TOPOLOGICAL COMPARISON THEOREMS FOR BREDON MOTIVIC COHOMOLOGY

J. HELLER, M. VOINEAGU, AND P. A. ØSTVÆR

Abstract. We prove equivariant versions of the Beilinson-Lichtenbaum conjecture for Bredon motivic cohomology of smooth complex and real varieties with an action of the group of order two. This identifies equivariant motivic and topological invariants in a large range of degrees.

Contents

1. Introduction

A major achievement of motivic homotopy theory is the proof of the Bloch-Kato conjecture relating Milnor K-theory to Galois cohomology [Voe03a], [Voe11], and consequently the solution of the Beilinson-Lichtenbaum conjecture by the results in [SV00, §7]. With finite mod-n coefficients and X a smooth scheme of finite type over a field k of characteristic coprime to n, one form of the Beilinson-Lichtenbaum conjecture asserts that the comparison map between motivic and étale cohomology

$$H^{p,q}_{\mathcal{M}}(X,\mathbb{Z}/n) \to H^{p}_{et}(X,\mu_n^{\otimes q}) \tag{1.1}$$

is an isomorphism when $p \leq q$ [SV00, Conjecture 6.8] and [VSF00, Chapter 1]. This "étale descent" property identifies a large range of motivic cohomology groups, a.k.a. higher Chow groups, with more computable étale cohomology groups. For a smooth complex variety X, the étale cohomology groups in (1.1) agree with the singular cohomology groups of its corresponding analytic space $X(\mathbb{C})$. Thus the Beilinson-Lichtenbaum conjecture provides a powerful link between algebro-geometric and topological invariants. This can further be enhanced to prove the Quillen-Lichtenbaum conjectures comparing the algebraic and hermitian K-theories

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of X with their analytic or étale counterparts, see e.g., [Sus95, Theorem 4.7], [Voe03a, Theorem 7.10] and [BKSØ15, Theorem 5.1].

A framework for invariants of smooth varieties equipped with a group scheme action has recently been organized in the subject of equivariant motivic homotopy theory [Del09], [HKO11], [HKØ15], [Her13], [Hoy15], [CJ14]. Of particular interest are actions by the group C_2 of order two governing the examples of hermitian K-theory and motivic Real cobordism [HKO11]. Bredon motivic cohomology introduced in [HVØ15, §5] is an equivariant generalization of motivic cohomology for finite group actions. This theory is now amenable to a homotopical analysis on account of the equivariant cancellation theorem shown in [HVØ15, Theorem 9.7].

We employ this setup to prove an equivariant Beilinson-Lichtenbaum comparison theorem for smooth complex varieties. Before stating our main theorem, we review the players involved. If M is a topological space with C_2-action and A is an abelian group, the Bredon cohomology groups $H^{a+p\sigma}_{C_2}(M,\underline{A})$ are an equivariant analog of singular cohomology groups. Here a,p are integers and σ stands for the sign representation, i.e., topological Bredon cohomology is graded by virtual representations of C_2. Now if X is a smooth complex C_2-scheme of finite type, the Bredon motivic cohomology groups $H^{a+p\sigma,b+q\sigma}_{C_2}(X,\underline{A})$ have a grading by a 4-tuple of integers. This 4-tuple is a pair of virtual representations $V=a+p\sigma$ and $W=b+q\sigma$ which are respectively the "cohomological degree" and the "weight" of the grading.

In Appendix A we construct a natural comparison map between Bredon motivic cohomology and its equivariant topological counterpart

$$H^{a+p\sigma,b+q\sigma}_{C_2}(X,\underline{A})\to H^{a+p\sigma}_{C_2}(X(\mathbb{C}),\underline{A}). \tag{1.2}$$

The map in (1.2) is induced by a Betti realization functor for the stable C_2-equivariant motivic homotopy category. In fact, the groups on the left hand side of (1.2) are represented by a C_2-equivariant Bredon motivic cohomology spectrum $\mathbf{M}\underline{A}$, whose Betti realization agrees with the C_2-equivariant Eilenberg-MacLane spectrum for the constant Mackey functor $\underline{A}$, cf. Definition 3.2, Section A.6, and Theorem A.31.

A virtual C_2-representation U determines a pair of integers via its dimension and the fixed point dimension. Write $\underline{\dim}(U):=(\dim(U),\dim(U^{C_2}))$ for this pair. Below we use the usual partial ordering on $\mathbb{Z}\oplus\mathbb{Z}$, namely $(m_1,m_2)\leq(n_1,n_2)$ provided $m_i\leq n_i$, and if n is an integer we write as well n for the pair (n,n). The Beilinson-Lichtenbaum comparison theorem says that the comparison map (1.1) is an isomorphism (resp. injection) for cohomological degrees less than or equal than the weight (resp. weight plus one). Our main result, which appears as Theorem 6.7 below, asserts roughly the same thing but for the comparison map (1.2), where cohomological degrees and weights are virtual representations rather than merely integers. A precise statement is the following.

