be a family of subgroups. In classical equivariant homotopy theory there is a G -space $\mathbf{E}_\bullet \mathcal{F}$ characterized by the property that a G -space X admits a unique map $X \rightarrow \mathbf{E}_\bullet \mathcal{F}$ if all of the stabilizers of X are in \mathcal{F} , and no maps from X to $\mathbf{E}_\bullet \mathcal{F}$ otherwise. The G -space $\mathbf{E}_\bullet \mathcal{F}$ formally exists as a presheaf on Sm_B^G and hence as a motivic G -space over B , but it doesn't have the correct universality property.

Example 3.1. Let $G \neq \{e\}$, $B = \mathrm{Spec}(k)$ a field, and L/k be a finite Galois extension such that $G \subseteq \mathrm{Gal}(L/k)$ and consider $\mathrm{Spec}(L)$ as a smooth G -scheme over k via the Galois action. Then $\mathrm{Spec}(L)^H = \emptyset$ for all $e \neq H \subseteq G$, i.e., it has a free G -action. However, we claim that

$$\mathrm{Map}_{\mathrm{Spc}^G(k)}(\mathrm{Spec}(L), \mathbf{E}_\bullet G) = \emptyset.$$

Indeed,

$$(\mathbf{E}_\bullet G)_0(\mathrm{Spec}(L)) = \mathrm{Hom}_{\mathrm{Sm}_k^G}(\mathrm{Spec}(L), \coprod_G \mathrm{Spec}(k)) = \emptyset,$$

and so the claim follows, since $(\mathbf{E}_\bullet G)_0(\mathrm{Spec}(L))$ surjects onto $\pi_0(\mathrm{Map}_{\mathrm{Spc}^G(k)}(\mathrm{Spec}(L), \mathbf{E}_\bullet G))$ (see e.g., [MV99, Corollary 2.3.22, Remark 3.2.5]).

Definition 3.2. Let \mathcal{F} be a family of subgroups in G . The *universal motivic \mathcal{F} -space over S* is the object $\mathbf{E}\mathcal{F}_S \in \mathcal{P}(\mathrm{Sm}_S^G)$ whose value on $X \in \mathrm{Sm}_S^G$ is

$$\mathbf{E}\mathcal{F}_S(X) = \begin{cases} \mathrm{pt} & X \in \mathrm{Sm}_S^G[\mathcal{F}] \\ \emptyset & \text{else.} \end{cases}$$

When the base S is understood, we simply write $\mathbf{E}\mathcal{F}$.

Proposition 3.3. *Let \mathcal{F} be a family of subgroups in G . The presheaf $\mathbf{E}\mathcal{F}$ is a motivic G -space.*

Proof. We need to check that $\mathbf{E}\mathcal{F}$ is Nisnevich excisive and \mathbb{A}^1 -homotopy invariant. From the definition we have that $\mathbf{E}\mathcal{F}(\emptyset) = \mathrm{pt}$. Given a Nisnevich square (2.11), the possible values of

$$\begin{array}{ccc} \mathbf{E}\mathcal{F}(X) & \longrightarrow & \mathbf{E}\mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathbf{E}\mathcal{F}(Y) & \longrightarrow & \mathbf{E}\mathcal{F}(V) \end{array}$$

are the squares

$$\begin{array}{cccccc} \emptyset & \longrightarrow & \emptyset & \longrightarrow & \emptyset & \longrightarrow & \mathrm{pt} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \emptyset & \longrightarrow & \emptyset, & \longrightarrow & \mathrm{pt} & \longrightarrow & \mathrm{pt}, & \longrightarrow & \mathrm{pt}, \\ \emptyset & \longrightarrow & \mathrm{pt}, & \longrightarrow & \mathrm{pt} & \longrightarrow & \mathrm{pt}, & \longrightarrow & \mathrm{pt}, \end{array}$$

which are all cartesian squares. It follows that $\mathbf{E}\mathcal{F}$ is Nisnevich excisive. Also $\mathbf{E}\mathcal{F}$ is \mathbb{A}^1 -homotopy invariant since $(X \times_S \mathbb{A}_S^1)^H = X^H \times_{S^H} \mathbb{A}_{S^H}^1$. \square

Totaro [Tot99] and Morel-Voevodsky [MV99, Section 4.2] construct a geometric model for the classifying space of an algebraic group. This is generalized by Hoyois in [Hoy16, Section 2], to construct a geometric model for certain equivariant classifying spaces. Similar considerations lead to geometric models for the spaces \mathbf{EF} .

Definition 3.4. A *system of approximations to \mathbf{EF}_S* is a diagram $(U_i)_{i \in I}$ which is a subdiagram of a diagram $(V_i)_{i \in I}$ of inclusions of G -equivariant vector bundles over S , where I is a filtered poset and subject to the following conditions:

1. Each U_i is in $\mathrm{Sm}_S[\mathcal{F}]$ and $U_i \subseteq V_i$ is an open subscheme.
2. For $i \in I$, there exists an element $2i \in I$ with the property that $2i \geq i$ and such that there is an isomorphism $V_{2i} \cong V_i \oplus V_i$ of G -vector bundles which identifies the inclusion $V_i \hookrightarrow V_{2i}$ with the inclusion $(\mathrm{id}, 0) : V_i \hookrightarrow V_i \oplus V_i$.
3. Under the isomorphism $V_{2i} \cong V_i \oplus V_i$, $(U_i \times V_i) \cup (V_i \times U_i) \subseteq U_{2i}$.
4. There is a Nisnevich cover $\{T_j \rightarrow S\}$ such that for any affine X in $\mathrm{Sm}_{T_j}^G[\mathcal{F}]$, there is an $i \in I$ such that $(U_i)_X \rightarrow X$ admits a section.

Example 3.5. Write $\rho = \rho_G$. Let $U_n \subseteq \mathbb{V}_B(n\rho)$ be the open invariant subscheme

$$U_n := \mathbb{V}(n\rho) \setminus \bigcup_{H \in \mathrm{co}(\mathcal{F})} \mathbb{V}(n\rho)^H.$$

The inclusions $\mathbb{V}(n\rho) \subseteq \mathbb{V}((n+1)\rho)$ induce maps $U_n \rightarrow U_{n+1}$.

Let $f : S \rightarrow B$ in Sch_B^G . Then $(f^*U_n)_{n \in \mathbb{N}}$ is a system of approximations to \mathbf{EF}_S . The conditions (1)-(3) are clear. To check the last condition we may assume that B and S are affine (since S is equivariant locally affine). Let $X \in \mathrm{Sm}_S^G[\mathcal{F}]$ be affine. Then for n sufficiently large, there is an equivariant closed immersion of B -schemes, $X \hookrightarrow \mathbb{V}(n\rho)$. For any $H \notin \mathcal{F}$, we have $X \cap (\mathbb{V}(n\rho)^H) = \emptyset$ which means that $X \rightarrow \mathbb{V}(n\rho)$ factors to give a map $X \rightarrow U_n$ over B . This defines the desired section $X \rightarrow f^*U_n$.

Example 3.6. Let $N \trianglelefteq G$ be a normal subgroup. The family $\mathcal{F}[N]$ consists of all subgroups not containing N . Write $W := \rho_G / \rho_{G/N}$ for the quotient representation (where $\rho_{G/N}$ is viewed as a G -representation via the quotient homomorphism $G \rightarrow G/N$). Let $U_n = \mathbb{V}_S(nW) \setminus \{0\}$. This defines a system of approximations to $\mathbf{EF}[N]_S$. The conditions (1)-(3) of the definition are clear. As in the previous example, to check the last condition, it suffices to assume that B and S are affine. Let $X \in \mathrm{Sm}_S^G[\mathcal{F}[N]]$ be affine. Then for n large enough, there is an equivariant closed immersion of B -schemes $X \hookrightarrow \mathbb{V}_B(n\rho_G)$. The preimage of 0 under the projection $p : \mathbb{V}(n\rho_G) \rightarrow \mathbb{V}(nW)$ is $\mathbb{V}(n\rho_{G/N})$. Since N is not contained in any stabilizer of X , $X \cap \mathbb{V}_B(n\rho_{G/N}) = \emptyset$, which implies that restriction of p to X factors through U_n . This defines the desired section $X \rightarrow U_n$.

If $(U_i)_{i \in I}$ is a system of approximations to \mathbf{EF}_S , define

$$U_\infty := \mathrm{colim}_I U_i \in \mathrm{Spc}^G(S).$$

Proposition 3.7. Let \mathcal{F} be a family of subgroups in G and $(U_i)_{i \in I}$ a system of approximations to \mathbf{EF}_S . Then there is an equivalence

$$U_\infty \xrightarrow{\simeq} \mathbf{EF}_S$$

in $\mathrm{Spc}^G(S)$.

Proof. To prove the result, it suffices to work Nisnevich locally on S , and so we may assume that $S = T_j$ in the last condition of Definition 3.4. For each i , there is a unique map $U_i \rightarrow \mathbf{EF}$ which induces the unique map $U_\infty \rightarrow \mathbf{EF}$. It suffices to show that $\mathrm{Sing}_{\mathbb{A}^1}(U_\infty)(X) \rightarrow \mathrm{Sing}_{\mathbb{A}^1}(\mathbf{EF})(X)$ is an equivalence for any affine X in

Sm_S^G . Both sides are empty if $X \notin \mathrm{Sm}_S^G[\mathcal{F}]$, so we just need to show that $\mathrm{Sing}_{\mathbb{A}^1}(U_\infty)(X)$ is contractible for any affine X in $\mathrm{Sm}_S^G[\mathcal{F}]$. By assumption there is a section of $U_\infty \times_S X \rightarrow X$ and so the result follows from [Hoy16, Lemma 2.6]. \square

Proposition 3.8. *Let \mathcal{F} be a family of subgroups. Let $f : S' \rightarrow S$ be a morphism in Sch_B^G . Then*

$$f^*(\mathbf{E}\mathcal{F}_S) \simeq \mathbf{E}\mathcal{F}_{S'}.$$

Proof. Follows from Proposition 3.7 together with Example 3.5. \square

Write $i : \mathrm{Sm}_S^G[\mathcal{F}] \subseteq \mathrm{Sm}_S^G$ for the inclusion of categories.

Proposition 3.9. *There is a canonical equivalence of endofunctors*

$$i_! i^* \simeq \mathbf{E}\mathcal{F} \times - : \mathrm{Spc}^G(S) \rightarrow \mathrm{Spc}^G(S)$$

and

$$i_! i^* \simeq \mathbf{E}\mathcal{F}_+ \wedge - : \mathrm{Spc}_\bullet^G(S) \rightarrow \mathrm{Spc}_\bullet^G(S).$$

Proof. We treat the unbased case, the based case then follows. Let $X \in \mathrm{Spc}^G(S)$. We have $(i^* X)(W) \simeq X(W)$ for $W \in \mathrm{Sm}_S^G[\mathcal{F}]$. In particular, the projection $\mathbf{E}\mathcal{F} \rightarrow \mathrm{pt}$ induces the equivalences $i^*(\mathbf{E}\mathcal{F} \times X) \simeq i^*(X)$ and thus

$$i_! i^*(\mathbf{E}\mathcal{F} \times -) \xrightarrow{\sim} i_! i^*(-).$$

Now $\mathbf{E}\mathcal{F}$ is equivalent to $\mathrm{colim}_i U_i$ where $U_i \in \mathrm{Sm}_S^G[\mathcal{F}]$. If $W \in \mathrm{Sm}_S^G$ then each $U_i \times W$ is in $\mathrm{Sm}_S^G[\mathcal{F}]$. It follows that if X is any object of $\mathrm{Spc}^G(S)$ then, writing X as a colimit of objects of Sm_S^G , we see that $\mathbf{E}\mathcal{F} \times X \simeq i_!(E)$ for some $E \in \mathrm{Spc}^{G,\mathcal{F}}(S)$. In particular, since $\eta : \mathrm{id} \simeq i^* i_!$ is an equivalence by Proposition 2.30, we have $i_! \eta : i_!(E) \simeq i_! i^* i_!(E)$. From the triangle identity for the unit and counit, we have $\epsilon i_! : i_! i^*(i_!(E)) \simeq i_!(E)$ is also an equivalence. It follows that we have equivalences

$$\mathbf{E}\mathcal{F} \times - \simeq i_! i^*(\mathbf{E}\mathcal{F} \times -) \simeq i_! i^*.$$

\square

Corollary 3.10. *Let \mathcal{F} be a family of subgroups.*

1. *The essential images of $\mathbf{E}\mathcal{F} \times -$ and $\mathbf{E}\mathcal{F}_+ \wedge -$ are respectively the subcategories $\mathrm{Spc}^{G,\mathcal{F}}(S) \subseteq \mathrm{Spc}^G(S)$ and $\mathrm{Spc}_\bullet^{G,\mathcal{F}}(S) \subseteq \mathrm{Spc}_\bullet^G(S)$.*

2. *The projection $\mathbf{E}\mathcal{F} \times X \rightarrow X$ is an equivalence for $X \in \mathrm{Spc}^G(S)$ if and only if $X \simeq i_!(X')$ for some $X' \in \mathrm{Spc}^{G,\mathcal{F}}(S)$.*

3. *The projection $\mathbf{E}\mathcal{F}_+ \wedge Y \rightarrow Y$ is an equivalence for $Y \in \mathrm{Spc}_\bullet^G(S)$ if and only if $Y \simeq i_!(\tilde{Y})$ for some $\tilde{Y} \in \mathrm{Spc}_\bullet^{G,\mathcal{F}}(S)$.*

4. *The canonical maps are equivalences*

$$\begin{aligned} \mathrm{Map}_{\mathrm{Spc}^G(S)}(\mathbf{E}\mathcal{F} \times X, X') &\simeq \mathrm{Map}_{\mathrm{Spc}^G(S)}(\mathbf{E}\mathcal{F} \times X, \mathbf{E}\mathcal{F} \times X') \\ \mathrm{Map}_{\mathrm{Spc}_\bullet^G(S)}(\mathbf{E}\mathcal{F}_+ \wedge Y, Y') &\simeq \mathrm{Map}_{\mathrm{Spc}_\bullet^G(S)}(\mathbf{E}\mathcal{F}_+ \wedge Y, \mathbf{E}\mathcal{F}_+ \wedge Y'). \end{aligned}$$

\square

Recall that $i_! : \mathrm{Spt}_{\mathcal{T}}^{G,\mathcal{F}}(S) \rightarrow \mathrm{Spt}_{\mathcal{T}}^G(S)$ is full and faithful with essential image the localizing tensor ideal generated by $T^{-\mathcal{E}} \otimes X_+$ where $X \in \mathrm{Sm}_S^G[\mathcal{F}]$ and $T^{\mathcal{E}} \in \mathcal{T}$, see Lemma 2.38.

Proposition 3.11. *There is an equivalence of colocalization endofunctors*

$$\mathbf{E}\mathcal{F}_+ \otimes - \simeq i_! i^* : \mathcal{Spt}_{\mathcal{F}}^G(S) \rightarrow \mathcal{Spt}_{\mathcal{F}}^G(S).$$

In particular there are natural equivalences

$$i_! i^* \simeq i_! i^*(\mathbf{1}_S) \otimes \text{id}$$

and

$$i_* i^* \simeq F_{\mathcal{Spt}_{\mathcal{F}}^G(S)}(\mathbf{E}\mathcal{F}_+, \text{id}).$$

Proof. The first statements follow from [Proposition 3.9](#). The last statement follows from the natural equivalences

$$\text{Map}_{\mathcal{Spt}_{\mathcal{F}}^G(S)}(X, i_* i^* Y) \simeq \text{Map}_{\mathcal{Spt}_{\mathcal{F}}^G(S)}(i_! i^* X, Y) \simeq \text{Map}_{\mathcal{Spt}_{\mathcal{F}}^G(S)}(\mathbf{E}\mathcal{F}_+ \otimes X, Y).$$

□

3.2 Filtrations by adjacent families

We recall the definition of adjacent families.

Definition 3.12. Let $\mathcal{F} \subseteq \mathcal{F}'$ be an inclusion of families of subgroups of G .

1. We say that \mathcal{F} and \mathcal{F}' are *adjacent* if there is a subgroup $H \leq G$ such that $\mathcal{F}' \setminus \mathcal{F} = \{(H)\}$.
2. If $N \trianglelefteq G$ is a normal subgroup, say that \mathcal{F} and \mathcal{F}' are *N -adjacent* if there is a subgroup $H \leq N$ such that $\mathcal{F}' \setminus \mathcal{F} = \{K \leq G \mid (K \cap N) = (H)\}$ (where as before, (A) denotes G -conjugacy class of a subgroup A).
3. Say that \mathcal{F} and \mathcal{F}' are *N -adjacent at $H \leq N$* if the families are N -adjacent and $\mathcal{F}' \setminus \mathcal{F}$ is the set of subgroups $K \leq G$ such that $(K \cap N) = (H)$.

Of course, if $N = G$ then N -adjacent families are exactly adjacent families. Since G is finite, one can always find a filtration

$$\emptyset = \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}_{\text{all}},$$

such that each pair $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ is N -adjacent. For example, one can be produced as follows. Define the sequence of families

$$\{e\} = \text{Fil}_0^G \subset \text{Fil}_1^G \subset \cdots \subset \text{Fil}_i^G \subset \cdots \subset \text{Fil}_N^G = \mathcal{F}_{\text{all}}$$

by setting $\text{Fil}_0 = \{e\}$ and inductively defining Fil_i^G by

$$\text{Fil}_i^G := \{H \leq G \mid \text{each proper subgroup } K < H \text{ is in } \text{Fil}_{i-1}^G\}.$$

(Since G is finite, this sequence terminates at a finite stage.) Each Fil_i^G is a family. More generally, define

$$\text{Fil}_i^{N \trianglelefteq G} := \{K \leq G \mid K \cap N \in \text{Fil}_i^N\}.$$

The families just defined aren't adjacent, but $\text{Fil}_i^{N \trianglelefteq G} \setminus \text{Fil}_{i-1}^{N \trianglelefteq G}$ is a finite union of conjugacy classes. Let $\{(H_i)\}$ be the set of these conjugacy classes. Then the families

$$\text{Fil}_{i-1}^{N \trianglelefteq G} \subseteq \text{Fil}_{i-1}^{N \trianglelefteq G} \cup \{(H_1)\} \subseteq \text{Fil}_{i-1}^{N \trianglelefteq G} \cup \{(H_1), (H_2)\} \subseteq \cdots \subseteq \text{Fil}_i^{N \trianglelefteq G}$$

are all N -adjacent.

In any case, a filtration $\emptyset \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{\text{all}}$ gives rise to a filtration

$$* \rightarrow \mathbf{E}\mathcal{F}_{0+} \wedge X \rightarrow \mathbf{E}\mathcal{F}_{1+} \wedge X \rightarrow \cdots \rightarrow \mathbf{E}\mathcal{F}_{n+} \wedge X \simeq X \quad (3.13)$$

of an object $X \in \mathcal{C}$ whenever \mathcal{C} is an $\text{Spc}_{\bullet}^G(S)$ -module (e.g., $\mathcal{Spt}^G(S)$).

To utilize this filtration, we need to analyze the filtration quotients, which we will do below in [Proposition 3.27](#) and [Proposition 4.12](#).

3.3 Universal spaces for pairs

Definition 3.14. Let $\mathcal{F} \subseteq \mathcal{F}'$ be a subfamily. Define the based motivic G -space $\mathbf{E}(\mathcal{F}', \mathcal{F})$ so that it sits in the cofiber sequence

$$\mathbf{E}\mathcal{F}_+ \rightarrow \mathbf{E}\mathcal{F}'_+ \rightarrow \mathbf{E}(\mathcal{F}', \mathcal{F}).$$

If $\mathcal{F}' = \mathcal{F}_{\text{all}}$ define $\tilde{\mathbf{E}}\mathcal{F} := \mathbf{E}(\mathcal{F}_{\text{all}}, \mathcal{F})$

Note that $\mathbf{E}(\mathcal{F}', \emptyset) \simeq \mathbf{E}(\mathcal{F}')_+$. At the other extreme, since $\mathbf{E}\mathcal{F}_{\text{all}} \simeq \text{pt}$, the space $\tilde{\mathbf{E}}\mathcal{F}$ sits in the cofiber sequence in $\text{Spc}_{\bullet}^G(S)$

$$\mathbf{E}\mathcal{F}_+ \rightarrow S^0 \rightarrow \tilde{\mathbf{E}}\mathcal{F}. \quad (3.15)$$

Proposition 3.16. Let $\mathcal{F} \subseteq \mathcal{F}'$ be a subfamily. There is a canonical equivalence $\mathbf{E}(\mathcal{F}', \mathcal{F}) \simeq \tilde{\mathbf{E}}\mathcal{F} \wedge \mathbf{E}\mathcal{F}'_+$. In particular, $\tilde{\mathbf{E}}\mathcal{F} \wedge \mathbf{E}\mathcal{F}_+ \simeq \text{pt}$.