Theorem 1.3. *Let X be a smooth complex scheme of finite type with C_2-action. Write $V=a+p\sigma$ and $W=b+q\sigma$. Then, the comparison map (1.2) is an isomorphism for $\underline{\dim}(V)\leq\underline{\dim}(W)$ and an injection when $\underline{\dim}(V)\leq\underline{\dim}(W)+1$.*

In particular, in this same range of degrees it follows that $H^{a+p\sigma,b+q\sigma}_{C_2}(X,\underline{A})$ is a finite abelian group.

We also establish an equivariant version of the Beilinson-Lichtenbaum comparison theorem in the case of smooth real C_2-schemes of finite type, see Theorem 7.9. In this case, if X is a finite-type C_2-scheme over $\mathbb{R}$, the space of complex points

$X(\mathbb{C})$ has a $C_2 \times C_2$-action. One copy of C_2 acts algebraically through the action on X and the other acts via complex conjugation. Thus Theorem 7.9 is a comparison result which relates C_2-equivariant Bredon motivic cohomology of X with $C_2 \times C_2$-equivariant topological Bredon cohomology of $X(\mathbb{C})$.

The nonequivariant Beilinson-Lichtenbaum comparison theorem is a fundamental ingredient in the proof of Theorem 6.7. We leverage this nonequivariant comparison result to an equivariant comparison by means of a motivic version of the isotropy separation cofiber sequence together with some computations in Borel motivic cohomology. We emphasize that these arguments rely crucially both on the representability of Bredon motivic cohomology in the stable equivariant motivic homotopy category as well as that it can be computed as sheaf hypercohomology. That the representable theory coincides with sheaf hypercohomology is basically a recasting of the homotopy invariance and equivariant cancellation theorems, established in [HVØ15], and is proved in Theorem 3.4 below.

The equivariant Beilinson-Lichtenbaum comparison theorem is a first important step towards understanding the Bredon motivic cohomology ring $H^{\star,\star}_{C_2}(k,\underline{\mathbb{Z}/2})$ of a field k. As in (1.2), the gradings are sums of C_2-representations. These invariants are fundamental for understanding key features of C_2-equivariant motivic homotopy theory, e.g., $H^{\star,\star}_{C_2}(k,\underline{\mathbb{Z}/2})$ forms part of the largely unknown C_2-equivariant motivic Steenrod algebra of cohomology operations. Another fundamental aspect of Bredon motivic cohomology is that the zeroth slice of the C_2-equivariant motivic sphere spectrum turns out to be the highly structured C_2-equivariant Bredon motivic cohomology spectrum $\mathbf{M}\underline{\mathbb{Z}}$ introduced in Definition 3.2, cf. [HØ16].

This paper is structured as follows. Sections 2-7 are devoted to the proof of our main result for the comparison map (1.2). The proof rests on techniques from equivariant motivic homotopy theory: roughly speaking, taking complex or real points induces a Betti realization functor which in turn gives rise to the comparison map for Bredon motivic cohomology. The details of these constructions are found in Appendix A. In Section 2 we develop the homotopical techniques that we need in the sequel; in particular, the motivic isotropy separation cofiber sequence plays a central role in our approach. The computational core of the paper lies in Sections 3-7. First, in Section 3, we establish that Bredon motivic cohomology is represented by $\mathbf{M}\underline{\mathbb{Z}}$ and show that it affords Thom classes for a certain class of equivariant vector bundles as well as Gysin sequences. In Section 4 we compare Bredon motivic cohomology with Edidin-Graham's equivariant higher Chow groups. In Section 5 we show that the generalized "geometric" Borel motivic cohomology ring $H^{\star,\star}_{C_2}(\mathbf{E}C_2,\underline{\mathbb{Z}/2})$ is periodic with periods $(\sigma-1,0)$ and $(\sigma-1,\sigma-1)$. These preliminary results are used to prove the complex and real comparison theorems for Bredon motivic cohomology in Section 6 and Section 7, respectively.

Notation and Conventions: Throughout k is a perfect field of characteristic $\mathrm{char}(k) \neq 2$. For a finite group G, let $G\mathrm{Sm}/k$ denote the category of smooth schemes of finite type over k with left G-actions and equivariant maps. We use the term *k-variety* synonymous with separated, finite type, scheme over k.

We write $\mathbb{A}(V) = \mathrm{Spec}(\mathrm{Sym}(V^\vee))$ for the affine scheme associated to a vector space V over k, and $\mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}(V^\vee))$ for the associated projective scheme.

The construction of the stable G-equivariant motivic homotopy category $\mathrm{SH}_G(k)$ is recalled in Appendix A. We write $[-,-]_G$ for maps in $\mathrm{SH}_G(k)$. We distinguish four sphere objects in the C_2-equivariant motivic homotopy category, see

Section 2.3. These are denoted S^1, S^σ, S^1_t, and S^σ_t. The sphere S^1 is the usual simplicial sphere and S^σ the simplicial sign representation sphere. The algebro-geometric sphere S^1_t is the pointed scheme $(\mathbb{G}_m, 1)$ equipped with trivial action and S^σ_t is the pointed scheme $(\mathbb{G}_m, 1)$ equipped with the inversion action, $x \mapsto x^{-1}$. We write $V = a + p\sigma$ for the C_2-representation which is the sum of a-copies of the trivial representation and p-copies of the sign representation, and define

$$S^{a+p\sigma,b+q\sigma} := S^{a-b} \wedge S^{(p-q)\sigma} \wedge S^b_t \wedge S^{q\sigma}_t. \tag{1.4}$$

We adopt the convention that $*$ refers to an integer grading of homotopy or cohomology groups while $\star$ refers to grading by representations.

2. Background

Equivariant motivic homotopy theory was introduced by Voevodsky [Del09] as a tool for understanding symmetric products and motivic Eilenberg-MacLane spaces. Stable equivariant motivic homotopy category was introduced by Hu-Kriz-Ormsby [HKO11] as part of their study of the homotopy limit problem for hermitian K-theory of fields. In this section we recall definitions and basic results about equivariant motivic homotopy theory. Technical details and a fairly complete, self-contained discussion are given in Appendix A.

2.1. Equivariant Nisnevich topology.

The equivariant Nisnevich topology was introduced by Voevodsky [Del09, §3]. See [HVØ15, §3] and [HKØ15, §2] for more details concerning the equivariant Nisnevich topology.

Definition 2.1. An *equivariant distinguished square*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ U \arrow[r, hook, "i"] & X \end{tikzcd} \tag{2.2}$$

is a cartesian square in $G\mathrm{Sch}/k$ such that p is étale, $i : U \subseteq X$ is an open embedding, and p induces an isomorphism of reduced schemes $(Y \setminus W)_{red} \cong (X \setminus U)_{red}$. An elementary Nisnevich cover is the cover $\{U \to X, Y \to X\}$ associated to an equivariant distinguished square. The *equivariant Nisnevich topology* on $G\mathrm{Sm}/k$ is the smallest Grothendieck topology containing the elementary Nisnevich covers.

Recall that the set-theoretic stabilizer S_x of a point $x \in X$ is the subgroup $S_x := \{g \in G \mid gx = x\}$. By [HVØ15, Proposition 3.5], an equivariant étale map $f : Y \to X$ is an equivariant Nisnevich cover if and only if for every $x \in X$ there is $y \in Y$ such that f induces an isomorphism $k(y) \cong k(x)$ on residue fields as well as an isomorphism $S_x \cong S_y$ on set-theoretic stabilizers.

Remark 2.3. Using this characterization of Nisnevich covers we see that every smooth G-scheme is locally affine. Every point $x \in X$ has an S_x-invariant affine neighborhood (take any affine neighborhood and consider the intersection of the translates by elements of S_x). Let U_x be such a neighborhood. Then $G \times^{S_x} U_x \to X$, where for a subgroup $H \subseteq G$ and H-scheme Z we write $G \times^H Z$ for $(G \times Z)/H$, is an equivariant Nisnevich neighborhood of $G{\cdot}x$ in the sense of [HKØ15, Section 2]. The collection $\{G \times^{S_x} U_x \to X\}$ is an "infinite" equivariant Nisnevich cover of X by smooth affine G-schemes admitting a finite subcover by [HKØ15, Remark 2.18].

2.2. **Motivic G-spaces and spectra.** A *motivic G-space* over k is a presheaf of simplicial sets on $G\mathrm{Sm}/k$. We write $G\mathrm{Spc}(k)$ and $G\mathrm{Spc}_\bullet(k)$ respectively for the categories of motivic G-spaces and pointed motivic G-spaces over k. The unstable equivariant motivic homotopy category is constructed following a pattern familiar from ordinary motivic homotopy theory; technical details are found in Appendix A. In brief, it is the homotopy category of a model structure which is constructed so that the following two relations hold.

(i) Any equivariant distinguished square (2.2) is a homotopy cocartesian square.
(ii) The projection $X \times \mathbb{A}^1 \to X$ is an equivariant motivic weak equivalence for any X in $G\mathrm{Sm}/k$.