Proof. This follows from the commutative diagram

$$\begin{array}{ccccc} \mathbf{E}\mathcal{F}_+ & \longrightarrow & \mathbf{E}\mathcal{F}'_+ & \longrightarrow & \mathbf{E}(\mathcal{F}', \mathcal{F}) \\ \sim \downarrow & & \downarrow \sim & & \downarrow \\ (\mathbf{E}\mathcal{F} \times \mathbf{E}\mathcal{F}')_+ & \longrightarrow & (\mathbf{E}\mathcal{F}_{\text{all}} \times \mathbf{E}\mathcal{F}')_+ & \longrightarrow & \tilde{\mathbf{E}}\mathcal{F} \wedge \mathbf{E}\mathcal{F}'_+ \end{array}$$

induced by inclusions of families, in which the rows are each cofiber sequences and the left and middle vertical arrows are equivalences. \square

Corollary 3.17. Let $X \in \text{Spc}_{\bullet}^G(S)$. The map $X \rightarrow \tilde{\mathbf{E}}\mathcal{F} \wedge X$ induced by $S^0 \rightarrow \tilde{\mathbf{E}}\mathcal{F}$, induces an equivalence

$$\text{Map}_{\text{Spc}_{\bullet}^G(S)}(\tilde{\mathbf{E}}\mathcal{F} \wedge X, \tilde{\mathbf{E}}\mathcal{F} \wedge Y) \simeq \text{Map}_{\text{Spc}_{\bullet}^G(S)}(X, \tilde{\mathbf{E}}\mathcal{F} \wedge Y).$$

Proof. This follows from the previous proposition, [Corollary 3.10](#), and the fiber sequence $\text{Map}(\tilde{\mathbf{E}}\mathcal{F} \wedge X, \tilde{\mathbf{E}}\mathcal{F} \wedge Y) \rightarrow \text{Map}(X, \tilde{\mathbf{E}}\mathcal{F} \wedge Y) \rightarrow \text{Map}(\mathbf{E}\mathcal{F}_+ \wedge X, \tilde{\mathbf{E}}\mathcal{F} \wedge Y)$. Alternatively, simply note that $\mathbf{E}\mathcal{F}_+ \otimes -$ is a colocalization, $\tilde{\mathbf{E}}\mathcal{F} \otimes -$ is a localization endofunctor. \square

Lemma 3.18. Let $\mathcal{F} \subseteq \mathcal{F}'$ be an inclusion of families and \mathcal{E} a family such that $\mathcal{E} \cap \mathcal{F} = \mathcal{E} \cap \mathcal{F}'$. Then the map $S^0 \rightarrow \tilde{\mathbf{E}}\mathcal{E}$ induces an equivalence of based motivic G -spaces

$$\mathbf{E}(\mathcal{F}', \mathcal{F}) \xrightarrow{\sim} \mathbf{E}(\mathcal{F}', \mathcal{F}) \wedge \tilde{\mathbf{E}}\mathcal{E}.$$

Proof. Smashing $\mathbf{E}\mathcal{E}_+$ with the defining cofiber sequence for $\mathbf{E}(\mathcal{F}', \mathcal{F})$ yields the cofiber sequence

$$\mathbf{E}\mathcal{F}_+ \wedge \mathbf{E}\mathcal{E}_+ \xrightarrow{f} \mathbf{E}\mathcal{F}'_+ \wedge \mathbf{E}\mathcal{E}_+ \rightarrow \mathbf{E}(\mathcal{F}', \mathcal{F}) \wedge \mathbf{E}\mathcal{E}_+.$$

Smashing $\mathbf{E}(\mathcal{F}', \mathcal{F})$ with the defining cofiber sequence for $\tilde{\mathbf{E}}\mathcal{E}$ yields the cofiber sequence

$$\mathbf{E}(\mathcal{F}', \mathcal{F}) \wedge \mathbf{E}\mathcal{E}_+ \rightarrow \mathbf{E}(\mathcal{F}', \mathcal{F}) \xrightarrow{i} \mathbf{E}(\mathcal{F}', \mathcal{F}) \wedge \tilde{\mathbf{E}}\mathcal{E}.$$

By the hypothesis and [Proposition 3.21](#), f is an equivalence, and so $\mathbf{E}(\mathcal{F}', \mathcal{F}) \wedge \mathbf{E}\mathcal{E}_+$ is contractible. This implies that i is an equivalence. \square

To continue the analysis, we pass to the ∞ -categorical stabilization (i.e., S^1 -spectra) of the various categories of motivic G -spaces. We write

$$\mathrm{Spt}_{S^1}^{G,\mathcal{E}}(S) := \mathrm{Stab}(\mathrm{Spc}_{\bullet}^{G,\mathcal{E}}(S)).$$

Recall that $\mathrm{Spt}_{S^1}^{G,\mathcal{E}}(S)$ can be identified with the category of \mathbb{A}^1 -invariant Nisnevich sheaves of spectra. We write $\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)$ for the spectrum of maps in a stable ∞ -category \mathcal{C} .

Let $u : \mathcal{C}_S \subseteq \mathcal{D}_S$ be as in (2.24). Then we have induced functors

$$\begin{array}{ccc} & \xrightarrow{u_!} & \\ \mathrm{Spt}_{S^1}(\mathcal{C}_S) & \xleftarrow{u^*} & \mathrm{Spt}_{S^1}(\mathcal{D}_S) \\ & \xrightarrow{u_*} & \end{array}$$

Since $u^* : \mathcal{P}(\mathcal{D}_S) \rightarrow \mathcal{P}(\mathcal{C}_S)$ is a symmetric monoidal left adjoint, it follows that $\mathrm{Spt}_{S^1}(\mathcal{D}_S) \rightarrow \mathrm{Spt}_{S^1}(\mathcal{C}_S)$ is as well. Since $u_!, u_* : \mathcal{P}(\mathcal{C}_S) \rightarrow \mathcal{P}(\mathcal{D}_S)$ are full and faithful by Proposition 2.30, it follows that $u_!, u_* : \mathrm{Spt}_{S^1}(\mathcal{C}_S) \rightarrow \mathrm{Spt}_{S^1}(\mathcal{D}_S)$ are as well.

We introduce a minor technical condition on pairs $\mathcal{F} \subseteq \mathcal{F}'$ over S , which we sometimes require.

Condition 3.19. Let $\mathcal{F} \subseteq \mathcal{F}'$. Suppose there is a normal subgroup $N \trianglelefteq G$ such that

1. N as well as the elements of $(\mathcal{F}' \cap \mathcal{F}[N]) \setminus \mathcal{F}$ act trivially on S , and
2. $\mathcal{F} \subseteq \mathcal{F}' \cap \mathcal{F}[N]$.

We will say that \mathcal{F} satisfies this condition if the pair $\mathcal{F} \subseteq \mathcal{F}_{\mathrm{all}}$ satisfies the condition.

Remark 3.20. The pair $\mathcal{F} \subseteq \mathcal{F}'$ satisfies Condition 3.19 in the following two important cases, which cover all of the cases relevant to this paper.

1. The base S has trivial action (in which case we take $G = N$).
2. The normal subgroup N acts trivially on S and $\mathcal{F} = \mathcal{F}' \cap \mathcal{F}[N]$.

Proposition 3.21. Let $\mathcal{F} \subseteq \mathcal{F}'$ be a subfamily satisfying Condition 3.19, $\mathcal{E} = \mathcal{F}' \setminus \mathcal{F}$, and $u : \mathrm{Sm}_S^G[\mathcal{E}] \subseteq \mathrm{Sm}_S^G$ the inclusion. Let $f : X_1 \rightarrow X_2$ be a map in $\mathrm{Spt}_{S^1}^G(S)$. Suppose that S has trivial action. Then the following are equivalent.

1. The map $f_* : X_1(W) \rightarrow X_2(W)$ is an equivalence, for any $W \in \mathrm{Sm}_S^G[\mathcal{E}]$.
2. The $u^*(f) : u^*(X_1) \rightarrow u^*(X_2)$ is an equivalence in $\mathrm{Spt}_{S^1}^{G,\mathcal{E}}(S)$.
3. The map

$$\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X_1 \rightarrow \mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X_2$$

is an equivalence in $\mathrm{Spt}_{S^1}^G(S)$.

Proof. The first two items are immediately equivalent.

To see that (3) implies (2), we have $u^*(\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X) \simeq u^*(\mathbf{E}(\mathcal{F}', \mathcal{F})) \otimes u^*(X)$ and it is straightforward that $u^*(\mathbf{E}(\mathcal{F}', \mathcal{F})) \simeq S^0$ in $\mathrm{Spt}_{S^1}^{G,\mathcal{E}}(S)$.

Now we show that (2) implies (3). First, we assume that all elements of \mathcal{E} act trivially on S . Filter the inclusion $\mathcal{F} \subseteq \mathcal{F}'$ by adjacent families and consider the resulting sequence

$$\mathbf{E}\mathcal{F}_+ = \mathbf{E}\mathcal{F}_{0+} \rightarrow \mathbf{E}\mathcal{F}_{1+} \rightarrow \mathbf{E}\mathcal{F}_{2+} \rightarrow \cdots \rightarrow \mathbf{E}\mathcal{F}_{n-1+} \rightarrow \mathbf{E}\mathcal{F}_{n+} = \mathbf{E}\mathcal{F}'_+.$$

An inductive argument shows that it suffices to establish that

$$\mathbf{E}(\mathcal{F}_r, \mathcal{F}_{r-1}) \otimes X_1 \rightarrow \mathbf{E}(\mathcal{F}_r, \mathcal{F}_{r-1}) \otimes X_2$$

is an equivalence for $1 \leq i \leq n$. Thus, we may assume that $\mathcal{F}', \mathcal{F}$ are adjacent, say $\mathcal{F}' \setminus \mathcal{F} = \{(H)\}$ and H acts trivially on S .

By [Proposition 3.9](#) and [Proposition 3.16](#), in order to show that the map displayed above is an equivalence, it suffices to show that

$$\underline{\text{Map}}_{\text{Spt}_{S^1}^G(S)}(W_+, \tilde{\mathbf{E}}\mathcal{F} \otimes f)$$

is an equivalence for any $p : W \rightarrow S$ in $\text{Sm}_S^G[\mathcal{F}']$, with W affine. From the gluing sequence [[Hoy17](#), Proposition 5.2], we have the cofiber sequence in $\text{Spt}_{S^1}^G(S)$

$$W(\mathcal{F})_+ \rightarrow W_+ \rightarrow p_{\#}i_*(W_+^{\mathcal{F}})$$

where $i : W^{\mathcal{F}} \subseteq W$ is the inclusion (see [Notation 2.7](#)). By [Proposition 3.16](#) we have

$$\underline{\text{Map}}_{\text{Spt}_{S^1}^G(S)}(-, \tilde{\mathbf{E}}\mathcal{F} \otimes f) \simeq \underline{\text{Map}}_{\text{Spt}_{S^1}^G(S)}(\tilde{\mathbf{E}}\mathcal{F} \otimes -, \tilde{\mathbf{E}}\mathcal{F} \otimes f).$$

We have $\tilde{\mathbf{E}}\mathcal{F} \otimes W(\mathcal{F})_+ \simeq \text{pt}$, using [Corollary 3.10](#) and [Proposition 3.16](#) and so we conclude that

$$\underline{\text{Map}}_{\text{Spt}_{S^1}^G(S)}(p_{\#}i_*(W_+^{\mathcal{F}}), \tilde{\mathbf{E}}\mathcal{F} \otimes f) \simeq \underline{\text{Map}}_{\text{Spt}_{S^1}^G(S)}(W_+, \tilde{\mathbf{E}}\mathcal{F} \otimes f).$$

The canonical map $G \times_{N(H)} W^H \rightarrow W^{\mathcal{F}}$ is an isomorphism. Indeed, it suffices to check that $W^H \cap W^{gHg^{-1}} = \emptyset$ if $H \neq gHg^{-1}$. Now, if $w \in W^H \cap W^{gHg^{-1}}$, then $\text{Stab}(w)$ contains both of these subgroups. But since $\text{Stab}(w) \in \mathcal{F}'$ and (H) is a maximal element of this poset, we have that $H = \text{Stab}(w) = gHg^{-1}$. In particular, $W^{\mathcal{F}}$ is smooth over S , since $W^H \rightarrow S$ is smooth. It follows from purity [[Hoy17](#), Proposition 5.7] that $p_{\#}i_*(W_+^{\mathcal{F}}) \simeq \text{Th}(\mathcal{N}_i)$. We show that if $V \rightarrow W^{\mathcal{F}}$ is an equivariant vector bundle, then $\underline{\text{Map}}_{\text{Spt}_{S^1}^G(S)}(\text{Th}(V), \tilde{\mathbf{E}}\mathcal{F} \otimes f)$ is an equivalence. For this, we may assume that H is normal and in particular that $W^{\mathcal{F}} = W^H$. Indeed, we have $\text{Th}(V) \simeq G_+ \times_{NH} \text{Th}(V|_{W^H})$, so via the induction-restriction adjunction, we can replace G by NH and $\mathcal{F}, \mathcal{F}'$ by $\mathcal{F}|_{NH}, \mathcal{F}'|_{NH}$, if necessary. Since G is linearly reductive, we can write $V \cong V' \oplus V^H$. Since $(V')^H = 0$, all stabilizers of $V' \setminus \{0\}$ are in \mathcal{F} and so $\tilde{\mathbf{E}}\mathcal{F} \otimes V' \setminus \{0\}_+ \simeq *$ which implies that

$$\tilde{\mathbf{E}}\mathcal{F} \otimes \text{Th}(V') \simeq \tilde{\mathbf{E}}\mathcal{F} \otimes W_+^{\mathcal{F}}.$$

Now it follows that

$$\begin{aligned} \underline{\text{Map}}_{\text{Spt}_{S^1}^G(S)}(\text{Th}(V), \tilde{\mathbf{E}}\mathcal{F} \otimes f) &\simeq \underline{\text{Map}}_{\text{Spt}_{S^1}^G(S)}(\text{Th}(V') \otimes \text{Th}(V^H), \tilde{\mathbf{E}}\mathcal{F} \otimes f) \\ &\simeq \underline{\text{Map}}_{\text{Spt}_{S^1}^G(S)}(W_+^{\mathcal{F}} \otimes \text{Th}(V^H), \tilde{\mathbf{E}}\mathcal{F} \otimes f) \\ &\simeq \underline{\text{Map}}_{\text{Spt}_{S^1}^{G,\varepsilon}(S)}(W_+^{\mathcal{F}} \otimes \text{Th}(V^H), u^*(\tilde{\mathbf{E}}\mathcal{F} \otimes f)) \\ &\simeq \underline{\text{Map}}_{\text{Spt}_{S^1}^{G,\varepsilon}(S)}(W_+^{\mathcal{F}} \otimes \text{Th}(V^H), u^*(f)), \end{aligned}$$

which is an equivalence, as needed.

Next we consider the case when $\mathcal{F} = \mathcal{F}' \cap \mathcal{F}[N]$, where N is a normal subgroup of G which acts trivially on S . Again we consider $p : W \rightarrow S$ in $\text{Sm}_S^G[\mathcal{F}']$ and the cofiber sequence in $\text{Spt}_{S^1}^G(S)$

$$W(\mathcal{F}[N])_+ \rightarrow W_+ \rightarrow p_{\#}i_*(W_+^N)$$

where now $i : W^N \subseteq W$ is the inclusion. Since $W(\mathcal{F}[N])_+ \in \text{Sm}_S^G[\mathcal{F}]$ we have that $\tilde{\mathbf{E}}\mathcal{F} \otimes W(\mathcal{F}[N])_+ \simeq \text{pt}$ and we conclude that

$$\underline{\text{Map}}_{\text{Spt}_{S^1}^G(S)}(p_{\#}i_*(W_+^N), \tilde{\mathbf{E}}\mathcal{F} \otimes f) \simeq \underline{\text{Map}}_{\text{Spt}_{S^1}^G(S)}(W_+, \tilde{\mathbf{E}}\mathcal{F} \otimes f).$$

Since N acts trivially on S , $W^N \rightarrow S$ is again smooth and so we have $p_{\#}i_*W^N \simeq \text{Th}(N_i)$ and a similar argument as in the previous paragraph shows that

$$\underline{\text{Map}}_{\text{Spt}_{S^1}^G}(\text{Th}(N_i), \tilde{\mathbf{E}}\mathcal{F} \otimes f)$$

is an equivalence.

Now we consider the general case. Let N be the normal subgroup as in [Condition 3.19](#) and consider the inclusions $\mathcal{F} \subseteq \mathcal{F}'' \subseteq \mathcal{F}'$, where we write $\mathcal{F}'' = \mathcal{F}' \cap \mathcal{F}[N]$. From the cofiber sequence

$$\mathbf{E}(\mathcal{F}'', \mathcal{F}) \rightarrow \mathbf{E}(\mathcal{F}', \mathcal{F}) \rightarrow \mathbf{E}(\mathcal{F}', \mathcal{F}'')$$

we see that it suffices to show that $\mathbf{E}(\mathcal{F}'', \mathcal{F}) \otimes f$ and $\mathbf{E}(\mathcal{F}', \mathcal{F}'') \otimes f$ are both equivalences. The case $\mathcal{F}'' \subseteq \mathcal{F}'$ is covered by the previous paragraph and since all elements of $\mathcal{F}'' \setminus \mathcal{F}$ act trivially on S , this case is covered by the first. □

Corollary 3.22. *Let $\mathcal{F} \subseteq \mathcal{F}'$ be a subfamily which satisfies [Condition 3.19](#). Write $\mathcal{E} = \mathcal{F}' \setminus \mathcal{F}$, and $u : \text{Sm}_S^G[\mathcal{E}] \subseteq \text{Sm}_S^G$ the inclusion. Then the map*

$$\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes u_!u^*X \rightarrow \mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X$$

is an equivalence for any $X \in \text{Spt}_{S^1}^G(S)$.

Proof. By [Proposition 3.21](#), it suffices to show that $u^*\epsilon : u^*(u_!u^*X) \rightarrow u^*(X)$ is an equivalence. This follows from the triangle identity for the unit and counit since the unit $\eta : \text{id} \simeq u^*u_!$ is an equivalence, as $u_!$ is full and faithful. □

3.4 Localization at a cofamily

Proposition 3.23. *Let \mathcal{F} be a family of subgroups which satisfies [Condition 3.19](#) and write $j : \text{Sm}_S^G[\text{co}(\mathcal{F})] \subseteq \text{Sm}_S^G$ for the inclusion. Then the natural equivalence $j^*(\tilde{\mathbf{E}}\mathcal{F} \otimes -) \simeq j^*$ induces an equivalence of localization endofunctors*

$$\tilde{\mathbf{E}}\mathcal{F} \otimes - \simeq j_*j^* : \text{Spt}_{S^1}^G(S) \rightarrow \text{Spt}_{S^1}^G(S).$$

Proof. We have natural equivalences

$$\begin{aligned} \underline{\text{Map}}(W, j_*j^*X) &\simeq \underline{\text{Map}}(j^*W, j^*X) \\ &\simeq \underline{\text{Map}}(j_!j^*W, \tilde{\mathbf{E}}\mathcal{F} \otimes X) \\ &\simeq \underline{\text{Map}}(\tilde{\mathbf{E}}\mathcal{F} \otimes j_!j^*W, \tilde{\mathbf{E}}\mathcal{F} \otimes X) \\ &\simeq \underline{\text{Map}}(\tilde{\mathbf{E}}\mathcal{F} \otimes W, \tilde{\mathbf{E}}\mathcal{F} \otimes X) \\ &\simeq \underline{\text{Map}}(W, \tilde{\mathbf{E}}\mathcal{F} \otimes X), \end{aligned}$$

where the second follows by adjunction and the equivalence $j^*X \simeq j^*(\tilde{\mathbf{E}}\mathcal{F} \otimes X)$, the fourth from [Corollary 3.22](#), and the third and fifth from [Corollary 3.17](#). □

Write $L_{\text{co}(\mathcal{F})}\text{Spt}_{\mathcal{F}}^G(S)$ for the essential image of $\tilde{\mathbf{E}}\mathcal{F} \wedge - : \text{Spt}_{\mathcal{F}}^G(S) \rightarrow \text{Spt}_{\mathcal{F}}^G(S)$. Since $\mathbf{E}\mathcal{F}_+ \wedge -$ is a colocalization endofunctor, $\tilde{\mathbf{E}}\mathcal{F} \wedge -$ is a localization endofunctor. In particular $\tilde{\mathbf{E}}\mathcal{F}$ is an idempotent object of $\text{Spt}_{\mathcal{F}}^G(S)$, see [[Lur17](#), Proposition 4.8.2.4] and so $\text{Spt}_{\mathcal{F}}^G(S) \rightarrow L_{\text{co}(\mathcal{F})}\text{Spt}_{\mathcal{F}}^G(S)$ is a symmetric monoidal localization.

We will abuse notation and terminology slightly by saying that a stabilizing set of spheres $\{T^\varepsilon \in \mathrm{Spc}_\bullet^G(S)\}$ is in $\mathrm{Spc}_\bullet^{G,\mathrm{co}(\mathcal{F})}(S)$ if it is in the essential image of $j_!$. Suppose that $\mathcal{T} \subseteq \mathrm{Sph}_B^G$ is a set of spheres such that $p^*\mathcal{T}$ is in $\mathrm{Spc}_\bullet^{G,\mathrm{co}(\mathcal{F})}(S)$. In this case, we can stabilize $\mathrm{Spc}_\bullet^{G,\mathrm{co}(\mathcal{F})}(S)$ with respect to \mathcal{T} and we define

$$\mathrm{Spt}_{\mathcal{T}}^{G,\mathrm{co}(\mathcal{F})}(S) := \mathrm{Spc}_\bullet^{G,\mathrm{co}(\mathcal{F})}(S)[(p^*\mathcal{T})^{-1}].$$

Equivalently, $\mathrm{Spt}_{\mathcal{T}}^{G,\mathrm{co}(\mathcal{F})}(S) \simeq \mathrm{Spt}_{S^1}^{G,\mathrm{co}(\mathcal{F})}(S)[(p^*\mathcal{T})^{-1}]$

Proposition 3.24. *Let \mathcal{F} be a family which satisfies [Condition 3.19](#) and write $j : \mathrm{Sm}_S^G[\mathrm{co}(\mathcal{F})] \subseteq \mathrm{Sm}_S^G$ for the inclusion.*

1. *The functor $j_! : \mathrm{Spt}_{S^1}^{G,\mathrm{co}(\mathcal{F})}(S) \rightarrow \mathrm{Spt}_{S^1}^G(S)$ is symmetric monoidal.*
2. *There is an equivalence $j_* \simeq \tilde{\mathbf{E}}\mathcal{F} \wedge j_!$. Moreover, j_* induces a symmetric monoidal equivalence $\bar{j}_* : \mathrm{Spt}_{S^1}^{G,\mathrm{co}(\mathcal{F})}(S) \rightarrow \mathrm{L}_{\mathrm{co}(\mathcal{F})}\mathrm{Spt}_{S^1}^G(S)$.*

Proof. Note that $S \in \mathrm{Sm}_S^G[\mathrm{co}(\mathcal{F})]$, so that $j_!$ preserves terminal objects. That it is symmetric monoidal then follows from the fact that j preserves products. Since $j_!$ is a fully faithful left adjoint, the unit map $\mathrm{id} \rightarrow j^*j_!$ is an equivalence. Precomposing $j_!$ with the equivalence $\tilde{\mathbf{E}}\mathcal{F} \wedge (-) \simeq j_*j^*$ of [Proposition 3.23](#), we obtain equivalences $\tilde{\mathbf{E}}\mathcal{F} \wedge j_!(-) \simeq j_*j^*j_! \simeq j_*$ and consequently a commutative diagram

$$\begin{array}{ccc} \mathrm{Spt}_{S^1}^{G,\mathrm{co}(\mathcal{F})}(S) & \xrightarrow{j_!} & \mathrm{Spt}_{S^1}^G(S) \\ & \searrow \bar{j}_* & \downarrow \tilde{\mathbf{E}}\mathcal{F} \wedge - \\ & & \mathrm{L}_{\mathrm{co}(\mathcal{F})}\mathrm{Spt}_{S^1}^G(S). \end{array}$$

In particular, the functor $\bar{j}_* : \mathrm{Spt}_{S^1}^{G,\mathrm{co}(\mathcal{F})}(S) \rightarrow \mathrm{L}_{\mathrm{co}(\mathcal{F})}\mathrm{Spt}_{S^1}^G(S)$ is a composite of symmetric monoidal functors, hence symmetric monoidal itself. It is fully faithful, since the composite functor $\mathrm{Spt}_{S^1}^{G,\mathrm{co}(\mathcal{F})}(S) \rightarrow \mathrm{L}_{\mathrm{co}(\mathcal{F})}\mathrm{Spt}_{S^1}^G(S) \subseteq \mathrm{Spt}_{S^1}^G(S)$ (also denoted j_*) is fully faithful by [Proposition 2.30](#). To see that j_* is also essential surjective, suppose given $X \in \mathrm{Spt}_{S^1}^G(S)$, and consider $j^*X \in \mathrm{Spt}_{S^1}^{G,\mathrm{co}(\mathcal{F})}(S)$. By the previous equivalence $\tilde{\mathbf{E}}\mathcal{F} \wedge j_!(-) \simeq j_*$ and [Proposition 3.23](#), we see that

$$\tilde{\mathbf{E}}\mathcal{F} \wedge j_!j^*X \simeq \tilde{\mathbf{E}}\mathcal{F} \wedge X$$

since both are equivalent to j_*j^*X . Thus $\bar{j}_* : \mathrm{Spt}_{S^1}^{G,\mathrm{co}(\mathcal{F})}(S) \rightarrow \mathrm{Spt}_{S^1}^{G,\mathrm{co}(\mathcal{F})}(S)$ is an equivalence of symmetric monoidal ∞ -categories. \square

Proposition 3.25. *Let \mathcal{F} be a family which satisfies [Condition 3.19](#) and write $j : \mathrm{Sm}_S^G[\mathrm{co}(\mathcal{F})] \subseteq \mathrm{Sm}_S^G$ for the inclusion. Suppose that $p^*\mathcal{T}$ is in $\mathrm{Spc}_\bullet^{G,\mathrm{co}(\mathcal{F})}(S)$.*

1. *$j_!$ induces a symmetric monoidal functor $j_! : \mathrm{Spt}_{\mathcal{T}}^{G,\mathrm{co}(\mathcal{F})}(S) \rightarrow \mathrm{Spt}_{\mathcal{T}}^G(S)$.*
2. *$\tilde{\mathbf{E}}\mathcal{F} \wedge j_!$ induces a symmetric monoidal equivalence*

$$\tilde{\mathbf{E}}\mathcal{F} \wedge j_! : \mathrm{Spt}_{\mathcal{T}}^{G,\mathrm{co}(\mathcal{F})}(S) \rightarrow \mathrm{L}_{\mathrm{co}(\mathcal{F})}\mathrm{Spt}_{\mathcal{T}}^G(S).$$

Proof. Since $j_!$ is a symmetric monoidal left adjoint, it induces a symmetric monoidal functor on $p^*\mathcal{T}$ -stabilizations.

By [Proposition 3.24](#) the induced map

$$\mathrm{Spt}_{\mathcal{T}}^{G,\mathrm{co}(\mathcal{F})}(S) \simeq \mathrm{Spt}_{S^1}^{G,\mathrm{co}(\mathcal{F})}(S)[p^*\mathcal{T}^{-1}] \rightarrow (\mathrm{L}_{\mathrm{co}(\mathcal{F})}\mathrm{Spt}_{S^1}^G(S))[p^*\mathcal{T}^{-1}]$$

is an equivalence in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes})$. The result now follows from the equivalence from [Lemma 2.40](#)

$$(\mathrm{L}_{\mathrm{co}(\mathcal{F})}\mathcal{S}\mathrm{pt}_{S^1}^G(S))[p^*\mathcal{T}^{-1}] \simeq \mathrm{L}_{\mathrm{co}(\mathcal{F})}(\mathcal{S}\mathrm{pt}_{S^1}^G(S))[p^*\mathcal{T}^{-1}].$$

□

3.5 Adjacent pairs

Lemma 3.26. *Suppose that $\mathcal{F} \subseteq \mathcal{F}'$ is N -adjacent at H and that H acts trivially on S . Let $X \in \mathrm{Sm}_S^G[\mathcal{F}' \setminus \mathcal{F}]$. Then the canonical map*

$$f : G \times_{N_G H} X^H \rightarrow X$$

is an isomorphism.

Proof. The map f is identified with the map $\coprod_{[g] \in G/N_G(H)} X^{gHg^{-1}} \rightarrow X$ induced by the inclusions $X^{gHg^{-1}} \subseteq X$. We show that the induced map f_s on the fiber over any $s \in S$, is an isomorphism. By [\[GD67, 17.9.5\]](#) this implies that f is an isomorphism as it is a map between smooth (in particular flat) finitely presented S -schemes. The stabilizer $\mathrm{Stab}(x)$ of any $x \in X_s$ contains a subgroup conjugate to H which implies that $\coprod X_s^{gHg^{-1}} \rightarrow X_s$ is surjective. On the other hand the closed subschemes $X_s^{gHg^{-1}} \subseteq X_s$ are pairwise disjoint since all stabilizers of X_s are in $\{K \leq G \mid (K \cap N) = (H)\}$. Indeed, if $\mathrm{Stab}(x)$ contains both H and gHg^{-1} , $\mathrm{Stab}(x) \cap N$ contains both of these subgroups, which means that $H = gHg^{-1}$. Thus $\coprod X_s^{gHg^{-1}} \rightarrow X_s$ is bijective, closed immersion. Since these are smooth over $\mathrm{Spec}(k(s))$, in particular reduced, the map f_s is an isomorphism. □

Let $H \leq N$ be a subgroup. Then $W_N H$ is a normal subgroup of $W_G H$ and we will often write $WH = W_G H / W_N H$ for the quotient of Weyl groups. We will often write $\mathbf{E}_{W_N H}(W_G H)$ instead of $\mathbf{E}\mathcal{F}(W_N H)$ for the universal $W_N H$ -free motivic $W_G H$ -motivic space, in order to emphasize the ambient group, where $\mathcal{F}(W_N H)$ is the family of subgroups $\{K \leq W_G H \mid K \cap W_N H = \{e\}\}$.

Proposition 3.27. *Suppose that $\mathcal{F} \subseteq \mathcal{F}'$ is N -adjacent at H . Then*

1. $(G \times_{N_G H} \mathbf{E}_{W_N H}(W_G H))_+ \wedge \mathbf{E}(\mathcal{F}', \mathcal{F}) \xrightarrow{\sim} \mathbf{E}(\mathcal{F}', \mathcal{F})$ is an equivalence in $\mathcal{S}\mathrm{pt}_{S^1}^G(S)$
2. $\mathbf{E}(\mathcal{F}', \mathcal{F})|_{N_G H} \xrightarrow{\sim} \widetilde{\mathbf{E}}\mathcal{F}[H] \wedge \mathbf{E}(\mathcal{F}', \mathcal{F})|_{N_G H}$ in $\mathcal{S}\mathrm{pt}_{S^1}^{N_G H}(S)$.

Proof. It suffices to prove these equivalences in the case $S = B$; the general case follows by applying f^* where for $f : S \rightarrow B$ is the structure map. For the first item, by [Proposition 3.21](#), it suffices to show that the projection

$$p : (G \times_{N_G H} \mathbf{E}_{W_N H}(W_G H))_+ \rightarrow S^0$$

induces equivalences $\mathrm{Map}_{\mathrm{Spc}^G(B)}(X, p)$, for any $X \in \mathrm{Sm}_B^G[\mathcal{F}' \setminus \mathcal{F}]$. But for such an X we have by [Lemma 3.26](#) that $X \cong G \times_{N_G H} X^H$, so we have

$$\begin{aligned} & \underline{\mathrm{Map}}_{\mathcal{S}\mathrm{pt}_{S^1}^G(B)}(X_+, (G \times_{N_G H} \mathbf{E}_{W_N H}(W_G H))_+) \\ & \simeq \underline{\mathrm{Map}}_{\mathcal{S}\mathrm{pt}_{S^1}^{N_G H}(B)}(X_+^H, (G \times_{N_G H} \mathbf{E}_{W_N H}(W_G H))_+) \\ & \simeq \underline{\mathrm{Map}}_{\mathcal{S}\mathrm{pt}_{S^1}^{W_G H}(B)}(X_+^H, (G \times_{N_G H} \mathbf{E}_{W_N H}(W_G H))_+^H) \\ & \simeq \underline{\mathrm{Map}}_{\mathcal{S}\mathrm{pt}_{S^1}^{W_G H}(B)}(X_+^H, (S^0)^H) \\ & \simeq \underline{\mathrm{Map}}_{\mathcal{S}\mathrm{pt}_{S^1}^{N_G H}(B)}(X_+^H, S^0) \\ & \simeq \underline{\mathrm{Map}}_{\mathcal{S}\mathrm{pt}_{S^1}^G(B)}((G \times_{N_G H} X^H)_+, S^0). \end{aligned}$$

For the second item, we have that $\mathcal{F}_{N_G H}[H] \cap \mathcal{F}'|_{N_G H} = \mathcal{F}_{N_G H}[H] \cap \mathcal{F}|_{N_G H}$ and so this follows from [Lemma 3.18](#). □

4 Fixed point functors

We define fixed point functors on motivic spaces and spectra. Throughout this section, $N \trianglelefteq G$ is a normal subgroup and we write $\pi : G \rightarrow G/N$ for the quotient homomorphism. Unless noted otherwise, we assume that N acts trivially on S .

4.1 Fixed point motivic spaces

We begin by extending the adjunction

$$\pi^{-1} : \mathrm{Sm}_S^{G/N} \rightleftarrows \mathrm{Sm}_S^G : (-)^N$$

to motivic G -spaces. Note that π^{-1} is full and faithful and induces an equivalence

$$\mathrm{Sm}_S^{G/N} \simeq \mathrm{Sm}_S^G[\mathrm{co}(\mathcal{F}[N])] \subseteq \mathrm{Sm}_S^G \quad (4.1)$$

with the subcategory of smooth G -schemes over S on which N acts trivially.

The functor π_* in the following propositions is the N -fixed point functor on motivic G -spaces. In keeping with standard notation, we sometimes write

$$(-)^N := \pi_*.$$

Proposition 4.2. *The functor $\pi^{-1} : \mathrm{Sm}_S^{G/N} \rightarrow \mathrm{Sm}_S^G$ induces adjoint pairs of functors*

$$\begin{aligned} \pi^* : \mathrm{Spc}^{G/N}(S) &\rightleftarrows \mathrm{Spc}^G(S) : \pi_*, \text{ and} \\ \pi^* : \mathrm{Spc}_\bullet^{G/N}(S) &\rightleftarrows \mathrm{Spc}_\bullet^G(S) : \pi_* \end{aligned}$$

such that the diagrams commute

$$\begin{array}{ccc} \mathrm{Sm}_S^{G/N} & \xrightarrow{\pi^{-1}} & \mathrm{Sm}_S^G & & \mathrm{Sm}_S^G & \xrightarrow{(-)^N} & \mathrm{Sm}_S^{G/N} \\ \downarrow & & \downarrow & \text{and} & \downarrow & & \downarrow \\ \mathrm{Spc}^{G/N}(S) & \xrightarrow{\pi^*} & \mathrm{Spc}^G(S) & & \mathrm{Spc}^G(S) & \xrightarrow{\pi_*} & \mathrm{Spc}^{G/N}(S) \\ (-)_+ \downarrow & & \downarrow (-)_+ & & (-)_+ \downarrow & & (-)_+ \downarrow \\ \mathrm{Spc}_\bullet^{G/N}(S) & \xrightarrow{\pi^*} & \mathrm{Spc}_\bullet^G(S) & & \mathrm{Spc}_\bullet^G(S) & \xrightarrow{\pi_*} & \mathrm{Spc}_\bullet^{G/N}(S). \end{array}$$

Moreover, π^* and π_* are symmetric monoidal and π_* preserve colimits.

Proof. Nisnevich topologies correspond under the equivalence (4.1), so the adjunction can be obtained as a special case of [Proposition 2.28](#), where we set $\pi^* := (\pi^{-1})_!$ and $\pi_* := (\pi^{-1})_*$. The remaining claims about the π^* and π_* are straightforward. \square

4.2 Fixed point spectra

We view G/N -equivariant vector bundles on S as G -equivariant vector bundles via the quotient homomorphism $\pi : G \rightarrow G/N$. Given a stabilizing subset $\mathcal{J} \subseteq \mathrm{Sph}_B^{G/N}$, we have the stabilizing subset

$$\pi^*\mathcal{J} = \{T^{\pi^*\mathcal{E}} \mid T^{\mathcal{E}} \in \mathcal{J}\} \subseteq \mathrm{Sph}_B^G$$

which for simplicity we usually write again \mathcal{J} . We call a G -sphere of the form $\pi^*(T^{\mathcal{E}})$ an N -trivial G -sphere. Recall also that we write $N\text{-triv} = \pi^*(\mathrm{Sph}_B^{G/N})$.

Proposition 4.3. *Let $\mathcal{T} \subseteq \mathrm{Sph}_B^{G/N}$ be a stabilizing set of spheres. There is an adjoint pair of symmetric monoidal functors*

$$\pi^* : \mathrm{Spt}_{\mathcal{T}}^{G/N}(S) \rightleftarrows \mathrm{Spt}_{\mathcal{T}}^G(S) : \pi_*$$

such that $\pi_*(\Sigma_{\mathcal{T}}^\infty Y) \simeq \Sigma_{\mathcal{T}}^\infty(Y^N)$ for $Y \in \mathrm{Spc}_{\bullet}^G(S)$.

Proof. The adjoint pair is the stabilization of the adjoint pair in [Proposition 4.2](#), using that $\pi^*(X \otimes T^\varepsilon) \simeq \pi^*(X) \otimes T^{\pi^*\varepsilon}$ and $(Y \otimes T^{\pi^*\varepsilon})^N \simeq Y^N \otimes T^\varepsilon$. \square

That the fixed point functor in the previous proposition commutes with stabilization is a consequence of the fact that we have only stabilized with respect to a set of N -trivial spheres. In general, fixed point functors do not commute with stabilization; rather, this is a key feature of the geometric fixed points functor, defined later.

Lemma 4.4. *Let $\mathcal{T} \subseteq \mathrm{Sph}_B^{G/N}$. The equivalence [\(4.1\)](#) induces inverse equivalences*

$$\tilde{\pi}^* : \mathrm{Spt}_{\mathcal{T}}^{G/N}(S) \simeq \mathrm{Spt}_{\mathcal{T}}^{G, \mathrm{co}(\mathcal{F}[N])}(S) : \tilde{\pi}_*$$

which fit into commutative diagrams

$$\begin{array}{ccc} \mathrm{Spt}_{\mathcal{T}}^{G/N}(S) & \xrightarrow{\tilde{\pi}^*} & \mathrm{Spt}_{\mathcal{T}}^{G, \mathrm{co}(\mathcal{F}[N])}(S) \\ & \searrow \pi^* & \downarrow j! \\ & & \mathrm{Spt}_{\mathcal{T}}^G(S) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Spt}_{\mathcal{T}}^{G, \mathrm{co}(\mathcal{F}[N])}(S) & \xrightarrow{\tilde{\pi}_*} & \mathrm{Spt}_{\mathcal{T}}^{G/N}(S) \\ & \downarrow j! & \nearrow \pi_* \\ & \mathrm{Spt}_{\mathcal{T}}^G(S) & \end{array}$$

Proof. Nisnevich topologies correspond under the equivalence [\(4.1\)](#), so that $\pi^* : \mathrm{Spc}_{\bullet}^{G/N}(S) \simeq \mathrm{Spc}_{\bullet}^{G, \mathrm{co}(\mathcal{F}[N])}(S)$ is an equivalence of symmetric monoidal ∞ -categories with inverse π_* . The first statement follows. The commutativity of the displayed diagrams is straightforward to check. \square

Let $\mathcal{T} \subseteq \mathrm{Sph}_B^G$ be a stabilizing set of G -spheres and $\mathcal{T}^N = \{T^{\varepsilon^N} \mid T^\varepsilon \in \mathcal{T}\}$ the associated stabilizing set of G/N -spheres. Continuing to overload notation, we simply write $\pi^* : \mathrm{Spt}_{\mathcal{T}^N}^{G/N}(S) \rightarrow \mathrm{Spt}_{\mathcal{T}}^G(S)$ again for composite

$$\mathrm{Spt}_{\mathcal{T}^N}^{G/N}(S) \xrightarrow{\iota^*} \mathrm{Spt}_{\mathcal{T}}^{G/N}(S) \xrightarrow{\pi^*} \mathrm{Spt}_{\mathcal{T}}^G(S).$$

Since π^* preserves colimits, we obtain the fixed-points adjunction

$$\pi^* : \mathrm{Spt}_{\mathcal{T}^N}^{G/N}(S) \rightleftarrows \mathrm{Spt}_{\mathcal{T}}^G(S) : \pi_*.$$

The stabilizing set of spheres \mathcal{T} does not appear in the notation for the fixed points functor π_* . We usually consider stabilization with respect to all spheres and will always make the domain of π_* explicit in other cases. When $\mathcal{T} = \mathrm{Sph}_B^G$, in keeping with standard notation, we will sometimes write

$$(-)^N := \pi_*.$$

We have also not made reference to the base scheme in the notation. We show in the next section, after proving the Adams isomorphism, that the fixed points functor is compatible with the various change-of-base functors in motivic homotopy.

4.3 Geometric fixed points

Recall that $\mathcal{F}[N] := \{H \leq G \mid N \not\subseteq H\}$ and that we write

$$\mathrm{L}_{\mathrm{co}(\mathcal{F}[N])}\mathcal{S}\mathrm{pt}_{\mathcal{F}}^G(S) \subseteq \mathcal{S}\mathrm{pt}_{\mathcal{F}}^G(S).$$

for the essential image of the endofunctor $\tilde{\mathbf{E}}\mathcal{F}[N] \otimes - : \mathcal{S}\mathrm{pt}_{\mathcal{F}}^G(S) \rightarrow \mathcal{S}\mathrm{pt}_{\mathcal{F}}^G(S)$. Write $W = \rho_G/\rho_{G/N}$. By [Example 3.6](#) we have

$$\tilde{\mathbf{E}}\mathcal{F}[N] \simeq \mathrm{colim}_n T^{nW}. \quad (4.5)$$

Lemma 4.6. *The map $S^0 \rightarrow T^W$ induces an equivalence in $\mathrm{Spc}_{\bullet}^G(S)$,*

$$\tilde{\mathbf{E}}\mathcal{F}[N] \simeq T^W \otimes \tilde{\mathbf{E}}\mathcal{F}[N].$$

Proof. This follows from (4.5) together with the fact that the cyclic permutation acts as the identity on $T^W \wedge T^W \wedge T^W$, see [[Hoy17](#), Lemma 6.3]. \square

Proposition 4.7. *The stabilization functor $\mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S) \rightarrow \mathcal{S}\mathrm{pt}^G(S)$ induces an equivalence*

$$\mathrm{L}_{\mathrm{co}(\mathcal{F}[N])}\mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S) \xrightarrow{\simeq} \mathrm{L}_{\mathrm{co}(\mathcal{F}[N])}\mathcal{S}\mathrm{pt}^G(S)$$

of symmetric monoidal stable ∞ -categories.

Proof. Write $\mathrm{L} = \mathrm{L}_{\mathrm{co}(\mathcal{F}[N])}$. We make use of the equivalences $\mathcal{S}\mathrm{pt}_{T^{\rho_{G/N}}}^G(S) \simeq \mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S)$ and $\mathcal{S}\mathrm{pt}_{T^{\rho_G}}^G(S) \simeq \mathcal{S}\mathrm{pt}^G(S)$ and the equivalence (4.5). Since $T^{\rho_G} = T^{\rho_{G/N}} \otimes T^W$, we have an equivalence

$$\mathcal{S}\mathrm{pt}^G(S) \simeq \mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S)[(T^W)^{-1}]$$

and under this equivalence $\mathrm{L}\mathcal{S}\mathrm{pt}^G(S) \simeq (\mathrm{L}\mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S))[(T^W)^{-1}]$ by [Lemma 2.40](#). But it follows from [Lemma 4.6](#) that Σ^W is an autoequivalence of $\mathrm{L}\mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S)$, and therefore the stabilization induces an equivalence

$$\mathrm{L}\mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S) \simeq (\mathrm{L}\mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S))[(T^W)^{-1}].$$

The result follows since $(\mathrm{L}\mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S))[(T^W)^{-1}] \simeq \mathrm{L}(\mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S))[(T^W)^{-1}]$. \square

Write ψ for the composite

$$\mathcal{S}\mathrm{pt}^{G/N}(S) \xrightarrow{\pi^*} \mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S) \rightarrow \mathcal{S}\mathrm{pt}^G(S) \rightarrow \mathrm{L}_{\mathrm{co}(\mathcal{F}[N])}\mathcal{S}\mathrm{pt}^G(S).$$

Proposition 4.8. *The functor $\psi : \mathcal{S}\mathrm{pt}^{G/N}(S) \rightarrow \mathrm{L}_{\mathrm{co}(\mathcal{F}[N])}\mathcal{S}\mathrm{pt}^G(S)$ is an equivalence of symmetric monoidal stable ∞ -categories.*

Proof. By [Proposition 4.7](#) this composite is equivalent to

$$\mathcal{S}\mathrm{pt}^{G/N}(S) \xrightarrow{\pi^*} \mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S) \xrightarrow{\tilde{\mathbf{E}}\mathcal{F}[N] \otimes -} \mathrm{L}_{\mathrm{co}(\mathcal{F}[N])}\mathcal{S}\mathrm{pt}_{N\text{-triv}}^G(S),$$

which is an equivalence by [Lemma 4.4](#) and [Proposition 3.25](#). \square

Definition 4.9. Let $N \trianglelefteq G$ be a normal subgroup. The *motivic geometric fixed points functor*

$$(-)^{\Phi^N} : \mathcal{S}\mathrm{pt}^G(S) \rightarrow \mathcal{S}\mathrm{pt}^{G/N}(S)$$

is the composite $\mathcal{S}\mathrm{pt}^G(S) \rightarrow \mathrm{L}_{\mathrm{co}(\mathcal{F}[N])}\mathcal{S}\mathrm{pt}^G(S) \xrightarrow{\psi^{-1}} \mathcal{S}\mathrm{pt}^{G/N}(S)$

Proposition 4.10. *The functor $(-)^{\Phi^N} : \mathrm{Spt}^G(S) \rightarrow \mathrm{Spt}^{G/N}(S)$ satisfies the following properties.*

1. *It is a symmetric monoidal left adjoint. Moreover, its right adjoint $\tilde{\mathbf{E}}\mathcal{F}[N] \otimes \pi^*$ is full and faithful.*
2. *There is a natural equivalence $(\Sigma^\infty X)^{\Phi^N} \simeq \Sigma^\infty(X^N)$ for $X \in \mathrm{Spc}_\bullet^G(S)$.*
3. *$(-)^{\Phi^N} \simeq (\tilde{\mathbf{E}}\mathcal{F}[N] \otimes -)^N$.*
4. *There is a natural transformation $(-)^N \rightarrow (-)^{\Phi^N}$.*
5. *There is a natural equivalence $\tilde{\mathbf{E}}\mathcal{F}[N] \otimes Y \simeq \tilde{\mathbf{E}}\mathcal{F}[N] \otimes Y^{\Phi^N}$ for $Y \in \mathrm{Spt}^G(S)$.*

Proof. Easy consequence of the results above. \square

In § 6.2 and § 6.4 below, we show that fixed points commutes with arbitrary base-change. We note here the easier fact that geometric fixed points commute with arbitrary base-change.