These relations have non-obvious consequences. For example, by the Whitehead theorem, a map inducing isomorphisms on equivariant Nisnevich sheaves of homotopy groups is an equivariant motivic weak equivalence, cf. [MV99, §3.2, Proposition 2.14]. Moreover, every G-equivariant vector bundle is an equivariant motivic weak equivalence [HKØ15, Proposition 4.10].

Let V be a representation of G, e.g., the regular representation $\rho_G = k[G]$. The associated *motivic representation sphere* is defined to be the pointed motivic G-space

$$T^V := \mathbb{P}(V \oplus 1)/\mathbb{P}(V).$$

For an integer $n \geq 0$ we use the smash product in $G\mathrm{Spc}_\bullet(k)$ to define

$$T^{nV} := (T^V)^{\wedge n}.$$

Consider the equivariant distinguished square

$$\begin{tikzcd}
\mathbb{A}(V) \setminus \{0\} \arrow[r] \arrow[d] & \mathbb{A}(V) \arrow[d] \\
\mathbb{P}(V \oplus 1) \setminus \mathbb{P}(1) \arrow[r] & \mathbb{P}(V \oplus 1).
\end{tikzcd}$$

It is in particular a homotopy cocartesian square. The inclusion of $\mathbb{P}(V \oplus 1) \setminus \mathbb{P}(1)$ into $\mathbb{P}(V \oplus 1)$ is equivariantly $\mathbb{A}^1$-homotopic to the inclusion $\mathbb{P}(V) \subseteq \mathbb{P}(V \oplus 1)$; the requisite deformation is given by $([x_0 : \cdots : x_{n+1}], t) \mapsto [x_0 : \cdots : x_n : tx_{n+1}]$. We conclude there is an equivariant motivic weak equivalence

$$T^V \simeq \mathbb{A}(V)/\mathbb{A}(V) \setminus \{0\}.$$

Also note that $T^{nV} \simeq T^{V^{\oplus n}}$ and more generally $T^V \wedge T^W \simeq T^{V \oplus W}$.

The stable equivariant motivic homotopy category $\mathrm{SH}_G(k)$ is the stabilization of $G\mathrm{Spc}_\bullet(k)$ with respect to the sphere T^{ρ_G}. We use symmetric T^{ρ_G}-spectra as a model for $\mathrm{SH}_G(k)$, see Section A.4. It is a tensor triangulated category with unit the sphere spectrum $\mathbf{1} = \Sigma^\infty_{T^{\rho_G}} S^0$. Here, $S^0 = \mathrm{Spec}(k)_+$ is the unit for the smash product in $G\mathrm{Spc}_\bullet(k)$. If X is an unbased motivic G-space, e.g., a smooth G-scheme, we have an associated based motivic G-space X_+, by adding a disjoint basepoint, and an associated suspension spectrum $\Sigma^\infty_{T^{\rho_G}} X_+$. When no confusion should arise, we sometimes simply write X for $\Sigma^\infty_{T^{\rho_G}} X_+$ and S^0 for the sphere spectrum.

2.3. **C_2-equivariant spheres.** When $G = C_2$ there are only two representations, the trivial representation and the sign representation which we write as σ. It is convenient to introduce the following sphere objects. The *sign Tate sphere* S^1_t is the pointed C_2-scheme $(\mathbb{G}_m, 1)$ where $\mathbb{G}_m$ is equipped with the action $x \mapsto x^{-1}$. The *simplicial sign representation sphere* S^σ is defined to be the unreduced suspension of C_2, i.e., it is the homotopy cofiber of ${C_2}_+ \to S^0$. We have as well the usual simplicial sphere S^1 and the Tate sphere S^1_t which is the pointed scheme $(\mathbb{G}_m, 1)$ considered with trivial action. As observed in [HKO11, §4.1], there is an equivariant motivic weak equivalence $S^\sigma \wedge S^\sigma_t \simeq T^\sigma$.

The indexing convention

$$S^{a+p\sigma,b+q\sigma} := S^{a-b} \wedge S^{(p-q)\sigma} \wedge S^b_t \wedge S^{q\sigma}_t$$

in (1.4) is a mixture between the convention standardly used in motivic homotopy theory and that used in classical equivariant homotopy theory. The translation between the convention of indexing here and the one in [HKO11, §4.1] is given by

$$S^{a+p\sigma,b+q\sigma} = S^{(a-b)+(p-q)\gamma+b\alpha+q\gamma\alpha}.$$

The convention we use in this paper has the feature that the effect of the complex Betti realization functor (constructed in Section A.5) is the first entry of the index,

$$\mathrm{Re}_{\mathbb{C}}(S^{a+p\sigma,b+q\sigma}) = S^{a+p\sigma}.$$

A real Betti realization (taking value in $\mathrm{SH}_{C_2\times\Sigma_2}$) is constructed in Section A.7. In this case,

$$\mathrm{Re}_{\mathbb{C},\,\Sigma_2}(S^{a+p\sigma,b+q\sigma}) = S^{a-b+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon},$$

where σ is the sign representation corresponding to the factor C_2 and ϵ is the sign representation corresponding to Σ_2.