Proposition 4.11. *Let $p : T \rightarrow S$ be a map in $\mathrm{Sch}_B^{G/N}$. There is a natural equivalence $p^*(Y^{\Phi^N}) \simeq (p^*Y)^{\Phi^N}$.*

Proof. Since $p^*\tilde{\mathbf{E}}\mathcal{F}[N]_S \simeq \tilde{\mathbf{E}}\mathcal{F}[N]_T$, we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spt}^{G/N}(S) & \xrightarrow{\pi^*} & \mathrm{Spt}^G(S) & \longrightarrow & \mathrm{L}_{\mathrm{co}(\mathcal{F}[N])}\mathrm{Spt}^G(S) \\ \downarrow p^* & & \downarrow p^* & & \downarrow p^* \\ \mathrm{Spt}^{G/N}(T) & \xrightarrow{\pi^*} & \mathrm{Spt}^G(T) & \longrightarrow & \mathrm{L}_{\mathrm{co}(\mathcal{F}[N])}\mathrm{Spt}^G(T), \end{array}$$

so that $p^*\psi \simeq \psi p^*$, which implies the result. \square

Recall that for a subgroup $H \leq N$ we write $\mathcal{W}H = \mathrm{W}_G H / \mathrm{W}_N H$ for the quotient of Weyl groups. We also write $\mathbf{E}_{\mathrm{W}_N H}(\mathrm{W}_G H)$ for the universal $\mathrm{W}_N H$ -free $\mathrm{W}_G H$ -motivic space (also denoted $\mathbf{E}\mathcal{F}(\mathrm{W}_N H)$). See § 3.2 for a recollection of N -adjacency.

Proposition 4.12. *Suppose that $\mathcal{F} \subseteq \mathcal{F}'$ is N -adjacent at the subgroup $H \leq N$. Then, for $X \in \mathrm{Spt}^G(S)$, there is a natural equivalence*

$$(\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X)^N \simeq G/N_+ \times_{\mathcal{W}H} (\mathbf{E}_{\mathrm{W}_N H}(\mathrm{W}_G H)_+ \otimes X^{\Phi^H})^{\mathrm{W}_N H}$$

Proof. Consider the commutative diagram of group homomorphisms

$$\begin{array}{ccccc} & & \mathrm{N}_G H & \xleftarrow{\lambda} & G \\ & \swarrow \pi' & \downarrow & & \downarrow \pi \\ \mathrm{W}_G H & \xrightarrow{\pi''} & \mathcal{W} & \xleftarrow{\bar{\lambda}} & G/N. \end{array}$$

There is a natural equivalence $\pi_* \lambda_! \simeq \bar{\lambda}_! \pi''_* \pi'_*$ since both are seen to be right adjoint to the restriction along $\mathrm{N}_G H \rightarrow G/N$. By Proposition 3.27, the canonical maps

$$\begin{array}{ccc} \lambda_! (\mathbf{E}_{\mathrm{W}_N H}(\mathrm{W}_G H)_+ \otimes \lambda^*(\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X)) & \xrightarrow{\sim} & \mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X \\ \downarrow \sim & & \\ \lambda_! (\mathbf{E}_{\mathrm{W}_N H}(\mathrm{W}_G H)_+ \otimes \tilde{\mathbf{E}}\mathcal{F}[H] \otimes \lambda^*(\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X)) & & \end{array}$$

are equivalences. Applying π_* , the result follows from the equivalence displayed above and the fact that $(-)^{\Phi}$ is symmetric monoidal. \square

4.4 Homotopy fixed points and the Tate construction

Homotopy fixed points together with the Tate spectrum, introduced in [GM95], play critical roles in computations and applications of equivariant homotopy theory. Using the constructions above, it is straightforward to define the motivic version of these functors and to establish the analogue of the Tate diagram for $G = C_p$.

Definition 4.13. Let $X \in \mathcal{Spt}^G(S)$ and $N \leq G$ a normal subgroup.

1. The *motivic homotopy fixed point spectrum* of X is

$$X^{\mathbf{h}N} := \pi_* i_* i^*(X) \simeq F(\mathbf{E}\mathcal{F}(N)_+, X)^N,$$

where $i : \mathcal{S}m_S^{G, N\text{-free}} \subseteq \mathcal{S}m_S^G$ is the inclusion.

2. the *motivic Tate spectrum* of X is

$$X^{\mathbf{t}N} := \left(\tilde{\mathbf{E}}\mathcal{F}(N) \wedge F(\mathbf{E}\mathcal{F}(N)_+, X) \right)^N.$$

Proposition 4.14 (Motivic Tate diagram). *Suppose that p is invertible on B . Let $X \in \mathcal{Spt}^{C_p}(B)$. There is a natural pushout square in $\mathcal{Spt}(B)$*

$$\begin{array}{ccc} X^{C_p} & \longrightarrow & X^{\Phi C_p} \\ \downarrow & & \downarrow \\ X^{\mathbf{h}C_p} & \longrightarrow & X^{\mathbf{t}C_p}. \end{array}$$

Proof. Consider the commutative diagram with exact rows

$$\begin{array}{ccccc} (\mathbf{E}C_{p+} \wedge X)^{C_p} & \longrightarrow & X^{C_p} & \longrightarrow & (\tilde{\mathbf{E}}C_p \wedge X)^{C_p} \\ \sim \downarrow & & \downarrow & & \downarrow \\ (\mathbf{E}C_{p+} \wedge F(\mathbf{E}C_{p+}, X))^{C_p} & \longrightarrow & F(\mathbf{E}C_{p+}, X)^{C_p} & \longrightarrow & (\tilde{\mathbf{E}}C_p \wedge F(\mathbf{E}C_{p+}, X))^{C_p}. \end{array}$$

The proposition follows by noting that the left vertical map is an equivalence. To see this, note that by [Proposition 3.11](#), the map $X \rightarrow F(\mathbf{E}C_{p+}, X)$ is identified with the natural transformation $i_! i^* \eta : i_! i^* \rightarrow i_! i^* i_* i^*$ where η is the unit $\text{id} \rightarrow i_* i^*$ of the adjunction. Since $i_!$ is full and faithful, so is i_* . In particular, the counit $i^* i_* \rightarrow \text{id}$ is an equivalence. That $i_! i^* \eta$ is an equivalence now follows from the triangle identity. \square

5 Quotient spectra

As in the previous section, we let $N \trianglelefteq G$ be a normal subgroup and $\pi : G \rightarrow G/N$ the quotient homomorphism. It turns out (as in classical equivariant homotopy) that the functor $\pi^* : \mathcal{Spt}^{G/N}(S) \rightarrow \mathcal{Spt}^G(S)$ does not have a left adjoint, except in the trivial case that $G = \{e\}$. It does however have a partial left adjoint, constructed in this section, defined on the full subcategory of N -free G -spectra.

5.1 Stabilization of free objects

We write $\mathcal{Spt}_\sigma^{G, N\text{-free}}(S) := \mathcal{Spt}_\sigma^{G, \mathcal{F}(N)}(S)$. In this section, we show that the stabilization

$$\lambda^* : \mathcal{Spt}_{N\text{-triv}}^{G, N\text{-free}}(S) \rightarrow \mathcal{Spt}^{G, N\text{-free}}(S)$$

is an equivalence of stable ∞ -categories. We are grateful to Tom Bachmann for suggestions which streamlined the argument we originally gave here.

Recall that the quotient functor induces an equivalence of categories $(-)/N : \mathrm{Sm}_S^G \simeq \mathrm{Sm}_{S/N}^{G/N}$ if N acts freely on S . Under this equivalence, Nisnevich topologies correspond, so we obtain an equivalence of symmetric monoidal ∞ -categories

$$\mathrm{Spc}_\bullet^G(S) \xrightarrow{\sim} \mathrm{Spc}_\bullet^{G/N}(S/N).$$

Lastly, stabilizing with respect to $\mathrm{Sph}_{S/N}^{G/N}$, we obtain an equivalence of symmetric monoidal ∞ -categories

$$\pi_! : \mathrm{Spt}_{N\text{-triv}}^G(S) \xrightarrow{\sim} \mathrm{Spt}^{G/N}(S/N).$$

Lemma 5.1. *If N acts freely on S , then $\lambda^* : \mathrm{Spt}_{N\text{-triv}}^{G,N\text{-free}}(S) \rightarrow \mathrm{Spt}^{G,N\text{-free}}(S)$ is an equivalence.*

Proof. It suffices to see that if $E \rightarrow S$ is an equivariant vector bundle on S then T^E is invertible in $\mathrm{Spt}_{N\text{-triv}}^{G,N\text{-free}}(S) \simeq \mathrm{Spt}_{N\text{-triv}}^G(S)$. Under the equivalence $\pi_! : \mathrm{Spt}_{N\text{-triv}}^G(S) \simeq \mathrm{Spt}^{G/N}(S/N)$ of symmetric monoidal ∞ -categories, we have that $\pi_!(T^E) \simeq T^{E/N}$, where $E/N \rightarrow S/N$ is the quotient, which is a G/N -equivariant vector bundle. In particular, $T^{E/N}$ is invertible, so the result follows. \square

Theorem 5.2. *Let $N \trianglelefteq G$ be a normal subgroup. The stabilization functor*

$$\lambda^* : \mathrm{Spt}_{N\text{-triv}}^{G,N\text{-free}}(S) \rightarrow \mathrm{Spt}^{G,N\text{-free}}(S)$$

is an equivalence of stable ∞ -categories.

Proof. Write λ_* for the right adjoint of λ^* . We first show that $\mathrm{id} \rightarrow \lambda_*\lambda^*$ is an equivalence, i.e., λ^* is fully faithful. By [Proposition 2.36](#) it suffices to check that $p^* \rightarrow p^*\lambda_*\lambda^*$ is an equivalence for any $p : X \rightarrow S$ in $\mathrm{Sm}_S^{G,N\text{-free}}$ (indeed the family of such p^* is conservative). But λ^* commutes with p^* , and since p is smooth, so does λ_* . Therefore this transformation is identified with $p^* \rightarrow \lambda_*\lambda^*p^*$ and since X is N -free, $\lambda_*\lambda^*p^* \simeq p^*$ by [Lemma 5.1](#) as required.

To see that λ^* is essentially surjective, by [Proposition 2.36](#), it suffices to show that $T^{-\mathcal{V}} \otimes X_+$ is in the image of λ^* , where $T^{\mathcal{V}} \in \mathrm{Sph}_B^G$ and $p : X \rightarrow S$ is in $\mathrm{Sm}_S^{G,N\text{-free}}$. But $T^{-\mathcal{V}} \otimes X_+ \simeq p_{\#}p^*(T^{-\mathcal{V}})$ and by [Lemma 5.1](#), we have $p^*(T^{-\mathcal{V}})$ is in the essential image of λ^* and since λ^* and $p_{\#}$ commute, we are done. \square

Remark 5.3. It follows that when S has N -free action, there is an induced equivalence of symmetric monoidal ∞ -categories

$$\pi_! : \mathrm{Spt}^G(S) \simeq \mathrm{Spt}^{G/N}(S/N).$$

Let $p : X \rightarrow S$ be a map. The exceptional pushforward on N -free G -spectra $p_! : \mathrm{Spt}^{G,N\text{-free}}(X) \rightarrow \mathrm{Spt}^{G,N\text{-free}}(S)$ can be defined as the composite

$$i^*p_!i_! : \mathrm{Spt}^{G,N\text{-free}}(X) \hookrightarrow \mathrm{Spt}^G(X) \rightarrow \mathrm{Spt}^G(S) \rightarrow \mathrm{Spt}^{G,N\text{-free}}(S).$$

If p is smooth, then $p_! \simeq p_{\#}\Sigma^{-\Omega_f}$, where Ω_f is the sheaf of differentials of X over S .

Corollary 5.4. *The stable ∞ -category $\mathrm{Spt}^{G,N\text{-free}}(S)$ is generated under colimits by any of the following sets.*

1. $\Sigma^{-k\rho_{G/N}}p_{\#}\mathbb{1}_X$ where $k \geq 0$ and $p : X \rightarrow S$ is in $\mathrm{Sm}_S^{G,N\text{-free}}$ with X affine.
2. $\Sigma^{-k\rho_{G/N}}p_!\mathbb{1}_X$ where $k \geq 0$ and $p : X \rightarrow S$ is in $\mathrm{Sm}_S^{G,N\text{-free}}$ with X affine.
3. $\Sigma^{-k\rho_{G/N}}i^*q_*\mathbb{1}_X$ where $k \geq 0$ and $q : X \rightarrow S$ is in Sch_S^G with q projective.
4. $\Sigma^{-k\rho_{G/N}}q_*\mathbb{1}_X$ where $k \geq 0$ and $q : X \rightarrow S$ is in $\mathrm{Sch}_S^{G,N\text{-free}}$.

Proof. The objects in (1) are generators of $\mathcal{Spt}_{N\text{-triv}}^{G,N\text{-free}}(S)$ by [Proposition 2.36](#).

If X is affine then there is a surjection $p^*(n\rho_G) \rightarrow \Omega_p$ for some $n > 0$. Let \mathcal{E} be the kernel of this map. Then in $\mathcal{Spt}^{G,N\text{-free}}(X)$ we have an equivalence $T^{\Omega_p} \wedge T^{\mathcal{E}} \simeq T^{n\rho_{G/N}}$. Let $r : \mathbb{V}(\mathcal{E}) \rightarrow X$ be the projection. We then have

$$\Sigma^{n\rho_{G/N}}(p \circ r)_!(\mathbb{1}_{\mathbb{V}(\mathcal{E})}) \simeq p\# \mathbb{1}_X.$$

Therefore the generators in (1) are contained in the category generated under colimits by the objects in (2).

Next, we check that the set in (2) is contained in the the category generated under colimits by the objects $\Sigma^{-k\rho_{G/N}} i^* q_* \mathbb{1}_X$ of (3).

Let $p : Y \rightarrow S$ be in $\mathcal{Sm}_S^{G,N\text{-free}}$. Since p is G -quasi-projective, there is an equivariant compactification

$$\begin{array}{ccccc} Y & \xleftarrow{u} & \bar{Y} & \xleftarrow{t} & Z \\ & \searrow p & \downarrow f & \swarrow g & \\ & & S & & \end{array}$$

where f is an equivariant projective morphism, u is an invariant open, and t is an invariant closed complement. From the gluing sequence, using the fact that f, g are proper, we obtain an exact sequence in $\mathcal{Spt}^G(S)$ of the form

$$\Sigma^{-k\rho_{G/N}} p! \mathbb{1}_Y \rightarrow \Sigma^{-k\rho_{G/N}} f_* \mathbb{1}_{\bar{Y}} \rightarrow \Sigma^{-k\rho_{G/N}} g_* \mathbb{1}_Z.$$

Applying i^* finishes the third case.

Write $\mathbf{EF}(N) \simeq \text{colim}_n U_n$ where $U_n \in \mathcal{Sm}_S^{G,N\text{-free}}$. Now, case (4) follows by noting that if $q : X \rightarrow S$ is projective, then $i_! i^* q_*(\mathbb{1}_X)$ is the colimit of $q_{n*}(\mathbb{1}_{X \times U_n})$ where $q_n : X \times U_n \rightarrow S$.

□

5.2 Quotient functors

Proposition 5.5. *Let $N \trianglelefteq G$ be a normal subgroup which acts trivially on S .*

1. *There are colimit preserving functors*

$$\begin{aligned} (-)/N &: \mathcal{Spc}^{G,N\text{-free}}(S) \rightarrow \mathcal{Spc}^{G/N}(S/N) \text{ and} \\ (-)/N &: \mathcal{Spc}_{\bullet}^{G,N\text{-free}}(S) \rightarrow \mathcal{Spc}_{\bullet}^{G/N}(S/N) \end{aligned}$$

such that the diagram commutes

$$\begin{array}{ccc} \mathcal{Sm}_S^{G,N\text{-free}} & \xrightarrow{(-)/N} & \mathcal{Sm}_{S/N}^{G/N} \\ \downarrow & & \downarrow \\ \mathcal{Spc}^{G,N\text{-free}}(S) & \xrightarrow{(-)/N} & \mathcal{Spc}^{G/N}(S/N) \\ (-)_+ \downarrow & & \downarrow (-)_+ \\ \mathcal{Spc}_{\bullet}^{G,N\text{-free}}(S) & \xrightarrow{(-)/N} & \mathcal{Spc}_{\bullet}^{G/N}(S/N). \end{array}$$

2. *The functor $(-)/N$ satisfies a projection formula: $(X \times \pi^* Y)/N \simeq (X/N) \times Y$ for $X \in \mathcal{Spc}^{G,N\text{-free}}(S)$ and $Y \in \mathcal{Spc}^{G/N}(S/N)$. Similarly, if A, B are based then $(A \wedge \pi^* B)/N \simeq (A/N) \wedge B$.*

Proof. Write $q : \mathrm{Sm}_S^G \rightarrow \mathrm{Sm}_{S/N}^{G/N}$ for the quotient functor, $q(W) = W/N$. Since q sends Nisnevich squares in $\mathrm{Sm}_S^{G, N\text{-free}}$ to Nisnevich squares in $\mathrm{Sm}_{S/N}^{G/N}$ and

$$q(W \times \mathbb{A}^1) \cong q(W) \times \mathbb{A}^1,$$

the functor $q^* : \mathcal{P}(\mathrm{Sm}_{S/N}^{G/N}) \rightarrow \mathcal{P}(\mathrm{Sm}_S^G)$ defined by precomposition restricts to a functor $q^* : \mathrm{Spc}^{G/N}(S/N) \rightarrow \mathrm{Spc}^{G, N\text{-free}}(S)$. Since q_* preserves limits, it admits a left adjoint

$$(-)/N : \mathrm{Spc}^{G, N\text{-free}}(S) \rightarrow \mathrm{Spc}^{G/N}(S/N)$$

with the stated properties. Similarly, since q_* preserves the terminal object, it induces a limit preserving functor $q_* : \mathrm{Spc}_\bullet^{G/N}(S/N) \rightarrow \mathrm{Spc}_\bullet^{G, N\text{-free}}(S)$ on based spaces and therefore admits a left adjoint $(-)/N : \mathrm{Spc}_\bullet^{G, N\text{-free}}(S) \rightarrow \mathrm{Spc}_\bullet^{G/N}(S/N)$. The last statement follows from [Proposition 2.2](#). \square

If $X \in \mathrm{Sm}_S^G$ doesn't have free action, the scheme X/G need not be smooth. It is still possible to define a quotient functor on motivic G -spaces. However, this functor does not stabilize to give a quotient functor on all of $\mathrm{Spt}^G(S)$.

Proposition 5.6. *Let $N \trianglelefteq G$ be a normal subgroup which acts trivially on S .*

1. *There is a colimit preserving functor*

$$\pi_! : \mathrm{Spt}^{G, N\text{-free}}(S) \rightarrow \mathrm{Spt}^{G/N}(S/N)$$

such that the diagram commutes

$$\begin{array}{ccc} \mathrm{Sm}_S^{G, N\text{-free}} & \xrightarrow{\pi_!} & \mathrm{Sm}_{S/N}^{G/N} \\ \downarrow & & \downarrow \\ \mathrm{Spt}^{G, N\text{-free}}(S) & \xrightarrow{\pi_!} & \mathrm{Spt}^{G/N}(S/N). \end{array}$$

The right adjoint of $\pi_!$ is the composite $i^ \pi^*$ where $i : \mathrm{Sm}_S^{G, N\text{-free}} \subseteq \mathrm{Sm}_S^G$ is the inclusion.*

2. *The functor $\pi_!$ satisfies a projection formula. That is, if $X \in \mathrm{Spt}^{G, N\text{-free}}(S)$ and $Y \in \mathrm{Spt}^{G/N}(S/N)$ then $\pi_!(X \otimes \pi^* Y) \simeq \pi_!(X) \otimes Y$.*

Proof. If $V \rightarrow S$ is G/N -equivariant vector bundle, we have $(\Sigma^V X)/N \simeq \Sigma^V X/N$ by [Proposition 5.5](#). It follows that $(-)/N$ extends to a colimit preserving functor

$$\pi_! : \mathrm{Spt}_{N\text{-triv}}^{G, N\text{-free}}(S) \rightarrow \mathrm{Spt}^{G/N}(S).$$

The first statement then follows from [Theorem 5.2](#). The second statement follows from [Proposition 5.5](#). \square

6 The motivic Adams isomorphism

In classical homotopy, the Adams isomorphism identifies the N -fixed points of an N -free G -spectrum X with the quotient of X by the N -action. This equivalence was established by Adams for $N = G$ in [[Ada84](#), Theorem 5.3] and generalized in [[LMSM86](#), Theorem II.7.1]. A recent modern take, in terms of orthogonal spectra, on this appears in [[RV16](#)]. In this section, we establish a version for N -free motivic G -spectra.

6.1 The Adams transformation

For the remainder of this section we suppose that S has trivial N -action and we let $\pi : G \rightarrow G/N$ denote the quotient homomorphism. As before, we write $i : \mathrm{Sm}_S^{G, N\text{-free}} \subseteq \mathrm{Sm}_S^G$ for the inclusion. The Adams isomorphism is a comparison of the two functors

$$\pi_!, \pi_* i_! : \mathcal{Spt}^{G, N\text{-free}}(S) \rightarrow \mathcal{Spt}^{G/N}(S).$$

We first construct a comparison transformation $\tau : \pi_! \rightarrow \pi_* i_!$. Consider the cartesian square of surjective homomorphisms

$$\begin{array}{ccc} G \times_{G/N} G & \xrightarrow{\mathrm{pr}_1} & G \\ \mathrm{pr}_2 \downarrow & & \downarrow \pi \\ G & \xrightarrow{\pi} & G/N. \end{array} \quad (6.1)$$

Write $G' = G \times_{G/N} G$ and $N' = \ker(\mathrm{pr}_2)$. Observe that if $X \in \mathrm{Sm}_S^{G, N\text{-free}}$ then $\mathrm{pr}_1^* X$ is in $\mathrm{Sm}_S^{G', N'\text{-free}}$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{Spt}^{G, N\text{-free}}(S) & \xrightarrow{i_!} & \mathcal{Spt}^G(S) \\ j^* \mathrm{pr}_1^* i_! \downarrow & & \downarrow \mathrm{pr}_1^* \\ \mathcal{Spt}^{G', N'\text{-free}}(S) & \xrightarrow{j_!} & \mathcal{Spt}^{G'}(S), \end{array}$$

of colimit preserving functors, where $j : \mathrm{Sm}_S^{G', N'\text{-free}} \subseteq \mathrm{Sm}_S^{G'}$ is the inclusion.