The following two fundamental homotopy cofiber sequences of pointed motivic C_2-spaces are useful for computations,

$$C_{2+} \to S^0 \to S^{\sigma,0} \tag{2.4}$$

and

$$(\mathbb{A}(n\sigma) \setminus \{0\})_+ \to S^0 \to S^{2n\sigma,n\sigma}. \tag{2.5}$$

Here, the first maps in these sequences are induced respectively by the projections $C_2 \to \mathrm{Spec}(k)$ and $\mathbb{A}(n\sigma) \setminus \{0\} \to \mathrm{Spec}(k)$.

2.4. **Motivic isotropy separation.** The isotropy separation cofiber sequence is a fundamental tool for analyzing equivariant homotopy types in classical equivariant homotopy theory. Our proof of the main Theorems 6.7 and 7.9 makes use of an appropriate motivic version. We shall restrict our attention to the group C_2. See [GH16] for a general discussion of motivic isotropy separation.

Recall that the classical topological isotropy separation cofiber sequence is

$$\mathrm{E}_\bullet C_{2+} \to S^0 \to \widetilde{\mathrm{E}}_\bullet C_2,$$

where the last term is defined by this sequence. A check of fixed points shows that a model for $\widetilde{\mathrm{E}}_\bullet C_2$ is given by

$$\widetilde{\mathrm{E}}_\bullet C_2 \simeq \operatorname*{colim}_n S^{n\sigma}. \tag{2.6}$$

The *geometric classifying space* $\mathbf{B}C_2$ for C_2 is defined as the quotient of

$$\mathbf{E}C_2 := \operatorname*{colim}_n \mathbb{A}(n\sigma) \setminus \{0\}$$

by the free C_2-action. This space plays an important role in nonequivariant motivic homotopy theory because it is a geometric model for the étale classifying space

[MV99, §4.2]. A similar interpretation of the geometric classifying space is also true in equivariant motivic homotopy theory as well, see [GH16].

The *motivic isotropy separation cofiber sequence* is the cofiber sequence

$$\mathbf{E}C_{2+} \to S^0 \to \widetilde{\mathbf{E}}C_2, \tag{2.7}$$

where the space $\widetilde{\mathbf{E}}C_2$ is defined by this cofiber sequence. Because of the definition of the geometric classifying space we have an equivariant motivic equivalence

$$\widetilde{\mathbf{E}}C_2 \simeq \operatorname*{colim}_n S^{2n\sigma,n\sigma}. \tag{2.8}$$

Proposition 2.9. *The maps $S^0 \to S^{2\sigma,\sigma}$ and $S^0 \to S^{\sigma,0}$ induce equivariant motivic equivalences $\widetilde{\mathbf{E}}C_2 \simeq S^{2\sigma,\sigma} \wedge \widetilde{\mathbf{E}}C_2$ and $\widetilde{\mathbf{E}}C_2 \simeq S^{\sigma,0} \wedge \widetilde{\mathbf{E}}C_2$.*

Proof. The first equivalence follows immediately from (2.8). Using Lemma A.9, and that $\mathbf{E}C_{2,+}$ is nonequivariantly contractible [MV99, Proposition 4.2.3], we have equivariant motivic equivalences

$$\mathbf{E}C_{2+} \wedge C_{2+} \simeq (\mathbf{E}C_{2+})^e \wedge C_{2+} \simeq C_{2+}.$$

It follows that $(C_2)_+ \wedge \widetilde{\mathbf{E}}C_2 \simeq *$. Thus the second equivalence follows from the cofiber sequence (2.4) and the equivariant motivic weak equivalence (2.6). □

3. Bredon motivic cohomology

In [HVØ15, §5] we introduced a Bredon motivic cohomology theory on $G\mathrm{Sm}/k$. Here we define an equivariant motivic spectrum which is a representing object for the Bredon motivic cohomology groups of loc. cit., and record some of its basic properties. To keep the exposition streamlined, we restrict our discussion to the group C_2.

3.1. Motivic complexes.

For X and Y smooth k-schemes of finite type we write $\mathrm{Cor}_k(X,Y)$ for the group of finite correspondences [SV00, §3.1]. If X and Y have an action by C_2 then C_2 acts on $\mathrm{Cor}_k(X,Y)$ as well. The category of equivariant correspondences $C_2\,\mathrm{Cor}_k$ has the same objects as $C_2\mathrm{Sm}/k$ and maps $\mathrm{Cor}_k(X,Y)^{C_2}$. A *presheaf with equivariant transfers* on $C_2\mathrm{Sm}/k$ is an additive presheaf on $C_2\,\mathrm{Cor}_k$.

If Y is a smooth C_2-scheme over k, write $\mathbb{Z}_{tr,C_2}(Y)$ for the free presheaf with equivariant transfers,

$$\mathbb{Z}_{tr,C_2}(Y)(X) := \mathrm{Cor}_k(X,Y)^{C_2}.$$

More generally, if A is an abelian group, we write $A_{tr,C_2}(Y) = \mathbb{Z}_{tr,C_2}(Y) \otimes A$. It is useful to extend the definition of $A_{tr,C_2}(-)$ to quotients of G-schemes. If $\mathcal{X} = \mathrm{colim}_i\, X_i$, where X_i are smooth C_2-schemes over k and the colimit is in the category of presheaves, then we define $A_{tr,C_2}(\mathcal{X}) := \mathrm{colim}_i\, A_{tr,C_2}(X_i)$, where the colimit is computed in the category of presheaves of abelian groups.

Defining $\mathbb{Z}_{tr,C_2}(X) \otimes^{tr} \mathbb{Z}_{tr,C_2}(Y) := \mathbb{Z}_{tr,C_2}(X \times Y)$ determines a symmetric monoidal product $\otimes^{tr}$ on the category of presheaves with equivariant transfers.

If W is a finite C_2-set, viewed as a smooth C_2-scheme over k, we have isomorphisms as presheaves on $C_2\mathrm{Sm}/k$, $\mathbb{Z}_{tr,C_2}(W) \cong \mathbb{Z}(W) := \mathbb{Z}\,\mathrm{Hom}_k(-,W)^{C_2}$. Moreover, if F is a presheaf with equivariant transfers, then we also have an isomorphism $F \otimes^{tr} \mathbb{Z}_{tr,C_2}(W) \cong F \otimes \mathbb{Z}(W)$ of presheaves on $C_2\mathrm{Sm}/k$.

If F is a presheaf of abelian groups we write $C_*F(X)$ for the simplicial abelian group $F(X \times \Delta^\bullet_k)$. We use the same notation for the associated cochain complex. Here, $\Delta^\bullet_k$ is the standard cosimplicial object in Sm/k.

Write $D^-(C_2\operatorname{Cor}_k)$ for the derived category of bounded above chain complexes of equivariant Nisnevich sheaves with equivariant transfers on $C_2\mathrm{Sm}/k$. According to [HVØ15, Lemma 5.12] the cone $\mathbb{Z}_{top}(\sigma)$ of the map $\mathbb{Z}_{tr,C_2}(\mathbb{Z}/2)\to\mathbb{Z}$ is invertible in $(D^-(C_2\operatorname{Cor}_k),\otimes^{tr})$. If $F_\bullet$ is a cochain complex of presheaves with equivariant transfers, write $F_\bullet[\sigma]=F_\bullet\otimes^{tr}\mathbb{Z}_{top}(\sigma)$.

Let $V=a+b\sigma$ be a representation of C_2 and A an abelian group. Define the *motivic Bredon complex*

$$\underline{A}(V):=a_\#(C_*A_{tr,C_2}(T^V))[-2a-2b\sigma],$$

where $a_\#$ denotes sheafification in the equivariant Nisnevich topology, see Definition 2.1. There are quasi-isomorphisms $a_\#(C_*A_{tr,C_2}(T))\simeq a_\#(C_*A_{tr,C_2}(S^1_t))[1]$ and $a_\#(C_*A_{tr,C_2}(T^\sigma))\simeq a_\#(C_*A_{tr,C_2}(S^\sigma_t))[\sigma]$, see [HVØ15, p.328]. It follows that there is a quasi-isomorphism

$$a_\#(C_*A_{tr,C_2}(T^V))\simeq a_\#(C_*A_{tr,C_2}(S^a_t\wedge S^{b\sigma}_t))[a+b\sigma]. \tag{3.1}$$

Now let A be a commutative ring. We construct a product pairing

$$\underline{A}(V)\otimes\underline{A}(W)\to\underline{A}(V\oplus W).$$

First, we have an associative pairing $A_{tr,C_2}(T^V)\otimes A_{tr,C_2}(T^W)\to A_{tr,C_2}(T^V\wedge T^W)$ of presheaves which is induced by the pairing

$$\begin{aligned}A_{tr,C_2}(\mathbb{P}(V\oplus 1))(U)\otimes & A_{tr,C_2}(\mathbb{P}(W\oplus 1))(U)\\ &\xrightarrow{\times} A_{tr,C_2}(\mathbb{P}((V\oplus 1)\times\mathbb{P}(W\oplus 1))(U\times U)\\ &\xrightarrow{\Delta^*} A_{tr,C_2}(\mathbb{P}(V\oplus 1)\times\mathbb{P}(W\oplus 1))(U).\end{aligned}$$