Proposition 6.2. *With notation as above, the diagram commutes*

$$\begin{array}{ccc} \mathcal{Spt}^{G, N\text{-free}}(S) & \xrightarrow{j^* \mathrm{pr}_1^* i_!} & \mathcal{Spt}^{G', N'\text{-free}}(S) \\ \downarrow \pi_! & & \downarrow \mathrm{pr}_{2!} \\ \mathcal{Spt}^{G/N}(S) & \xrightarrow{\pi^*} & \mathcal{Spt}^G(S). \end{array}$$

Proof. We have a transformation, $\nu : \mathrm{pr}_{2!} j^* \mathrm{pr}_1^* i_! \rightarrow \pi^* \pi_!$, defined as the composite

$$\mathrm{pr}_{2!} j^* \mathrm{pr}_1^* i_! \rightarrow \mathrm{pr}_{2!} j^* \mathrm{pr}_1^* i_! i^* \pi^* \pi_! \rightarrow \mathrm{pr}_{2!} j^* \mathrm{pr}_2^* \pi^* \pi_! \rightarrow \pi^* \pi_!,$$

where the first arrow is induced by the unit of the adjunction $(\pi_!, i^* \pi^*)$ and the last by the counit of $(\mathrm{pr}_{2!}, j^* \mathrm{pr}_2^*)$. To check this is an equivalence, by [Corollary 5.4](#) it suffices to check that ν is an equivalence on $\Sigma^{-\rho_{G/N}} X_+$ where $X \in \mathrm{Sm}_S^{G, N\text{-free}}$. Via the projection formula, [Proposition 5.6](#), we see that $\nu_{\Sigma^{-\rho_{G/N}} X_+}$ is identified with $\Sigma^{-\rho_{G/N}} \nu_{X_+}$ and so it suffices to check ν is an equivalence on X_+ . But this case follows from the isomorphism of G -schemes $(\mathrm{pr}_1^{-1} X)/N' \cong \pi^{-1}(X/N)$. \square

Remark 6.3. It is sometimes convenient to write again pr_1^* for $j^* \mathrm{pr}_1^* i_!$ so that the equivalence $\pi^* \pi_! \simeq \mathrm{pr}_{2!} j^* \mathrm{pr}_1^* i_!$ can be expressed compactly as

$$\pi^* \pi_! \simeq \mathrm{pr}_{2!} \mathrm{pr}_1^*.$$

Since pr_1^* takes N -free spectra to $\ker(\mathrm{pr}_2)$ -free spectra, this is only a minor overloading of notation and no confusion should arise.

We obtain a transformation

$$\hat{\tau} : \pi^* \pi_! \rightarrow i_! \quad (6.4)$$

via

$$\pi^* \pi_! \simeq \text{pr}_{2!} j^* \text{pr}_1^* i_! \rightarrow \text{pr}_{2!} j^* \Delta_! \Delta^* \text{pr}_1^* i_! \simeq i_!,$$

where $\Delta : G \hookrightarrow G \times_{G/N} G$ is the diagonal and we use that $\Delta_! \simeq \Delta_*$ by the Wirthmueller isomorphism [Proposition 2.41](#).

Definition 6.5. The *Adams transformation*

$$\tau : \pi_! \rightarrow \pi_* i_!$$

is the transformation induced by adjunction from $\hat{\tau} : \pi^* \pi_! \rightarrow i_!$ constructed above.

In a certain sense, the Adams transformation is “smashing”, as made precise in the proposition below. First note that there is a canonical transformation

$$\pi_* i_! i^* (\mathbf{1}_S) \otimes \text{id} \rightarrow \pi_* i_! i^* \pi^* \quad (6.6)$$

between endofunctors of $\text{Spt}^{G/N}(S)$ obtained as the adjoint of

$$\pi^* \pi_* i_! i^* (\mathbf{1}_S) \otimes \pi^* \rightarrow i_! i^* (\mathbf{1}_S) \otimes \pi^* \pi_! \simeq i_! i^* \pi^*.$$

Since $i_! i^* \simeq i_! i^* (\mathbf{1}_S) \otimes \text{id} \simeq \mathbf{1}_{\mathbf{E}\mathcal{F}(N)} \otimes \text{id}$ by [Proposition 3.11](#), the transformation (6.6) can equivalently be written as $\pi_* (\mathbf{1}_{\mathbf{E}\mathcal{F}(N)}) \otimes \text{id} \rightarrow \pi_* (\mathbf{1}_{\mathbf{E}\mathcal{F}(N)}) \otimes \pi^*$.

Proposition 6.7. Let $X \in \text{Spt}^{G,N\text{-free}}(S)$. The diagram

$$\begin{array}{ccccc} \pi_! X & \xrightarrow{\tau} & \pi_* i_! X & \longrightarrow & \pi_* i_! i^* \pi^* \pi_! X \\ \downarrow & & & & \uparrow \\ \pi_! i^* (\mathbf{1}_S) \otimes \pi_! X & \xrightarrow{\tau \otimes \text{id}} & \pi_* i_! i^* (\mathbf{1}_S) \otimes \pi_! X & & \end{array}$$

commutes, where the left vertical map is obtained from the op-lax monoidality of $\pi_!$, the right one is (6.6), and the top right horizontal arrow comes from the unit of the adjunction $(\pi_!, i^* \pi^*)$.

Proof. Consider the following diagram

$$\begin{array}{ccc} \pi^* \pi_! X & \longrightarrow & \pi^* \pi_! i^* (\mathbf{1}_S) \otimes \pi^* \pi_! X \\ \uparrow \sim & & \sim \uparrow \\ \text{pr}_{2!} j^* \text{pr}_1^* i_! X & \longrightarrow & \text{pr}_{2!} j^* \text{pr}_1^* i_! i^* (\mathbf{1}_S) \otimes \text{pr}_{2!} j^* \text{pr}_1^* i_! X \\ \downarrow & & \downarrow \\ \text{pr}_{2!} j^* \Delta_! \Delta^* \text{pr}_1^* i_! X & \longrightarrow & \text{pr}_{2!} j^* \Delta_! \Delta^* \text{pr}_1^* i_! i^* (\mathbf{1}_S) \otimes \text{pr}_{2!} j^* \text{pr}_1^* i_! X \\ \sim \downarrow & & \downarrow \simeq \\ i_! X & \longrightarrow & i_! i^* (\mathbf{1}_S) \otimes \text{pr}_{2!} j^* \text{pr}_1^* i_! X \\ \downarrow & & \downarrow \sim \\ i_! i^* \pi^* \pi_! X & \xrightarrow{\sim} & i_! i^* (\mathbf{1}_S) \otimes \pi^* \pi_! X \end{array}$$

$\hat{\tau}$ (left curved arrow), $\hat{\tau}_{i^* \mathbf{1}_S} \otimes \text{id}$ (right curved arrow)

The functors π^* , pr_1^* , j^* are symmetric monoidal, $i_!$ is non-unital symmetric monoidal, and $i^* (\mathbf{1}_S) = \mathbf{1}_{\mathbf{E}\mathcal{F}(N)}$ is the unit of $\text{Spt}^{G,N\text{-free}}(S)$. It is straightforward to check that the top square commutes. Using the natural equivalence $\Delta_! \Delta^* \simeq \Delta_! \Delta^* (\mathbf{1}_S) \otimes \text{id}$, it is straightforward to check that the remaining squares commute. This implies the result by adjointness. \square

6.2 Changing the base

Our next goal is to verify that the Adams transformation τ is compatible with the various change of base functors in motivic homotopy. First, however, we recall some basic facts about manipulating natural transformations and adjunctions used below.

Let

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f^*} & \mathcal{B} \\ g^* \downarrow & \phi \swarrow & \downarrow k^* \\ \mathcal{C} & \xrightarrow{h^*} & \mathcal{D} \end{array}$$

be a diagram of ∞ -categories, where $\phi : k^* f^* \rightarrow h^* g^*$ is a natural transformation. Suppose that f^* , h^* admit respective left adjoints $f_!$ and $h_!$. The *left mate* of ϕ is a natural transformation

$$\begin{array}{ccc} \mathcal{A} & \xleftarrow{f_!} & \mathcal{B} \\ g^* \downarrow & \phi_L \swarrow & \downarrow k^* \\ \mathcal{C} & \xleftarrow{h_!} & \mathcal{D}. \end{array}$$

Explicitly, $\phi_L : h_! k^* \rightarrow g^* f_!$ is defined to be the composite

$$h_! k^* \rightarrow h_! k^* f^* f_! \xrightarrow{\phi} h_! h^* g^* f_! \rightarrow g^* f_!.$$

Similarly, if g^* , k^* admit respective right adjoints g_* and k_* then its *right mate* is a transformation

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f^*} & \mathcal{B} \\ g_* \uparrow & \phi_R \searrow & \uparrow k_* \\ \mathcal{C} & \xrightarrow{h^*} & \mathcal{D}. \end{array}$$

Explicitly, $\phi_R : f^* g_* \rightarrow k_* h^*$ is defined to be the composite

$$f^* g_* \rightarrow k_* k^* f^* g_* \xrightarrow{\phi} k_* h^* g^* g_* \rightarrow k_* h^*.$$

If ψ and ϕ are natural transformations, then $\psi \simeq \phi_L$ if and only if $\phi \simeq \psi_R$.

Lemma 6.8. *Suppose we are given a diagram of ∞ -categories.*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f^*} & \mathcal{B} \\ g^* \downarrow & \phi \swarrow & \downarrow k^* \\ \mathcal{C} & \xrightarrow{h^*} & \mathcal{D}. \end{array}$$

where f^* , h^* have left adjoints and g^* , k^* have right adjoints. Then ϕ_L is an equivalence if and only if ϕ_R is an equivalence.

Proof. This is a straightforward check. □

Remark 6.9. It often happens that ϕ is invertible. Care should be taken to not confuse the mates of ϕ with those of ϕ^{-1} (assuming all requisite adjoints exist). For example, it is not the case the mates of ϕ are equivalences exactly when the mates of ϕ^{-1} are equivalences.

We will need to know that units and counits of adjunctions are compatible across equivalences induced by mates.

Lemma 6.10. *Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ and $L' : \mathcal{C}' \rightleftarrows \mathcal{D}' : R'$ be two adjoint pairs and $F : \mathcal{C} \rightarrow \mathcal{C}'$ and $G : \mathcal{D} \rightarrow \mathcal{D}'$ functors. Let $\phi : FR \rightarrow R'G$ and $\psi : L'F \rightarrow GL$ be mates. Write η, ϵ for the unit and counit of (L, R) and η', ϵ' for the unit and counit of (L', R') . Then for $X \in \mathcal{D}, Y \in \mathcal{C}$, the diagrams*

$$\begin{array}{ccccc} L'R'GX & \xleftarrow{L'\phi} & L'FRX & \xrightarrow{\psi R} & GLRX \\ & \searrow \epsilon'G & & \swarrow G\epsilon & \\ & & GX & & \end{array}$$

and

$$\begin{array}{ccccc} & & FY & & \\ & \swarrow F\eta & & \searrow \eta'F & \\ FRLY & \xrightarrow{\phi L} & R'GLY & \xleftarrow{R'\psi} & R'L'FY. \end{array}$$

commute.

Proof. The first claim follows from the commutativity of the diagram

$$\begin{array}{ccccccc} L'FRX & \xrightarrow{L'F\eta R} & L'FRRLX & \xrightarrow{L'\phi LR} & L'R'GLRX & \xrightarrow{\epsilon'GLR} & GLRX \\ & \searrow \text{id} & \downarrow L'FR\epsilon & & \downarrow L'R'G\epsilon & & \downarrow G\epsilon \\ & & L'FRX & \xrightarrow{L'\phi} & L'R'GX & \xrightarrow{\epsilon'G} & GX \end{array}$$

since the top composite is ψR . The second claim follows from the commutativity of the diagram

$$\begin{array}{ccccccc} R'L'FY & \xrightarrow{R'L'F\eta} & R'L'FRLY & \xrightarrow{R'L'\phi L} & R'L'R'GLY & \xrightarrow{\epsilon GL} & R'GLY \\ \eta'F \uparrow & & \eta'FRL \uparrow & & \eta'R'GL \uparrow & \nearrow \text{id} & \\ FY & \xrightarrow{F\eta} & FRLY & \xrightarrow{\phi L} & R'GLY, & & \end{array}$$

where the top composite is $R'\psi$. □

Let $p : T \rightarrow S$ be a map in Sch_B^G , on which N acts trivially. Write $\pi : G \rightarrow G/N$ for the quotient. We will use i both to denote the inclusion $\text{Sm}_S^{G, N\text{-free}} \subseteq \text{Sm}_S^G$ as well as the inclusion $\text{Sm}_T^{G, N\text{-free}} \subseteq \text{Sm}_T^G$. Fix equivalences

$$\alpha : \pi^* p^* \xrightarrow{\sim} p^* \pi^*$$

and

$$\gamma : i_! p^* \simeq p^* i_!.$$

The right mate of α is a transformation $\alpha_R : p^* \pi_* \rightarrow \pi_* p^*$. We write ν for the right mate of α_R ,

$$\nu = (\alpha_R)_R : \pi_* p_* \xrightarrow{\sim} p_* \pi_*,$$

which is an equivalence $\alpha \simeq (\alpha_R)_L$ by [Lemma 6.8](#), since $\alpha \simeq (\alpha_R)_L$ is an equivalence. Write $\nu' = \nu(\gamma^{-1})_R$ which is an equivalence

$$\nu' : \pi_* i_! p_* \xrightarrow{\sim} p_* \pi_* i_!.$$

We have an equivalence

$$\alpha^{-1}\gamma_R : p^*i^*\pi^* \simeq i^*\pi^*p^*.$$

Write $\beta = (\alpha^{-1}\gamma_R)_L$ for the left mate of $\alpha^{-1}\gamma_R$. Then β is an equivalence

$$\beta : \pi_!p^* \simeq p^*\pi_!$$

and write $\phi = (\beta^{-1})_R$, which is a transformation

$$\phi : \pi_!p_* \rightarrow p_*\pi_!.$$

If p is smooth, then α has a left mate $\alpha_L : p_{\#}\pi^* \xrightarrow{\sim} \pi^*p_{\#}$ which is an equivalence. It follows that α_R is an equivalence by Lemma 6.8. In Corollary 6.37 we see that α_R is more generally an equivalence even when p is not smooth. Write

$$\bar{\alpha} : p_{\#}\pi_* \rightarrow \pi_*p_{\#}$$

for the left mate of $\alpha_R^{-1} : \pi_*p^* \simeq p^*\pi_*$ and

$$\bar{\alpha}\gamma : p_{\#}\pi_*i_! \rightarrow \pi_*i_!p_{\#}$$

for the left mate of $\alpha_R^{-1}\gamma$.

For the next lemmas it is convenient to fix some further exchange transformations. Let $\kappa_i : \mathrm{pr}_i^*p^* \simeq p^*\mathrm{pr}_i^*$ and $\lambda : j^*p^* \simeq p^*j^*$. Set $\nu = ((\lambda\kappa_2)^{-1})_L : \mathrm{pr}_{2!}p^* \simeq p^*\mathrm{pr}_{2!}$.

We use the following basic consequence of the fact that $\mathrm{Spt}^{G,\mathcal{F}}(S)$ is the value of a functor $\mathrm{Sch}_B^{\circlearrowleft}[\cdot]^{\mathrm{op}}$. Let $f : T \rightarrow S$ be a scheme map and $\phi : G \rightarrow K$ a homomorphism. The exchange $\phi^*f^* \simeq f^*\phi^*$ expresses the fact that $(\mathrm{id}, f)(\phi, \mathrm{id})$ and $(\phi, \mathrm{id})(\mathrm{id}, f)$ are both equal to (ϕ, f) . In particular, these exchanges can be chosen compatibly. That is, if $\phi : G \rightarrow K$ and $\psi : K \rightarrow H$ are homomorphisms, then the diagram

$$\begin{array}{ccccc} \phi^*\psi^*f^* & \xrightarrow{\sim} & \phi^*f^*\psi^* & \xrightarrow{\sim} & f^*\phi^*\psi^* \\ \sim \downarrow & & & & \downarrow \sim \\ (\psi\phi)^*f^* & \xrightarrow{\sim} & & \xrightarrow{\sim} & f^*(\psi\phi)^* \end{array}$$

commutes, and similarly for a composite of scheme maps.

Lemma 6.11. *Let $X \in \mathrm{Spt}^{G,N\text{-free}}(S)$. The following diagram commutes:*

$$\begin{array}{ccc} \pi^*\pi_!p^*X & \xrightarrow[\sim]{\alpha\beta} & p^*\pi^*\pi_!X \\ \hat{\tau}p^* \downarrow & & \downarrow p^*\hat{\tau} \\ i_!p^*X & \xrightarrow[\sim]{\gamma} & p^*i_!X. \end{array}$$

Proof. We have to see that each rectangle of the diagram below commutes

$$\begin{array}{ccccc} \pi^*\pi_!p^*X & \xrightarrow{\beta} & \pi^*p^*\pi_!X & \xrightarrow{\alpha} & p^*\pi^*\pi_!X \\ \sim \uparrow & & & & \uparrow \sim \\ \mathrm{pr}_{2!}j^*\mathrm{pr}_{1!}i_!p^*X & \xrightarrow{\kappa_1\gamma} & \mathrm{pr}_{2!}j^*p^*\mathrm{pr}_{1!}i_!X & \xrightarrow{\nu\lambda} & p^*\mathrm{pr}_{2!}j^*\mathrm{pr}_{1!}i_!X \\ \downarrow & & & & \downarrow \\ \mathrm{pr}_{2!}j^*\Delta_!\Delta^*\mathrm{pr}_{1!}i_!p^*X & \longrightarrow & \mathrm{pr}_{2!}j^*\Delta_!\Delta^*p^*\mathrm{pr}_{1!}i_!X & \longrightarrow & p^*\mathrm{pr}_{2!}j^*\Delta_!\Delta^*p^*\mathrm{pr}_{1!}i_!X \\ \downarrow \simeq & & & & \downarrow \simeq \\ i_!p^* & \xrightarrow{\gamma} & & \xrightarrow{\gamma} & p^*i_!. \end{array}$$

To see the top diagram commutes, consider the following diagram

$$\begin{array}{ccccc}
\mathrm{pr}_{2!}j^*\mathrm{pr}_1^*i_!p^*X & \longrightarrow & \mathrm{pr}_{2!}j^*\mathrm{pr}_1^*p^*i_!X & \longrightarrow & p^*\mathrm{pr}_{2!}j^*\mathrm{pr}_1^*i_!X \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow \\
\mathrm{pr}_{2!}j^*\mathrm{pr}_1^*i_!i^*\pi^*\pi_!p^*X & \longrightarrow & \mathrm{pr}_{2!}j^*\mathrm{pr}_1^*i_!p^*i^*\pi^*\pi_!X & \longrightarrow & \mathrm{pr}_{2!}j^*\mathrm{pr}_1^*p^*i_!i^*\pi^*\pi_!X \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{pr}_{2!}j^*\mathrm{pr}_2^*\pi^*\pi_!p^*X & \longrightarrow & \mathrm{pr}_{2!}j^*\mathrm{pr}_2^*p^*\pi^*\pi_!X & \longrightarrow & p^*\mathrm{pr}_{2!}j^*\mathrm{pr}_2^*\pi^*\pi_!X \\
\downarrow & & \downarrow & & \downarrow \\
\pi^*\pi_!p^*X & \longrightarrow & & \longrightarrow & p^*\pi^*\pi_!X.
\end{array}$$

The outer composites of this diagram yield the diagram of the lemma and a straightforward inspection suffices to see that most of the pieces of this diagram commute. The remaining pieces involve either moving p^* across the unit for the adjunction $(\pi_!, i^*\pi^*)$ or across the counit for the adjunction $(\mathrm{pr}_{2!}, j^*\mathrm{pr}_2^*)$. In either case, that this results in a commutative diagram follows from [Lemma 6.10](#), since the pairs of exchange equivalences $\pi_!p^* \simeq p^*\pi_!$, $p^*i^*\pi^* \simeq i^*\pi^*p^*$ and $\mathrm{pr}_{2!}p^* \simeq p^*\mathrm{pr}_{2!}$, $p^*j^*\mathrm{pr}_2^* \simeq j^*\mathrm{pr}_2^*p^*$ are mates.

The argument for commutativity of the remaining squares is similar. \square

Proposition 6.12. *Let $X \in \mathrm{Spt}^{G,N\text{-free}}(S)$. The diagram*

$$\begin{array}{ccc}
\pi_!p^*X & \xrightarrow{\beta} & p^*\pi_!X \\
\tau p^* \downarrow & & \downarrow p^*\tau \\
\pi_*i_!p^*X & \xleftarrow{\gamma^{-1}\alpha_R} & p^*\pi_*i_!X.
\end{array}$$

commutes.