Let $A_{*,*}$ be a bisimplicial abelian group. We write $A_{*,*}$ as well for the associated cochain complex. By the Eilenberg-Zilber theorem [GJ99, Theorem IV.2.4], taking totalizations and diagonals yields a natural quasi-isomorphism of chain complexes $\operatorname{Tot}(A_{*,*})\to\operatorname{diag}(A_{*,*})$. We thus obtain the natural pairing

$$\begin{aligned}\operatorname{Tot}&\left(A_{tr,C_2}(T^V)(U\times\Delta^\bullet_k)\otimes A_{tr,C_2}(T^W)(U\times\Delta^\bullet_k)\right)\\ &\to\operatorname{diag}\left(A_{tr,C_2}(T^V)(U\times\Delta^\bullet_k)\otimes A_{tr,C_2}(T^W)(U\times\Delta^\bullet_k)\right)\\ &\to A_{tr,C_2}(T^V\wedge T^W)(U\times\Delta^\bullet_k)\xleftarrow{\cong} A_{tr,C_2}(T^{V\oplus W})(U\times\Delta^\bullet_k).\end{aligned}$$

This induces our desired pairing upon sheafification. Now the quasi-isomorphism obtained from the Eilenberg-Zilber theorem is homotopy associative, so the pairing is also homotopy associative.

3.2. **Stable representability.** Let A be a commutative ring. Let $\mathcal{F}$ be a presheaf of sets. we may consider $A_{tr,C_2}(\mathcal{F})$ as a presheaf of sets and therefore as a based motivic C_2-space, the basepoint is 0. We have a canonical map $\gamma:\mathcal{F}\to A_{tr,C_2}(\mathcal{F})$ of motivic C_2-spaces and there is a pairing $\mu: A_{tr,C_2}(X)\wedge A_{tr,C_2}(Y)\to A_{tr,C_2}(X\times Y)$ of motivic C_2-spaces.

Definition 3.2. The *motivic Bredon cohomology spectrum* $\mathbf{M}\underline{A}$, is defined by letting $\mathbf{M}\underline{A}_n:=A_{tr,C_2}(T^{n\rho_{C_2}})$ with structure maps

$$\begin{aligned}A_{tr,C_2}(T^{n\rho_{C_2}})\wedge T^{\rho_{C_2}}&\xrightarrow{\mathrm{id}\wedge\gamma}A_{tr,C_2}(T^{n\rho_{C_2}})\wedge A_{tr,C_2}(T^{\rho_{C_2}})\\ &\xrightarrow{\mu}A_{tr,C_2}(T^{(n+1)\rho_{C_2}}).\end{aligned}$$

The symmetric group Σ_n acts on $\mathbf{M}\underline{A}_n$ by permuting the factors of $T^{\rho_{C_2}}$. The iterated structure maps $\mathbf{M}\underline{A}_n \wedge T^{k\rho_{C_2}} \to \mathbf{M}\underline{A}_{n+k}$ are $(\Sigma_n \times \Sigma_k)$-equivariant. This means that $\mathbf{M}\underline{A}$ is a symmetric motivic C_2-spectrum, cf. Definition A.10. There are pairings $\mathbf{M}\underline{A}_n \wedge \mathbf{M}\underline{A}_k \to \mathbf{M}\underline{A}_{n+k}$ which give $\mathbf{M}\underline{A}$ the structure of a commutative ring spectrum (i.e., a commutative monoid in the category of equivariant symmetric motivic spectra $\mathrm{Spt}^{\Sigma}_{C_2}(k)$). Finally, we note that the spectrum $\mathbf{M}\underline{A}$ is stably equivalent to $\mathbf{M}\underline{\mathbb{Z}} \wedge \mathbb{S}A$ where $\mathbb{S}A$ is the Moore spectrum associated to A.

Definition 3.3. The *Bredon motivic cohomology* of a motivic C_2-spectrum E, with coefficients in the abelian group A, is defined by

$$\widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(\mathsf{E},\underline{A}) := [\mathsf{E}, S^{a+p\sigma,b+q\sigma} \wedge \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)}.$$

If X is a smooth C_2-scheme, then its unreduced Bredon motivic cohomology is defined via its suspension spectrum by setting

$$H^{a+p\sigma,b+q\sigma}_{C_2}(X,\underline{A}) := \widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(\Sigma^{\infty}_{T^{\rho_{C_2}}} X_+,\underline{A}).$$

Next we verify that the definition of Bredon motivic cohomology which we have just given agrees with the one given in [HVØ15]. (Note, however, that the indexing we use in the present paper is slightly different than in loc. cit.). The fact that Bredon motivic cohomology defined in the stable equivariant motivic homotopy category is equal to the hypercohomology groups of the motivic complexes, plays a crucial role in our arguments in the later sections. This fact relies on the homotopy invariance and equivariant cancellation theorems for presheaves with equivariant transfers proved in [HVØ15, Theorem 8.12, Theorem 9.7].