Proof. To see this, consider the diagram

$$\begin{array}{ccccc}
\pi^*\pi_!p^*X & \xrightarrow{\pi^*\beta} & \pi^*p^*\pi_!X & \xrightarrow{\alpha} & p^*\pi^*\pi_!X \\
\pi^*\tau p^* \downarrow & & \downarrow \pi^*p^*\tau & & \downarrow p^*\pi^*\tau \\
\pi^*\pi_*i_!p^*X & \xleftarrow{\pi^*\gamma^{-1}\alpha_R} & \pi^*p^*\pi_*i_!X & \xrightarrow{\alpha} & p^*\pi^*\pi_*i_!X. \\
\downarrow & & \downarrow & & \downarrow \\
i_!p^*X & \xleftarrow{\sim} & & \xrightarrow{\sim} & p^*i_!X.
\end{array}$$

By adjointness, it suffices to see that the top left square commutes. The right square commutes and the lower rectangle commutes by [Lemma 6.10](#), so it suffices to see the outer diagram commutes. This follows from [Lemma 6.11](#). \square

Proposition 6.13. *Let $Y \in \mathrm{Spt}^{G,N\text{-free}}(T)$. The following diagram commutes:*

$$\begin{array}{ccc}
p_{\#}\pi_!Y & \xrightarrow{\beta_L} & \pi_!p_{\#}Y \\
p_{\#}\tau \downarrow & & \downarrow \tau p_{\#} \\
p_{\#}\pi_*i_!Y & \xrightarrow{\overline{\gamma\alpha}} & \pi_*i_!p_{\#}Y.
\end{array}$$

Proof. The diagram of the lemma is the composite of the squares

$$\begin{array}{ccccccc}
p_{\#}\pi_!Y & \longrightarrow & p_{\#}\pi_!p^*p_{\#}Y & \xrightarrow{\beta} & p_{\#}p^*\pi_!p_{\#}Y & \longrightarrow & \pi_!p_{\#}Y \\
p_{\#}\tau \downarrow & & \downarrow p_{\#}\tau p^*p_{\#} & & \downarrow p_{\#}p^*\tau p_{\#} & & \downarrow \tau p_{\#} \\
p_{\#}\pi_*i_!Y & \longrightarrow & p_{\#}\pi_*i_!p^*p_{\#}Y & \xrightarrow{(\gamma\alpha_R)^{-1}} & p_{\#}p^*\pi_*i_!p_{\#}Y & \longrightarrow & \pi_*i_!p_{\#}Y.
\end{array}$$

The first and the third square commute by functoriality. The second square commutes by [Proposition 6.12](#). \square

Proposition 6.14. *Let $Y \in \text{Spt}^{G,N\text{-free}}(T)$. The following diagram commutes:*

$$\begin{array}{ccc}
\pi_!p_*Y & \xrightarrow[\sim]{\phi} & p_*\pi_!Y \\
\downarrow \tau p_* & & \downarrow p_*\tau \\
\pi_*i_!p_*Y & \xrightarrow[\sim]{\nu'} & p_*\pi_*i_!Y.
\end{array}$$

Proof. Consider the following diagram. By adjointness, we need to see that the combined top rectangles commute:

$$\begin{array}{ccccccc}
\pi_!p^*p_*Y & \xrightarrow[\sim]{\beta} & p^*\pi_!p_*Y & \xrightarrow{p^*\phi} & p^*p_*\pi_!Y & \xrightarrow{\epsilon\pi_!} & \pi_!Y \\
& & \downarrow p^*\tau p_* & & \downarrow p^*p_*\tau & & \downarrow \tau \\
& & p^*\pi_*i_!p_*Y & \xrightarrow[\sim]{\nu'} & p^*p_*\pi_*i_!Y & \xrightarrow{\epsilon\pi_*i_!} & \pi_*i_!Y \\
& \searrow \tau p^*p_* & \downarrow \gamma^{-1}\alpha_R & & \nearrow \pi_*i_!\epsilon & & \\
& & \pi_*i_!p^*p_*Y & & & &
\end{array}$$

Here ϵ is the counit of the adjunction (p^*, p_*) . The left triangle commutes by [Proposition 6.12](#), the bottom triangle commutes by [Lemma 6.10](#), and the top composite is equivalent to $\pi_!\epsilon$, also by [Lemma 6.10](#). It follows that the outer diagram commutes, and since β is an equivalence the combined rectangles commute as desired. \square

Lemma 6.15. *Let $f : T \rightarrow S$ be a map in Sch_B^G . The diagram*

$$\begin{array}{ccc}
\text{Spt}^{G,N\text{-free}}(T) & \xrightarrow{i_!} & \text{Spt}^G(T) \\
f_* \downarrow & & \downarrow f_* \\
\text{Spt}^{G,N\text{-free}}(S) & \xrightarrow{i_!} & \text{Spt}^G(S)
\end{array}$$

commutes.

Proof. The functor $f_* : \text{Spt}^{G,N\text{-free}}(T) \rightarrow \text{Spt}^{G,N\text{-free}}(S)$ may be computed as the composite

$$\text{Spt}^{G,N\text{-free}}(T) \xrightarrow{i_!} \text{Spt}^G(T) \xrightarrow{f_*} \text{Spt}^G(S) \xrightarrow{i^*} \text{Spt}^{G,N\text{-free}}(S).$$

We have $i_!i^* \simeq i_!i^*(\mathbf{1}) \otimes -$ and $i_!i^*(\mathbf{1}) \simeq \text{colim}_n U_n$, where U_n is dualizable over S , by [Corollary 6.32](#).

Therefore

$$\begin{aligned}
i_! i^* f_* i_! &\simeq \operatorname{colim}_n F(D(U_n), f_* i_!) \simeq f_* F(f^* D(U_n), i_!) \\
&\simeq \operatorname{colim}_n f_* F(D f^* U_n, i_!) \\
&\simeq f_* \operatorname{colim}_n f^* U_n \otimes i_! \\
&\simeq f_*(i_! i^*(\mathbb{1}) \otimes i_!) \\
&\simeq f_* i_!,
\end{aligned}$$

where we use that $f^* i_! i^*(\mathbb{1}) \simeq i_! i^*(\mathbb{1})$ by [Proposition 3.8](#) since $i_! i^*(\mathbb{1}) \simeq \mathbf{EF}(N)$. \square

6.3 τ is an equivalence

In this subsection, we show that the Adams transformation τ is an equivalence. By [Corollary 5.4](#), it suffices to show that τ is an equivalence on $\Sigma^{-k\rho_{G/N}} q_{\#} \mathbb{1}_X$ where $q : X \rightarrow S$ is in $\operatorname{Sm}_S^{G, N\text{-free}}$, which in turn would follow by showing that τ is an equivalence on all $q_{\#} \mathbb{1}_X$. Unfortunately, we do not know how to show this directly. Instead our strategy is to first show that it is an equivalence on those $q_{\#} \mathbb{1}_X$ which are *dualizable*. This immediately implies that τ is an equivalence on the subcategory of $\operatorname{Spt}^{G, N\text{-free}}(S)$ generated under colimits by (N -trivial desuspensions of) such $q_{\#} \mathbb{1}_X$. Of course, this is only a proper subcategory of $\operatorname{Spt}^{G, N\text{-free}}(S)$, unless S is the spectrum of a field of characteristic zero. However, since $\mathbb{1}_{\mathbf{EF}(N)}$ has dualizable skeleta by [Corollary 6.32](#), we can at least conclude that

$$\tau : \pi_! \mathbb{1}_{\mathbf{EF}(N)} \rightarrow \pi_* \mathbb{1}_{\mathbf{EF}(N)}$$

is an equivalence. This now allows us to define an inverse τ^{-1} using [Proposition 6.7](#) and conclude that τ is an equivalence in general.

Lemma 6.16. *Let $\mathcal{E} \in \operatorname{Spt}^{G, N\text{-free}}(S)$ and suppose that $i_! \mathcal{E}$ is dualizable. Then \mathcal{E} is also dualizable and $i_! F_{\operatorname{Spt}^{G, N\text{-free}}(S)}(\mathcal{E}, \mathcal{D}) \simeq F_{\operatorname{Spt}^G(S)}(i_! \mathcal{E}, i_! \mathcal{D})$.*

In particular, the dual of \mathcal{E} , regarded as an object of $\operatorname{Spt}^{G, N\text{-free}}(S)$, agrees via $i_!$, with the dual of \mathcal{E} , regarded as an object of $\operatorname{Spt}^G(S)$.

Proof. This first claim follows formally from the fact that $\mathcal{E} \simeq i^* i_! \mathcal{E}$ and that i^* is a symmetric monoidal functor. The second claim follows by applying $i_!$ to the equivalence

$$i^* F_{\operatorname{Spt}^G(S)}(i_! \mathcal{E}, i_! \mathcal{D}) \simeq F_{\operatorname{Spt}^{G, N\text{-free}}(S)}(\mathcal{E}, \mathcal{D})$$

and that $i_! i^* \simeq i_! i^*(\mathbb{1}_S) \otimes \operatorname{id}$, see [Proposition 3.11](#), together with the commutative square

$$\begin{array}{ccc}
i_! i^* \mathbb{1}_S \otimes D_S(i_! \mathcal{E}) \otimes i_! \mathcal{D} & \xrightarrow{\sim} & i_! i^* \mathbb{1}_S \otimes F(i_! \mathcal{E}, i_! \mathcal{D}) \\
\sim \downarrow & & \downarrow \\
D_S(i_! \mathcal{E}) \otimes i_! \mathcal{D} & \xrightarrow{\sim} & F(i_! \mathcal{E}, i_! \mathcal{D})
\end{array}$$

where the vertical maps are induced by the projection $i_! i^*(\mathbb{1}_S) \rightarrow \mathbb{1}_S$ and the left hand vertical map is an equivalence since $i_! i^* \mathbb{1}_S \otimes i_! \mathcal{D} \rightarrow i_! \mathcal{D}$ is an equivalence.

The last statement follows since $F_{\operatorname{Spt}^G(S)}(i_! \mathcal{E}, \mathbb{1}_S) \simeq F_{\operatorname{Spt}^{G, N\text{-free}}(S)}(i_! \mathcal{E}, i_!(i^* \mathbb{1}_S))$. \square

Lemma 6.17. *Let $p : Y \rightarrow S$ be a smooth equivariant map. For any $\mathcal{E} \in \text{Spt}^G(Y)$, the diagram*

$$\begin{array}{ccc} D_S(p\#\mathcal{E}) \otimes_S p\#\mathcal{E} & \xrightarrow{\text{ev}} & \mathbf{1}_S \\ \sim \uparrow & & \nearrow \text{ev}' \\ p\#(p^*D_S(p\#\mathcal{E}) \otimes_Y \mathcal{E}) & & \end{array}$$

commutes, where the vertical equivalence is the projection formula and the diagonal map is the composite

$$\begin{aligned} p\#(p^*D_S(p\#\mathcal{E}) \otimes_Y \mathcal{E}) &\rightarrow p\#(D_Y(p^*p\#\mathcal{E}) \otimes_Y \mathcal{E}) \rightarrow p\#(D_Y(\mathcal{E}) \otimes_Y \mathcal{E}) \\ &\xrightarrow{p\#\text{ev}} p\#\mathbf{1}_Y \simeq p\#p^*\mathbf{1}_S \rightarrow \mathbf{1}_S, \end{aligned}$$

of the canonical map followed by maps induced by the unit and counit of the adjunction $(p\#, p^*)$, respectively.

Proof. The composite of the vertical and horizontal map is adjoint to the composite map in the diagram

$$\begin{array}{ccccc} p^*D_S(p\#\mathcal{E}) \otimes_Y \mathcal{E} & \longrightarrow & p^*(D_S(p\#\mathcal{E}) \otimes_Y p\#\mathcal{E}) & \longrightarrow & p^*\mathbf{1}_S \simeq \mathbf{1}_Y, \\ & \searrow & \sim \uparrow & & \uparrow \text{ev} \\ & & p^*D_Y(p\#\mathcal{E}) \otimes_Y p^*p\#\mathcal{E} & \longrightarrow & D_Y(p^*p\#\mathcal{E}) \otimes_Y p^*p\#\mathcal{E}, \end{array}$$

where the left diagonal is induced by the unit of the adjunction $(p\#, p^*)$ and the square commutes by [Lemma 6.10](#) (with $F = p^* = G$, $R = F(p\#\mathcal{E}, -)$, and $R' = F(p^*p\#\mathcal{E}, 0)$). Applying $p\#$ everywhere, we obtain the commutative diagram

$$\begin{array}{ccccccc} p\#(p^*D_S(p\#\mathcal{E}) \otimes_Y \mathcal{E}) & \longrightarrow & p\#(p^*D_S(p\#\mathcal{E}) \otimes_Y p^*p\#\mathcal{E}) & & & & \\ \downarrow & & \downarrow & & \searrow & & \\ p\#(D_Y(p^*p\#\mathcal{E}) \otimes_Y \mathcal{E}) & \longrightarrow & p\#(D_Y(p^*p\#\mathcal{E}) \otimes_Y p^*p\#\mathcal{E}) & \longrightarrow & p\#p^*\mathbf{1}_S & \longrightarrow & \mathbf{1}_S, \\ & \searrow & & & \nearrow & & \\ & & & & p\#(D_Y(\mathcal{E}) \otimes \mathcal{E}) & & \end{array}$$

where the lower piece of the diagram commutes for formal reasons: given a map $\varphi : M \rightarrow N$ in any symmetric monoidal category, the induced diagram

$$\begin{array}{ccc} D(N) \otimes M & \xrightarrow{D(\varphi) \otimes \text{id}_M} & D(M) \otimes M \\ \text{id}_{D(N)} \otimes \varphi \downarrow & & \downarrow \\ D(N) \otimes N & \longrightarrow & \mathbf{1} \end{array}$$

commutes. This proves the claim. \square

Let $q : X \rightarrow S$ be an object in $\text{Sm}_S^{G, N\text{-free}}$ and write $\overline{X} = X/N$, $f : X \rightarrow \overline{X}$ for the quotient. Since N acts trivially on S , the structure map factors through the quotient and we have the diagram in Sm_S^G

$$\begin{array}{ccc} X & \xrightarrow{f} & \overline{X} \\ & \searrow q & \downarrow p \\ & & S, \end{array} \tag{6.18}$$

where f is finite étale. Our first goal is to establish [Theorem 6.25](#), which says that when $q_{\#}\mathbb{1}_X$ is dualizable, its dual is computed as $\pi_!D_S(q_{\#}\mathbb{1}_X)$.

We define a candidate evaluation map

$$\epsilon : \pi_!D_S(p_{\#}f_{\#}\mathbb{1}_X) \otimes_S p_{\#}\mathbb{1}_{\overline{X}} \rightarrow \mathbb{1}_S \quad (6.19)$$

as follows:

$$\begin{aligned} \pi_!D_S(p_{\#}f_{\#}\mathbb{1}_X) \otimes_S p_{\#}\mathbb{1}_{\overline{X}} &\xrightarrow{\sim} p_{\#}p^*\pi_!D_S(p_{\#}f_{\#}\mathbb{1}_X) \\ &\rightarrow p_{\#}\pi_!D_{\overline{X}}(p^*p_{\#}f_{\#}\mathbb{1}_X) \\ &\rightarrow p_{\#}\pi_!f_{\#}\mathbb{1}_X \simeq p_{\#}\mathbb{1}_{\overline{X}} \\ &\rightarrow \mathbb{1}_S. \end{aligned}$$

Here the first equivalence is the projection formula, the second arrow is the equivalence $p^*\pi_! \simeq \pi_!p^*$ together with the exchange $p^*D_S \rightarrow D_{\overline{X}}p^*$, the third arrow is induced by the unit of the adjunction $(p_{\#}, p^*)$ together with the equivalence $D_{\overline{X}}(f_{\#}\mathbb{1}_X) \simeq f_{\#}\mathbb{1}_X$, and the last arrow is induced by the counit of the adjunction $(p_{\#}, p^*)$.

Remark 6.20. The adjoint of ϵ is the map

$$e : \pi_!D_S(q_{\#}\mathbb{1}_X) \rightarrow D_S(p_{\#}\mathbb{1}_{\overline{X}}). \quad (6.21)$$

The map e can also be described as the adjoint of the map

$$e' : D_S(q_{\#}\mathbb{1}_X) \rightarrow \pi^*D_S(p_{\#}\mathbb{1}_{\overline{X}}),$$

which is the dual of the composite $p_{\#}\mathbb{1}_{\overline{X}} \rightarrow p_{\#}f_*f^*\mathbb{1}_{\overline{X}} \simeq p_{\#}f_{\#}\mathbb{1}_X$.

Note that the evaluation map for $q_{\#}\mathbb{1}_X$ factors canonically as

$$D_S(q_{\#}\mathbb{1}_X) \otimes_S q_{\#}\mathbb{1}_X \rightarrow i_!\mathbb{1}_{\mathbf{E}\mathcal{F}(N)} \rightarrow \mathbb{1}_S,$$

where $\mathbb{1}_{\mathbf{E}\mathcal{F}(N)} \in \mathrm{Spt}^{G, N\text{-free}}(S)$ is the unit.

Lemma 6.22. *The following diagram commutes:*

$$\begin{array}{ccc} \pi_!(D_S(q_{\#}\mathbb{1}_X) \otimes_S q_{\#}\mathbb{1}_X) &\longrightarrow & \pi_!\mathbb{1}_{\mathbf{E}\mathcal{F}(N)} \\ \downarrow & & \downarrow \\ \pi_!(D_S(q_{\#}\mathbb{1}_X)) \otimes_S p_{\#}\mathbb{1}_{\overline{X}} &\xrightarrow{\epsilon} & \mathbb{1}_S. \end{array}$$

Proof. The left vertical map comes from the op-lax monoidality of $\pi_!$. It may also be described as the map induced by $p_{\#}f_{\#}\mathbb{1}_X \rightarrow p_{\#}\mathbb{1}_{\overline{X}}$ together with the equivalence

$$\pi_!(D_S(p_{\#}f_{\#}\mathbb{1}_X) \otimes_S p_{\#}\mathbb{1}_{\overline{X}}) \simeq \pi_!D_S(p_{\#}f_{\#}\mathbb{1}_X) \otimes_S p_{\#}\mathbb{1}_{\overline{X}}.$$

Consider the following diagram

$$\begin{array}{ccccc} p_{\#}\pi_!(p^*D(q_{\#}\mathbb{1}_X) \otimes_{\overline{X}} f_{\#}\mathbb{1}_X) &\longrightarrow & p_{\#}\pi_!(D(f_{\#}\mathbb{1}_X) \otimes_{\overline{X}} f_{\#}\mathbb{1}_X) &\longrightarrow & p_{\#}\mathbb{1}_{\overline{X}} \longrightarrow \mathbb{1}_S \\ \downarrow & & \downarrow & \nearrow & \\ p_{\#}\pi_!(p^*D(q_{\#}\mathbb{1}_X) \otimes_{\overline{X}} \mathbb{1}_{\overline{X}}) &\longrightarrow & p_{\#}\pi_!(D(f_{\#}\mathbb{1}_X) \otimes_{\overline{X}} \mathbb{1}_{\overline{X}}), & & \end{array}$$

where the top and bottom horizontal arrows of the square are the composite of the exchange $p^*D_S \rightarrow D_{\overline{X}}p^*$ with the map induced by the unit of the adjunction $(p_{\#}, p^*)$. Via [Lemma 6.17](#) and the equivalences $\pi_!p_{\#} \simeq p_{\#}\pi_!$, we see that the top row is identified with the composite of the top horizontal and right vertical arrows in the diagram of the lemma. The composite around the lower part of this diagram is identified with the composite of the left vertical and bottom horizontal arrows of the diagram in the lemma. That this diagram commutes follows from the next lemma together with the equivalence $D_{\overline{X}}(f_{\#}\mathbb{1}_X) \simeq f_{\#}\mathbb{1}_X$. \square

Lemma 6.23. *The following diagram commutes*

$$\begin{array}{ccc} f_{\#}\mathbb{1}_X \otimes_{\overline{X}} f_{\#}\mathbb{1}_X & \xrightarrow{\quad\quad\quad} & f_{\#}\mathbb{1}_X \\ & \searrow & \nearrow \sim \\ & f_{\#}\mathbb{1}_X \otimes_{\overline{X}} \mathbb{1}_{\overline{X}} & \end{array},$$

where the horizontal map is the projection formula $f_{\#}\mathbb{1}_X \otimes_{\overline{X}} f_{\#}\mathbb{1}_X \simeq f_{\#}f^*f_{\#}\mathbb{1}_X$ followed by the counit (using the ambidexterity equivalence $f_{\#} \simeq f_*$).

Proof. Let $p_i : X \times_{\overline{X}} X \rightarrow X$ be the projection to the i th factor and $\Delta : X \rightarrow X \times_{\overline{X}} X$ the diagonal. Under the ambidexterity equivalence $f_{\#} \simeq f_*$, the counit of the adjunction (f^*, f_*) is the arrow $f^*f_{\#} \rightarrow \text{id}$, defined as the composite

$$f^*f_{\#} \simeq p_{2\#}p_1^* \rightarrow p_{2\#}\Delta_*\Delta^*p_1^* \simeq p_{2\#}\Delta_* \simeq \text{id},$$

see [[Hoy17](#), Theorem 6.9]. In particular, together with the equivalence $f_{\#}p_{2\#} \simeq f_{\#}p_{1\#}$, we see the morphism $f_{\#}f^*f_{\#}\mathbb{1}_X \rightarrow f_{\#}\mathbb{1}_X$ in $\text{Spt}_{\bullet}^G(\overline{X})$ can be represented by the projection

$$f_{\#}(X \times_{\overline{X}} X)_+ \rightarrow f_{\#} \frac{(X \times_{\overline{X}} X)_+}{(X \times_{\overline{X}} X \setminus \Delta(X))_+} \simeq f_{\#}X_+,$$

where $X \times_{\overline{X}} X$ is an X -scheme via the *first* coordinate. The lemma follows from the commutativity of the diagram where the rows are the cofiber sequences in $\text{Spc}_{\bullet}^G(\overline{X})$ associated to the closed immersions $\Delta(X) \subseteq X \times_{\overline{X}} X$ and $\Delta(f) \subseteq X \times_{\overline{X}} \overline{X}$, the diagonal and the graph of f , respectively:

$$\begin{array}{ccccc} (X \times_{\overline{X}} X \setminus \Delta(X))_+ & \longrightarrow & (X \times_{\overline{X}} X)_+ & \xrightarrow{p_1} & X_+ \\ \downarrow & & \downarrow & & \downarrow \\ (X \times_{\overline{X}} \overline{X} \setminus \Delta(f))_+ & \longrightarrow & (X \times_{\overline{X}} \overline{X})_+ & \xrightarrow{\sim} & X_+. \end{array}$$

\square

Now we suppose that $q_{\#}\mathbb{1}_X$ is dualizable in $\text{Spt}^{G, N\text{-free}}$ and G is isomorphic to a semi-direct product of N and G/N . We define a ‘‘coevaluation’’

$$\eta : \mathbb{1}_S \rightarrow p_{\#}\mathbb{1}_{\overline{X}} \otimes_S \pi_!D_S(p_{\#}f_{\#}\mathbb{1}_X) \tag{6.24}$$

as follows. Since G is isomorphic to a semi-direct product of G/N and N , there is a map $\mathbb{1}_S \rightarrow \pi_!\mathbb{1}_{\mathbf{E}\mathcal{F}(N)}$ which splits the canonical map $\pi_!\mathbb{1}_{\mathbf{E}\mathcal{F}(N)} \rightarrow \mathbb{1}_S$, obtained by taking the N -quotient of the (unstable) map $N'_+ \rightarrow \mathbb{1}_{\mathbf{E}\mathcal{F}(N)}$, where N' is the G -set $G/(G/N)$. Now, since $q_{\#}\mathbb{1}_X$ is dualizable in $\text{Spt}^G(S)$ it is dualizable

in $\text{Spt}^{G, N\text{-free}}(S)$ by [Lemma 6.16](#) and the duals in both categories agree under the standard inclusion. This means we have a coevaluation map

$$\text{coev} : \mathbb{1}_{\mathbf{E}\mathcal{F}(N)} \rightarrow p_{\#} f_{\#} \mathbb{1}_X \otimes_S D_S(p_{\#} f_{\#} \mathbb{1}_X).$$

Now, η is defined as the composite

$$\begin{aligned} \mathbb{1}_S &\rightarrow \pi_! \mathbb{1}_{\mathbf{E}\mathcal{F}(N)} \xrightarrow{\pi_! \text{coev}} \pi_!(p_{\#} \mathbb{1}_X \otimes_S D_S(p_{\#} f_{\#} \mathbb{1}_X)) \\ &\rightarrow p_{\#} \mathbb{1}_{\overline{X}} \otimes_S \pi_! D_S(p_{\#} f_{\#} \mathbb{1}_X). \end{aligned}$$

Theorem 6.25. *Let $q : X \rightarrow S$ be an object of $\text{Sm}_S^{G, N\text{-free}}$ and $f : X \rightarrow \overline{X}$, $p : \overline{X} \rightarrow S$ as in [\(6.18\)](#). Suppose that $q_{\#} \mathbb{1}_X \in \text{Spt}^G(S)$ is dualizable. Then $p_{\#} \mathbb{1}_{\overline{X}}$ is dualizable in $\text{Spt}^{G/N}(S)$ and [\(6.21\)](#) is an equivalence*

$$\pi_! D_S(q_{\#} \mathbb{1}_X) \xrightarrow{\sim} D_S(\pi_! q_{\#} \mathbb{1}_X).$$

Proof. First we explain why it suffices to assume that G is a semi-direct product of N and G/N . Suppose that the result is true in this case. Given an arbitrary G and N and let X be a dualizable N -free smooth G -scheme over S , consider the cartesian square [\(6.1\)](#). Since X is dualizable over S , as a smooth G -scheme, the $\ker(\text{pr}_2)$ -free $G \times_{G/N} G$ -scheme $\text{pr}_1^* X$ is dualizable over S . Since $G \times_{G/N} G$ is a semi-direct product of G and $\ker(\text{pr}_2)$, our hypothesis implies that the composite is an equivalence

$$\text{pr}_{2!} \text{pr}_1^* D_S(q_{\#} \mathbb{1}_X) \xrightarrow{\sim} \text{pr}_{2!} D_S(\text{pr}_1^* q_{\#} \mathbb{1}_X) \xrightarrow{e} D_S(\text{pr}_{2!} \text{pr}_1^* q_{\#} \mathbb{1}_X),$$

where we again write e for an instance of [\(6.21\)](#) for the group $G \times_{G/N} G$ and we use that $\text{pr}_1^* D_S(q_{\#} \mathbb{1}_X) \simeq D_S(\text{pr}_1^* q_{\#} \mathbb{1}_X)$ since $q_{\#} \mathbb{1}_X$ is dualizable and pr_1^* is symmetric monoidal. We claim that this implies that $p_{\#} \mathbb{1}_{\overline{X}}$ is dualizable in $\text{Spt}^{G/N}(S)$. Indeed, $\text{pr}_{2!} \text{pr}_1^* q_{\#} \mathbb{1}_X \simeq \pi^* \pi_! q_{\#} \mathbb{1}_X$ is dualizable in $\text{Spt}^G(S)$ and therefore $\Phi^N(\pi^* \pi_! q_{\#} \mathbb{1}_X) \simeq \pi_! q_{\#} \mathbb{1}_X \simeq p_{\#} \mathbb{1}_{\overline{X}}$ is dualizable in $\text{Spt}^{G/N}(S)$ as Φ^N is symmetric monoidal. In this case ϵ and η defined above are evaluation and coevaluation maps for the duality pairing.

The diagram

$$\begin{array}{ccc} \pi^* \pi_! D_S(q_{\#} \mathbb{1}_X) & \xrightarrow{\pi^* e} & \pi^* D_S(\pi_! q_{\#} \mathbb{1}_X) & \xrightarrow{\sim} & D_S(\pi^* \pi_! q_{\#} \mathbb{1}_X) \\ \uparrow \sim & & & & \downarrow \sim \\ \text{pr}_{2!} \text{pr}_1^* D_S(q_{\#} \mathbb{1}_X) & \xrightarrow{\sim} & \text{pr}_{2!} D_S(\text{pr}_1^* q_{\#} \mathbb{1}_X) & \xrightarrow{e} & D_S(\text{pr}_{2!} \text{pr}_1^* q_{\#} \mathbb{1}_X) \end{array} \quad (6.26)$$

commutes. This follows by adjointness from the commutativity of the diagram

$$\begin{array}{ccc} \pi^* \pi_! D_S(q_{\#} \mathbb{1}_X) \otimes \pi^* \pi_! q_{\#} \mathbb{1}_X & \longrightarrow & \mathbb{1}_S \\ \sim \uparrow & \nearrow & \\ \text{pr}_{2!} \text{pr}_1^* D_S(q_{\#} \mathbb{1}_X) \otimes \text{pr}_{2!} \text{pr}_1^* q_{\#} \mathbb{1}_X & & \end{array}$$

where the horizontal and diagonal arrows come from [\(6.19\)](#). We thus find that

$$\pi^* \pi_! D_S(q_{\#} \mathbb{1}_X) \xrightarrow{\pi^* e} \pi^* D_S(\pi_! q_{\#} \mathbb{1}_X)$$

is an equivalence. The functor π^* is conservative since $\Phi^N \circ \pi^* \simeq \text{id}$, see [Proposition 4.10](#). It follows that e is an equivalence.

It remains to establish the result under that assumption that G is a semi-direct product. We show that in this case the evaluation and coevaluation maps [\(6.19\)](#) and [\(6.24\)](#) satisfy the triangle identities. To compactify

notation, we write $X = p_{\#}f_{\#}\mathbb{1}_X$, $\bar{X} = p_{\#}\mathbb{1}_{\bar{X}}$, $\mathbb{1} = \mathbb{1}_S$, and $\mathbb{1}_{\mathbf{E}\mathcal{F}} = \mathbb{1}_{\mathbf{E}\mathcal{F}(N)}$. We have to check that the two composites below are the identity

$$\begin{aligned} \bar{X} \otimes \mathbb{1} &\xrightarrow{\text{id} \otimes \eta} \bar{X} \otimes \pi_1 D(X) \otimes \bar{X} \xrightarrow{\epsilon \otimes \text{id}} \mathbb{1} \otimes \bar{X} \\ \mathbb{1} \otimes \pi_1 D(X) &\xrightarrow{\eta \otimes \text{id}} \pi_1 D(X) \otimes \bar{X} \otimes \pi_1 D(X) \xrightarrow{\text{id} \otimes \epsilon} \pi_1 D(X) \otimes \mathbb{1}. \end{aligned}$$

To establish the first identity we observe that we have a commutative diagram

$$\begin{array}{ccccc} \bar{X} \otimes \mathbb{1} & \xrightarrow{\text{id} \otimes \eta} & \bar{X} \otimes \pi_1 D(X) \otimes \bar{X} & \xrightarrow{\epsilon \otimes \text{id}} & \mathbb{1} \otimes \bar{X} \\ & \searrow \text{id} \otimes \pi_1(\text{coev}) & \uparrow & \swarrow & \uparrow \\ \bar{X} \otimes \pi_1 \mathbb{1}_{\mathbf{E}\mathcal{F}} & \rightarrow & \bar{X} \otimes \pi_1(D(X) \otimes X) & \rightarrow & \pi_1 \mathbb{1}_{\mathbf{E}\mathcal{F}} \otimes \bar{X} \\ \uparrow \text{id} & \nearrow & \uparrow & \nwarrow & \uparrow \\ \pi_1(X \otimes \mathbb{1}_{\mathbf{E}\mathcal{F}}) & \xrightarrow{\pi_1(\text{id} \otimes \text{coev})} & \pi_1(X \otimes D(X) \otimes X) & \xrightarrow{\pi_1(\text{ev} \otimes \text{id})} & \pi_1(\mathbb{1}_{\mathbf{E}\mathcal{F}} \otimes X) \end{array}$$

id

in which $X \simeq X \otimes \mathbb{1}_{\mathbf{E}\mathcal{F}}$ and the upper righthand square commutes by Lemma 6.22. The second identity is established by a similar diagram. \square

The following well-known result is purely categorical, and applies to our particular functor π^* , as π^* is symmetric monoidal by Proposition 4.3 and satisfies the projection formula of Proposition 5.6. See [FHM03] for detailed derivations of these results in the 1-categorical context. In the ∞ -categorical context, the natural transformations are derived via adjunction in exactly the same way, and whether or not a natural transformation is a natural equivalence is a property of the induced natural transformation between homotopy 1-categories.

Lemma 6.27. *Let $\pi^* : \mathcal{C} \rightarrow \mathcal{D}$ be a symmetric monoidal functor of closed symmetric monoidal ∞ -categories which admits both a left adjoint $\pi_!$ and a right adjoint π_* and satisfies the following projection formula: for all objects $Y \in \mathcal{C}$ and $Z \in \mathcal{D}$, the map*

$$\pi_!(Z \otimes \pi^* Y) \rightarrow (\pi_! Z) \otimes Y,$$

induced by the unit map $Z \rightarrow \pi^ \pi_! Z$, is an equivalence. Then π^* is closed symmetric monoidal; that is, the map*

$$\pi^* F_{\mathcal{C}}(Y, X) \rightarrow F_{\mathcal{D}}(\pi^* Y, \pi^* X),$$

induced by the evaluation $Y \otimes F_{\mathcal{C}}(Y, X) \rightarrow X$ is an equivalence.

Proof. Let Z be an object of \mathcal{D} . We have a commutative square

$$\begin{array}{ccc} \text{Map}(Z, \pi^* F(Y, X)) & \longrightarrow & \text{Map}(Z, F(\pi^* Y, \pi^* X)) \\ \downarrow & & \downarrow \\ \text{Map}((\pi_! Z) \otimes Y, X) & \longrightarrow & \text{Map}(\pi_!(Z \otimes \pi^* Y), X) \end{array}$$

in which the vertical maps are equivalences by adjunction. But the bottom horizontal map is induced by the equivalence $\pi_!(Z \otimes \pi^* Y) \rightarrow (\pi_! Z) \otimes Y$, so the top horizontal map is an equivalence as well. Since Z is arbitrary, the result follows. \square

Corollary 6.28. *For any given object X of \mathcal{C} , there is a canonical equivalence of functors*

$$F(\pi_!(-), X) \simeq \pi_* F(-, \pi^* X) : \mathcal{D}^{\text{op}} \rightarrow \mathcal{C}.$$

In particular, specializing to the case in which X is the unit of \mathcal{C} , we obtain an equivalence $D\pi_! \xrightarrow{\sim} \pi_ D$.*

Proof. The adjoint of the composite natural transformation of functors

$$\pi^* F_{\mathcal{C}}(\pi_!(-), X) \rightarrow F_{\mathcal{D}}(\pi^* \pi_!(-), \pi^* X) \rightarrow F_{\mathcal{D}}(-, \pi^* X)$$

is the desired natural transformation

$$F_{\mathcal{C}}(\pi_!(-), X) \rightarrow \pi_* F_{\mathcal{D}}(-, \pi^* X).$$

Let Z be any object of \mathcal{C} . This transformation is an equivalence if and only if it is an equivalence upon evaluation at any object Y of \mathcal{D} . We have equivalences

$$\text{Map}(Z, F(\pi_! Y, X)) \simeq \text{Map}(Z \otimes \pi_! Y, X)$$

and

$$\text{Map}(Z, \pi_* F(Y, \pi^* X)) \simeq \text{Map}(\pi^* Z \otimes Y, \pi_* X) \simeq \text{Map}(\pi_!(\pi^* Z \otimes Y), X).$$

Hence the result follows from the projection formula $\pi_!(\pi^* Z \otimes Y) \simeq Z \otimes \pi_! Y$. \square

We now return to the special case of our particular symmetric monoidal functor π^* , and we write $h : D_S \pi_! \xrightarrow{\sim} \pi_* D_S$ for the canonical equivalence of [Corollary 6.28](#) above.

Proposition 6.29. *Let $q : X \rightarrow S$ be an object of $\text{Sm}_S^{G, N\text{-free}}$ such that that $q_{\#} \mathbb{1}_X \in \text{Spt}^G(S)$ is dualizable. The diagram*

$$\begin{array}{ccc} D_S D_S(\pi_! q_{\#} \mathbb{1}_X) & \xrightarrow{\sim} \xrightarrow{D(e)} & D_S(\pi_! D_S(q_{\#} \mathbb{1}_X)) & \xrightarrow{\sim} \xrightarrow{h} & \pi_* D_S D_S(q_{\#} \mathbb{1}_X) \\ \text{can} \uparrow \sim & & & & \sim \uparrow \pi_* \text{can} \\ \pi_! q_{\#} \mathbb{1}_X & \xrightarrow{\tau} & & & \pi_* q_{\#} \mathbb{1}_X \end{array}$$

commutes, where e is the map from [Remark 6.20](#).

Proof. Write $D = D_S$ and $X = q_{\#} \mathbb{1}_X$. The commutativity of the diagram of the proposition is equivalent, by adjointness, to that of

$$\begin{array}{ccc} \pi^* D D(\pi_! X) & \xrightarrow{\pi^* h D(e)} & \pi^* \pi_* D D(X) & \xrightarrow{\epsilon} & D D(X) \\ \pi^* \text{can} \uparrow & & & & \uparrow \text{can} \\ \pi^* \pi_! X & \xrightarrow{\hat{\tau}} & & & X, \end{array}$$

where $\epsilon : \pi^* \pi_* \rightarrow \text{id}$ is the counit. The outer composite of this diagram agrees with the outer composite of

the following diagram below, which we will show to be commutative:

$$\begin{array}{c}
\begin{array}{ccccc}
& \pi^* DD(\pi_1 X) & \xrightarrow{De} & \pi^* D(\pi_1 D(X)) & \xrightarrow{\epsilon h} & DD(X) \\
& \uparrow \pi^* \text{can} & & \downarrow \pi^* D \rightarrow D\pi^* & & \downarrow D(\text{id} \rightarrow \pi^* \pi_1) \\
\pi^* \pi_1 X & \xrightarrow{\text{can}} & D(\pi^* D(X)) & \xrightarrow{De} & D(\pi^* \pi_1 D(X)) & \xrightarrow{D\beta} & D(\text{pr}_{2!} \Delta_! \Delta^* \text{pr}_1^* D(X)) \\
& & \uparrow D(\pi^* D \rightarrow D\pi^*) & & \uparrow & & \uparrow D(\text{pr}_1^* D \rightarrow D\text{pr}_1^*) \\
& & DD(\pi^* \pi_1 X) & & D(\text{pr}_{2!} \text{pr}_1^* D(X)) & \xrightarrow{D\beta} & D(\text{pr}_{2!} \Delta_! \Delta^* \text{pr}_1^* D(X)) \\
& & \uparrow & & \uparrow & & \uparrow D\gamma \\
& & DD(\text{pr}_{2!} \text{pr}_1^* X) & \xrightarrow{De} & D(\text{pr}_{2!} D(\text{pr}_1^* X)) & \xrightarrow{\phi} & D(\text{pr}_{2!} D(\Delta_! \Delta^* \text{pr}_1^* X)) \\
& & \uparrow \text{can} & & \downarrow \phi & & \downarrow De \\
& & \text{pr}_{2!} \text{pr}_1^* X & \xrightarrow{\phi} & \text{pr}_{2!} \Delta_! \Delta^* \text{pr}_1^* X & \xrightarrow{\sim} & X
\end{array}
\end{array}$$

④
③
⑤
②
①

Here, pr_i and Δ are as in 6.1, $\phi : \text{id} \rightarrow \Delta_! \Delta^*$ arises from the Wirthmueller isomorphism $\Delta_! \simeq \Delta_*$ together with the unit $\text{id} \rightarrow \Delta_* \Delta^*$, $\beta : \Delta_! \Delta^* \rightarrow \text{id}$ is the counit, and we identify $\pi_* \pi_1 X \simeq \text{pr}_{2!} \pi_1^* X$ as in Remark 6.3. The map γ is induced by the composite of exchanges

$$\Delta_! D \xrightarrow{\psi} D\Delta_* \quad \text{and} \quad D\Delta^* \xrightarrow{\psi_R} \Delta^* D$$

as follows: First, the exchange ψ is the equivalence fitting in the commutative diagram

$$\begin{array}{ccc}
\Delta_! D & \xrightarrow{\psi} & D\Delta_* \\
\sim \downarrow & & \downarrow \sim \\
\Delta_* D & \xleftarrow{\sim} & D\Delta_!
\end{array}$$

where the vertical equivalences are the Wirthmueller isomorphism and the bottom one is the standard equivalence. In particular ψ is an equivalence. The exchange ψ_R is its right mate and ψ_R is an equivalence on dualizable objects, so in particular ψ_R applied to X is invertible. By Lemma 6.10, the diagram

$$\begin{array}{ccccccc}
& & \gamma & & & & \\
& \swarrow & & \searrow & & \swarrow & \\
D\Delta_! \Delta^* q_{\#} \mathbb{1}_X & \xleftarrow{\sim} & D\Delta_* \Delta^* q_{\#} \mathbb{1}_X & \xleftarrow{\psi} & \Delta_! D\Delta^* q_{\#} \mathbb{1}_X & \xrightarrow{\psi_R} & \Delta_! \Delta^* Dq_{\#} \mathbb{1}_X \\
& & \searrow & & \swarrow & & \\
& & D(\text{unit}) & & \text{counit}_D & & \\
& & & & Dq_{\#} \mathbb{1}_X & &
\end{array} \tag{6.30}$$

commutes, where γ is by definition the upper horizontal composite. This implies that the subdiagram marked ② above commutes.

Next to see that $\textcircled{1}$ commutes, we need to see that the diagram

$$\begin{array}{ccc} \mathrm{pr}_{2!}D(\Delta!\Delta^*\mathrm{pr}_2^*q_{\#}\mathbb{1}_X) & \xrightarrow{e} & D(\mathrm{pr}_{2!}\Delta!\Delta^*\mathrm{pr}_2^*q_{\#}\mathbb{1}_X) \\ \gamma'\uparrow & & \sim\uparrow \\ \mathrm{pr}_{2!}\Delta!\Delta^*\mathrm{pr}_2^*D(q_{\#}\mathbb{1}_X) & \xrightarrow{\sim} & D(q_{\#}\mathbb{1}_X) \end{array}$$

commutes, where γ' is the composite of γ with the exchange $\mathrm{pr}_1^*D \rightarrow D\mathrm{pr}_1^*$. Using [Remark 6.20](#) and that $\Delta!\Delta^*\mathrm{pr}_1^*q_{\#}\mathbb{1}_X \simeq m_{\#}\mathbb{1}_{G'/G \times X}$, where $m : G'/G \times X \rightarrow S$ is the G' -equivariant structure map, we see that the arrow $e : \mathrm{pr}_{2!}D(\Delta!\Delta^*\mathrm{pr}_1^*q_{\#}\mathbb{1}_X) \rightarrow D(\mathrm{pr}_{2!}\Delta!\Delta^*\mathrm{pr}_1^*q_{\#}\mathbb{1}_X)$ is obtained as the adjoint of the map

$$D(\Delta!\Delta^*\mathrm{pr}_1^*q_{\#}\mathbb{1}_X) \simeq D(\Delta_*\Delta^*\mathrm{pr}_2^*q_{\#}\mathbb{1}_X) \xrightarrow{D(\mathrm{unit})} D(\mathrm{pr}_2^*q_{\#}\mathbb{1}_X) \simeq \mathrm{pr}_2^*D(q_{\#}\mathbb{1}_X).$$

It now follows, using [\(6.30\)](#), that this diagram, and hence $\textcircled{1}$, commutes.

That $\textcircled{3}$ commutes follows from the commutativity of the diagram for $W \in \mathrm{Spt}^{G, N\text{-free}}(S)$

$$\begin{array}{ccc} W & \longrightarrow & \pi^*\pi_1 W \\ \sim\downarrow & & \uparrow\sim \\ \mathrm{pr}_{2!}\Delta!\Delta^*\mathrm{pr}_1^*W & \xrightarrow{\beta} & \mathrm{pr}_{2!}\mathrm{pr}_1^*W. \end{array}$$

This commutativity can be checked for W the suspension spectrum of a smooth N -free G -scheme over S , which is straightforward to verify. Using [Lemma 6.10](#), we see that the subdiagram $\textcircled{4}$ commutes. That subdiagram $\textcircled{5}$ follows by applying D to the diagram [\(6.26\)](#). The remaining subdiagrams are easily seen to commute. \square

Next, we verify that $\mathbf{E}\mathcal{F}[N]_+$ is a colimit of dualizable spectra. Let \mathcal{F} be a family. If $X \in \mathrm{Sm}_B^G$, recall that we write

$$X^{\mathcal{F}} = \cup_{H \in \mathrm{co}(\mathcal{F})} X^H \text{ and } X(\mathcal{F}) = X \setminus X^{\mathcal{F}}.$$

Write $f : X \rightarrow B$ and $g : X(\mathcal{F}) \rightarrow B$ for the structure maps.

Lemma 6.31. *Suppose that $f_{\#}(\mathbb{1}_X)$ is dualizable in $\mathrm{Spt}^G(B)$. Then $g_{\#}(\mathbb{1}_{X(\mathcal{F})})$ is dualizable.*

Proof. Filter the inclusion $\mathcal{F} \subseteq \mathcal{F}_{\mathrm{all}}$ by adjacent families $\mathcal{F} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_n = \mathcal{F}_{\mathrm{all}}$. Write $\alpha_i : X(\mathcal{F}_i) \subseteq X$ and $\beta_i : X(\mathcal{F}_i)^{\mathcal{F}_{i-1}} \subseteq X(\mathcal{F}_i)$ for the inclusions and write $f_i : X(\mathcal{F}_i) \rightarrow B$ for the induced structure map. From the gluing sequence, we obtain exact sequences

$$f_{i\#}(\mathbb{1}_{X(\mathcal{F}_i)}) \rightarrow f_{i+1\#}(\mathbb{1}_{X(\mathcal{F}_{i+1})}) \rightarrow f_{i+1\#}\beta_{i+1*}(\mathbb{1}_{X(\mathcal{F}_{i+1})^{\mathcal{F}_i}}).$$

Let $H_i \leq G$ be a subgroup such that $\mathcal{F}_{i+1} \setminus \mathcal{F}_i = \{(H_i)\}$. Then since $X(\mathcal{F}_{i+1})^{\mathcal{F}_i}$ is concentrated at (H_i) , we have that $X(\mathcal{F}_{i+1})^{\mathcal{F}_i} \cong G \times_{NH_i} X(\mathcal{F}_{i+1})^{H_i}$ by [Lemma 3.26](#). We then have that

$$f_{i+1\#}\beta_{i+1*}(\mathbb{1}_{X(\mathcal{F}_{i+1})^{\mathcal{F}_i}}) \simeq G_+ \wedge_{NH_i} (f'_{H_i})_{\#}(\mathbb{1}_{X(\mathcal{F}_{i+1})^{H_i}}),$$

where $f'_{H_i} : X(\mathcal{F}_{i+1})^{H_i} \rightarrow B$ is the structure map.

Now suppose that $f_{i+1\#}(\mathbb{1}_{X(\mathcal{F}_{i+1})})$ is dualizable. It follows that $(f'_{H_i})_{\#}(\mathbb{1}_{X(\mathcal{F}_{i+1})^{H_i}})$ is also dualizable, since this is obtained by applying the geometric fixed points functor. Therefore $f_{i+1\#}\beta_{i+1*}(\mathbb{1}_{X(\mathcal{F}_{i+1})^{\mathcal{F}_i}})$ is dualizable and we conclude that $f_{i\#}(\mathbb{1}_{X(\mathcal{F}_i)})$ is dualizable as well. Since we have assumed that $f_{\#}(\mathbb{1}_X) = f_{n\#}(\mathbb{1}_{X(\mathcal{F}_n)})$ is dualizable, the result follows by (finite) induction. \square

Corollary 6.32. *Let $S \in \text{Sch}_B^G$. Given a family \mathcal{F} , $\mathbb{1}_{\mathbf{EF}} \in \mathcal{Spt}^G(S)$ can be expressed as*

$$\mathbb{1}_{\mathbf{EF}} \simeq \text{colim}_{n \in \mathbb{N}} q_n \# \mathbb{1}_{U_n}$$

where $q_n : U_n \rightarrow S$ is in $\text{Sm}_S^G[\mathcal{F}]$ and $q_n \# \mathbb{1}_{U_n}$ is dualizable in $\mathcal{Spt}^G(S)$.

Proof. It suffices to show this when $S = B$. Use the previous lemma together with the presentation of [Example 3.5](#). \square

Corollary 6.33. *Suppose that $q \# \mathbb{1}_X \in \mathcal{Spt}^G(S)$ is dualizable where $q : X \rightarrow S$ be an object of $\text{Sm}_S^{G, N\text{-free}}$. Then the Adams transformation is an equivalence*

$$\tau : \pi_! q \# \mathbb{1}_X \xrightarrow{\sim} \pi_* i_! q \# \mathbb{1}_X.$$

In particular,

$$\tau : \pi_! \mathbb{1}_{\mathbf{EF}(N)} \xrightarrow{\sim} \pi_* i_! \mathbb{1}_{\mathbf{EF}(N)}$$

Proof. The first statement is immediate from the previous proposition. The second statement then follows, using [Corollary 6.32](#). \square

Next we will define a transformation

$$\nu : \pi_* i_! \rightarrow \pi_!,$$

which we will show is inverse to τ . We begin with the following observation. Recall that there is a transformation $\pi_* i_! i^*(\mathbb{1}_S) \otimes \text{id} \rightarrow \pi_* i_! i^* \pi^*$, see [\(6.6\)](#).

Lemma 6.34. *The map [\(6.6\)](#) is an equivalence.*

Proof. Let $Y \in \mathcal{Spt}^{G/N}(S)$ and write $\mathbf{EF}(N) \simeq \text{colim}_n U_n$ with $q_n \# \mathbb{1}_{U_n}$ dualizable over S , see [Corollary 6.32](#). Then [\(6.6\)](#) evaluated on Y is the colimit of the transformations

$$\pi_*(q_n \# \mathbb{1}_{U_n}) \otimes Y \rightarrow \pi_*(q_n \# \mathbb{1}_{U_n} \otimes \pi^* Y).$$

We see that each of these transformations is an equivalence since each fits into the commutative diagram

$$\begin{array}{ccccc} \pi_*(DD(q_n \# \mathbb{1}_{U_n}) \otimes \pi^* Y) & \xleftarrow{\sim} & \pi_* F(D(q_n \# \mathbb{1}_{U_n}), \pi^* Y) & \xleftarrow{\sim} & F(\pi_! D(q_n \# \mathbb{1}_{U_n}), Y) \\ \uparrow & & & & \uparrow \sim \\ \pi_* DD(q_n \# \mathbb{1}_{U_n}) \otimes Y & \xleftarrow{\sim} & \pi_! q_n \# \mathbb{1}_{U_n} \otimes Y & \xleftarrow{\sim} & F(D(\pi_! q_n \# \mathbb{1}_{U_n}), Y). \end{array}$$

\square

Now define $\nu : \pi_* i_! \rightarrow \pi_!$ as the composite

$$\begin{aligned} \pi_* i_! &\xrightarrow{\text{unit}} \pi_* i_! i^* \pi^* \pi_! \simeq \pi_*(i_! i^*(\mathbb{1}_S) \otimes \pi^* \pi_!) \\ &\xrightarrow{\sim} \pi_*(i_! i^*(\mathbb{1}_S)) \otimes \pi_! \xrightarrow{\tau^{-1}} \pi_! i^*(\mathbb{1}_S) \otimes \pi_! \rightarrow \pi_!. \end{aligned}$$

Lemma 6.35. *The composite $\pi_! \xrightarrow{\tau} \pi_* i_! \xrightarrow{\nu} \pi_!$ is an equivalence.*

Proof. The composite $v\tau$ agrees with the composite around the following diagram

$$\begin{array}{ccccccc}
\pi_! & \xrightarrow{\tau} & \pi_* i_! & \longrightarrow & \pi_* i_! i^* \pi^* \pi_! & \longrightarrow & \pi_! \\
\downarrow & & & & \downarrow \sim & & \uparrow \\
\pi_! i^* (\mathbb{1}_S) \otimes \pi_! & \xrightarrow[\tau \otimes \text{id}]{\sim} & \pi_* i_! i^* (\mathbb{1}_S) \otimes \pi_! & \xrightarrow[\tau^{-1} \otimes \text{id}]{\sim} & \pi_! i^* (\mathbb{1}_S) \otimes \pi_! & &
\end{array}$$

□

We are now in a position to prove that the Adams transformation $\tau : \pi_! \rightarrow \pi_* i_!$, as defined in [Definition 6.5](#), is an equivalence. Recall that τ is obtained via adjunction from the natural transformation (6.4), where $i_! : \text{Spt}^{G, N\text{-free}}(S) \rightarrow \text{Spt}^G(S)$ is the inclusion functor, $\pi_! : \text{Spt}^{G, N\text{-free}}(S) \rightarrow \text{Spt}^{G/N}(S)$ is the quotient functor, and $\pi_* : \text{Spt}^G(S) \rightarrow \text{Spt}^{G/N}(S)$ is the fixed point functor.

Theorem 6.36 (Adams Isomorphism). *The Adams transformation $\tau : \pi_! \rightarrow \pi_* i_!$ is an equivalence.*

Proof. By [Corollary 5.4](#), it suffices to show that τ is an equivalence on $\Sigma^{-\mathcal{V}} q_* \mathbb{1}_X$ where \mathcal{V} is an N -trivial representation and $q : X \rightarrow S$ is an N -free (not necessarily smooth) G -scheme over S . Write $f : X \rightarrow \bar{X}$ for the quotient and $p : \bar{X} \rightarrow S$ for the induced map. Consider the diagram

$$\begin{array}{ccccc}
\pi_! p_* f_* \mathbb{1}_X & \xrightarrow{\tau} & \pi_* i_! p_* f_* \mathbb{1}_X & \xrightarrow{v} & \pi_! p_* f_* \mathbb{1}_X \\
\downarrow & & \downarrow \sim & & \downarrow \\
p_* \pi_! f_* \mathbb{1}_X & \xrightarrow[\tau]{\sim} & p_* \pi_* i_! f_* \mathbb{1}_X & \xrightarrow[v]{\sim} & p_* \pi_! f_* \mathbb{1}_X.
\end{array}$$

The left-hand square commutes by [Proposition 6.14](#) and [Lemma 6.15](#) and the right-hand square commutes by a similar argument. Both horizontal composites are equivalences by [Lemma 6.35](#), and therefore the outer vertical arrows are equivalences as the middle one is. Finally, $\tau : \pi_! f_* \mathbb{1}_X \rightarrow \pi_* i_! f_* \mathbb{1}_X$ is an equivalence by [Corollary 6.33](#) (since $f_{\#} \mathbb{1}_X \simeq f_* \mathbb{1}_X$, by ambidexterity [[Hoy17](#)], as f is finite étale). □

6.4 Applications

In this section, we present a few applications of the Adams isomorphism.

Let BG denote the classifying stack of the group G and $f : BG \rightarrow B(G/N)$ the resulting proper map of stacks. We have an equivalence $\text{Spt}(BG) \simeq \text{Spt}^G(S)$ and from this perspective the fixed point functor π_* becomes identified with the pushforward functor f_* . The following base change results are an instance of the proper and the smooth-proper base change formula, but with the curious feature that f is not a representable morphism. These don't follow immediately from the six functor formalism in [[Hoy17](#)], precisely because f is not representable.

We use the names for exchange morphisms given in the first part of the previous subsection.

Corollary 6.37 (Proper base change). *Let $p : T \rightarrow S$ be a morphism in $\text{Sch}_B^{G/N}$. The exchange*

$$\alpha_R : p^* \pi_* \rightarrow \pi_* p^*$$

is an equivalence.

Proof. Choose a filtration $\emptyset = \mathcal{F}_{-1} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_n = \mathcal{F}_{\text{all}}$ such that each pair $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ is N -adjacent, see [§ 3.2](#). This gives rise to the filtration of $X \in \text{Spt}^G(S)$,

$$* \simeq \mathbf{E}\mathcal{F}_{-1+} \otimes X \rightarrow \mathbf{E}\mathcal{F}_{0+} \otimes X \rightarrow \dots \rightarrow \mathbf{E}\mathcal{F}_{n-1+} \otimes X \rightarrow \mathbf{E}\mathcal{F}_{n+} \otimes X \simeq X.$$

It thus suffices to check that α_R is an equivalence on each filtration quotient $\mathbf{E}(\mathcal{F}_{i+1}, \mathcal{F}_i) \otimes X$. Suppose that $\mathcal{F} \subseteq \mathcal{F}'$ is N -adjacent at $H \leq N$. By [Proposition 4.12](#), we find that

$$\begin{aligned} p^* \pi_*(\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X) &\simeq p^*(G/N_+ \times_{\mathcal{W}} (\mathbf{E}\mathcal{F}(\mathbf{W}_N H)_+ \otimes X^{\Phi H})^{\mathbf{W}_N H}) \\ &\simeq (G/N_+ \times_{\mathcal{W}} p^*((\mathbf{E}\mathcal{F}(\mathbf{W}_N H)_+ \otimes X^{\Phi H})^{\mathbf{W}_N H})) \\ &\simeq (G/N_+ \times_{\mathcal{W}} (p^*(\mathbf{E}\mathcal{F}(\mathbf{W}_N H)_+ \otimes X^{\Phi H}))^{\mathbf{W}_N H}) \\ &\simeq (G/N_+ \times_{\mathcal{W}} (\mathbf{E}\mathcal{F}(\mathbf{W}_N H)_+ \otimes (p^*X)^{\Phi H})^{\mathbf{W}_N H}) \\ &\simeq \pi_* p^*(\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X). \end{aligned}$$

Here, the third equivalence follows from [Theorem 6.36](#) and [Proposition 6.12](#), since $\mathbf{E}\mathcal{F}(\mathbf{W}_N H)_+ \otimes X^{\Phi H}$ is $\mathbf{W}_N H$ -free. The fourth follows from [Proposition 4.11](#). \square

Corollary 6.38 (Smooth-proper base change). *Let $p : T \rightarrow S$ be a smooth morphism in $\text{Sch}_B^{G/N}$. The exchange*

$$\bar{\alpha} : p_{\#} \pi_* \rightarrow \pi_* p_{\#}$$

is an equivalence.

Proof. Similar to the proof of proper base change. \square

Corollary 6.39 (Projection formula). *Let $S \in \text{Sch}_B^{G/N}$. There is a canonical equivalence*

$$\pi_*(X) \otimes Y \simeq \pi_*(X \otimes \pi^*(Y))$$

for $X \in \text{Spt}^G(S)$ and $Y \in \text{Spt}^{G/N}(S)$.

Proof. Consider the map $\pi_*(X) \otimes Y \rightarrow \pi_*(X \otimes \pi^*(Y))$ adjoint to the map

$$\pi^*(\pi_*(X) \otimes Y) \simeq \pi^* \pi_*(X) \otimes \pi^* Y \rightarrow X \otimes \pi^* Y,$$

where the equivalence follows from the symmetric monoidality of π^* and the map is given by tensoring the counit $\pi^* \pi_* X \rightarrow X$ with $\pi^* Y$.

Let $\mathcal{F} \subseteq \mathcal{F}'$ be an N -adjacent pair of families, say at $H \leq N$. $\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X$. Then by [Proposition 4.12](#)

$$(\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X)^N \simeq G/N_+ \times_{\mathcal{W}} (\mathbf{E}\mathcal{F}(\mathbf{W}_N H)_+ \otimes X^{\Phi H})^{\mathbf{W}_N H}.$$

Under this equivalence, the transformation is identified with $G/N_+ \times_{\mathcal{W}} -$ applied to the transformation

$$(\mathbf{E}\mathcal{F}(\mathbf{W}_N H)_+ \otimes X^{\Phi})^{\mathbf{W}_N H} \otimes Y \rightarrow (\mathbf{E}\mathcal{F}(\mathbf{W}_N H)_+ \otimes \Phi^H X \otimes \pi^* Y)^{\mathbf{W}_N H}.$$

That this is an equivalence follows from the commutativity of the diagram

$$\begin{array}{ccc} \pi_! W \otimes V & \longrightarrow & \pi_!(W \otimes \pi^* V) \\ \tau \otimes \text{id} \downarrow & & \downarrow \tau \\ \pi_* W \otimes V & \longrightarrow & \pi_*(W \otimes \pi^* V) \end{array}$$

where $W \in \text{Spt}^{G, N\text{-free}}(S)$, $V \in \text{Spt}^{G/N}(S)$; the commutativity can be checked by an argument similar to ones above, e.g., that in [Proposition 6.7](#).

The general case follows by choosing a filtration $\emptyset = \mathcal{F}_{-1} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_n = \mathcal{F}_{\text{all}}$ such that each pair $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ is N -adjacent and considering the induced filtration $\mathbf{E}\mathcal{F}_i \otimes X$ on X . \square

7 Splitting motivic G -spectra à la tom Dieck

Throughout this section, $N \trianglelefteq G$ is a normal subgroup and we assume that N acts trivially on S .

For a subgroup $H \leq N$ we write $\mathcal{W}H = W_G H / W_N H$ for the quotient of Weil groups. Let $\mathcal{F}(W_N H)$ be the family of subgroups $\{K \leq W_G H \mid K \cap W_N H = \{e\}\}$. As before, we write $\mathbf{E}_{W_N H}(W_G H) = \mathbf{E}\mathcal{F}(W_N H)$ for the universal $W_N H$ -free $W_G H$ -motivic space, to emphasize the ambient group.

Definition 7.1. Let X be a motivic G -spectrum and $H \leq G$ a subgroup. A *splitting of X at H* is a map f_H in $\text{Spt}^{W_G H}(S)$,

$$X^H \begin{array}{c} \xleftarrow{f_H} \\ \longrightarrow \end{array} X^{\Phi H}$$

which splits the canonical map $X^H \rightarrow X^{\Phi H}$. An N -*splitting* of X is a choice of splitting of X at each subgroup $H \leq N$.

Example 7.2. 1. Let $Y \in \text{Sp}_{\bullet}^G(S)$. The suspension spectrum $\Sigma^\infty Y$ has a canonical splitting at any subgroup $H \leq G$, defined as follows. Write $\pi : N_G H \rightarrow W_G H$ for the quotient. The counit of the adjunction $\pi^* \dashv (-)^H$ on based motivic $N_G H$ -spaces yields the map $\pi^*(Y^H) \rightarrow Y$ of spaces and thus a map of $N_G H$ -spectra $\pi^* \Sigma^\infty(Y^H) \rightarrow \Sigma^\infty Y$. Its adjoint is the map

$$\Sigma^\infty(Y^H) \rightarrow (\Sigma^\infty Y)^H.$$

By [Proposition 4.10](#), this induces the desired splitting.

2. If $X \in \text{Spt}^{G/N}(S)$ then $\phi^*(X)$ is split.

3. If X and Y are split at H then $X \otimes Y$ is canonically split via the composition $(X \otimes Y)^{\Phi H} \simeq X^{\Phi H} \otimes Y^{\Phi H} \rightarrow X^H \otimes Y^H \rightarrow (X \otimes Y)^H$.

Write $i : \text{Sm}_S^{G, N\text{-free}} \subseteq \text{Sm}_S^G$ for the inclusion.

Definition 7.3. The *motivic homotopy orbit point spectrum* of X is

$$X_{\mathbf{h}N} := \pi_* i_{!} i^*(X) \simeq (\mathbf{E}\mathcal{F}(N)_+ \wedge X) / N$$

Let X be an N -split motivic G -spectrum. Let $H \leq N$ be a subgroup and consider the composition, where for notational brevity, we write simply $\mathbf{E}_H = \mathbf{E}_{W_N H}(W_G H)$. Define the map $\Theta_{X, H}$ as the following composite, where the maps are explained below,

$$\begin{aligned} G/N_+ \rtimes_{\mathcal{W}H} (X^{\Phi H})_{\mathbf{h}W_N H} &\simeq G/N_+ \rtimes_{\mathcal{W}H} (\mathbf{E}_{H_+} \otimes X^{\Phi H})^{W_N H} \\ &\simeq G/N_+ \rtimes_{\mathcal{W}H} ((\mathbf{E}_{H_+} \otimes X)^{\Phi H})^{W_N H} \\ &\xrightarrow{(f_H)_*} G/N_+ \rtimes_{\mathcal{W}H} ((\mathbf{E}_{H_+} \otimes X)^H)^{W_N H} \\ &\simeq ((G_+ \rtimes_{N_G H} \mathbf{E}_{H_+}) \otimes X)^N \\ &\longrightarrow X^N. \end{aligned}$$

The map f_H is the splitting of X at H and the last map is induced by the projection

$$G_+ \rtimes_{N_G H} \mathbf{E}_{H_+} \simeq (G \times_{N_H G} \mathbf{E}_H)_+ \rightarrow S^0.$$

The first equivalence comes from the Adams isomorphism. The second comes from the monoidality of geometric fixed points and that H acts trivially on $\mathbf{E}\mathcal{F}(W_N H)_+$. The fourth map, which is an equivalence, comes from a canonical exchange of functors (see proof of [Proposition 4.12](#)) together with the projection

formula for induction-restriction. Note that this map only depends on the G -conjugacy class of the subgroup H .

Now define the map of motivic G/N -spectra

$$\Theta_X : \bigoplus_{(H)} G/N_+ \times_{\mathcal{W}H} (X^{\Phi H})_{\mathbf{h}W_{NH}} \rightarrow X^N,$$

where the index is over the set of G -conjugacy classes of subgroups of N , to be the sum over the maps $\Theta_{X,H}$ defined above. The map Θ_X is natural with respect to maps of N -split spectra which are compatible with splitting.

Theorem 7.4 (Motivic tom Dieck splitting). *Let $X \in \mathcal{Spt}^G(S)$ be an N -split motivic G -spectrum. The map*

$$\Theta_X : \bigoplus_{(H)} G/N_+ \times_{\mathcal{W}H} (X^{\Phi H})_{\mathbf{h}W_{NH}} \rightarrow X^N$$

is an equivalence of motivic G/N -spectra. In particular, for integers a and b , there is a canonical isomorphism

$$\pi_{a,b}^G(\mathbb{1}_B) \cong \bigoplus_{(H)} \pi_{a,b}(\mathbf{B}WH_+).$$

Proof. Since G is finite there is a sequence of families

$$\emptyset = \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}_{\text{all}}$$

such that each pair $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ is N -adjacent, see § 3.2. This gives rise to the filtration of the identity functor

$$* \simeq \mathbf{E}\mathcal{F}_{-1+} \otimes - \rightarrow \mathbf{E}\mathcal{F}_{0+} \otimes - \rightarrow \cdots \rightarrow \mathbf{E}\mathcal{F}_{n-1+} \otimes - \rightarrow \mathbf{E}\mathcal{F}_{n+} \otimes - \simeq \text{id}.$$

It suffices to show that to show that $\Theta_{X \otimes \mathbf{E}(\mathcal{F}', \mathcal{F})}$ is an equivalence whenever $\mathcal{F} \subseteq \mathcal{F}'$ is an N -adjacent pair.

But if $\mathcal{F} \subseteq \mathcal{F}'$ is N -adjacent at $H \leq N$, then all summands of the domain of Θ vanish except the summand corresponding to the conjugacy class (H) and $\Theta_{X \otimes \mathbf{E}(\mathcal{F}', \mathcal{F})}$ is an equivalence by Proposition 4.12. \square

Remark 7.5. Let Fin_G denote the category of finite G -sets. It is not difficult to see that the functor $c : \text{Fin}_G \rightarrow \text{Sm}_B^G$, defined by $A \mapsto \coprod_A B$, induces a functor $c^* : \mathcal{Spt}^G \rightarrow \mathcal{Spt}^G(B)$, which is colimit preserving and symmetric monoidal. Segal [Seg71] showed that $\pi_0^G(\mathbb{1}) \cong A(G)$ is the Burnside ring and so we obtain a canonical ring map $c^* : A(G) \rightarrow \pi_{0,0}^G(\mathbb{1}_B)$. We also have a canonical ring map $\pi^* : \pi_{0,0}(\mathbb{1}_B) \rightarrow \pi_{0,0}^G(\mathbb{1}_B)$ induced by the projection $\pi : G \rightarrow \{e\}$. We thus obtain a ring map

$$A(G) \otimes \pi_{0,0}(\mathbb{1}_B) \rightarrow \pi_{0,0}^G(\mathbb{1}_B).$$

Using the splitting theorem above and that $\pi_{0,0}^G(\mathbf{B}WH_+) \cong \pi_{0,0}(\mathbf{B}WH) \oplus \pi_{0,0}(\mathbb{1}_B)$, we see that this is an injective ring map.

When $B = \text{Spec}(k)$ is the spectrum of a perfect field, Morel [Mor12] shows that $\pi_{0,0}(\mathbb{1}_k) \cong GW(k)$ is the Grothendieck-Witt ring of symmetric bilinear forms. We thus have in this case a canonical (injective) ring map

$$A(G) \otimes GW(k) \rightarrow \pi_{0,0}^G(\mathbb{1}_k).$$

This map is in general not surjective, since $\pi_{0,0}(\mathbf{B}WH)$ is in general nonzero, see [Mor06].

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