If $\mathcal{X} = \operatorname{colim} X_i$ is a colimit (in presheaves) of smooth C_2-schemes over k, write $\mathbb{Z}(\mathcal{X}) := \operatorname{colim}_i \mathbb{Z}(X_i)$ and

$$H^{a}_{C_2 Nis}(\mathcal{X},\underline{A}(V)[p\sigma]) := \operatorname{Ext}^{a}_{C_2 Nis}(\mathbb{Z}(\mathcal{X}),\underline{A}(V)[p\sigma]).$$

Theorem 3.4. *Let $V = b+q\sigma$ be a virtual representation of C_2, W a representation such that $V \oplus W$ is a representation, X a smooth C_2-scheme of finite type over k, and A an abelian group. Then there is a natural isomorphism*

$$H^{a+p\sigma,b+q\sigma}_{C_2}(X,\underline{A}) \cong H^{a}_{C_2 Nis}(T^{W} \wedge X_+,\underline{A}(W \oplus V)[p\sigma]).$$

Proof. We assume that $V = a + b\sigma$ is an actual representation; the more general case of a virtual representation is similar. Let $\mathbf{M}\underline{A} \to \mathbf{M}\underline{A}'$ be a levelwise motivic fibrant replacement, i.e., for each n, $\mathbf{M}\underline{A}'_n$ is motivic fibrant and $\mathbf{M}\underline{A}_n \to \mathbf{M}\underline{A}'_n$ is a motivic weak equivalence (see Section A.4 for details on the motivic model structure). We claim that $\mathbf{M}\underline{A}'$ is already a fibrant motivic C_2-spectrum. Indeed, using Theorem A.6, the map

$$\pi_i(\mathbf{M}\underline{A}'_n)(X) \to \pi_i(\Omega_{T^{\rho_{C_2}}} \mathbf{M}\underline{A}'_{n+1})(X)$$

is naturally identified with the map

$$H^{-i}_{C_2 Nis}(X,\underline{A}(n\rho_{C_2})) \to H^{-i}_{C_2 Nis}(T^{\rho_{C_2}} \wedge X_+,\underline{A}((n+1)\rho_{C_2})).$$

This is the map of the equivariant Cancellation Theorem [HVØ15, Theorem 9.8], which is an isomorphism for all i. Therefore, $\mathbf{M}\underline{A}'_n \to \Omega_{T^{\rho_{C_2}}} \mathbf{M}\underline{A}'_{n+1}$ is a weak equivalence of motivic C_2-spaces, which implies that $\mathbf{M}\underline{A}'$ is an $\Omega_{T^{\rho_{C_2}}}$-spectrum and so is a fibrant motivic C_2-spectrum, cf. Section A.4.

Let i, j, k, l be nonnegative integers such that

$$S^{(i+k)+(j+l)\sigma,k+l\sigma} \wedge S^{a+p\sigma,b+q\sigma} \simeq T^{m\rho_{C_2}}$$

for some $m \geq 0$. In particular, $m = i + a - b = j + p - q = k + b = l + q$. We have $H^{a+p\sigma,b+q\sigma}_{C_2}(X, \underline{A}) = [X_+, S^{a+p\sigma,b+q\sigma} \wedge \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)}$ and

$$\begin{aligned}
[X_+, & S^{a+p\sigma,b+q\sigma} \wedge \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)} \\
&\cong [S^{(i+k)+(j+l)\sigma,k+l\sigma} \wedge X_+, T^{m\rho_{C_2}} \wedge \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)} \\
&\cong [S^{i} \wedge S^{j\sigma} \wedge S^{k}_t \wedge S^{l\sigma}_t \wedge X_+, T^{m\rho_{C_2}} \wedge \mathbf{M}\underline{A}']_{\mathrm{SH}_{C_2}(k)} \\
&\cong [S^{i} \wedge S^{j\sigma} \wedge S^{k}_t \wedge S^{l\sigma}_t \wedge X_+, (\mathbf{M}\underline{A}')_m]_{\mathrm{H}_{\bullet,C_2}(k)} \\
&\cong H^{-i}_{C_2Nis}(S^{j\sigma} \wedge S^{k}_t \wedge S^{l\sigma}_t \wedge X_+, C_*A_{tr,C_2}(T^{m+m\sigma})) \\
&\cong H^{-i}_{C_2Nis}(S^{j\sigma} \wedge S^{k}_t \wedge S^{l\sigma}_t \wedge X_+, C_*A_{tr,C_2}(S^{m}_t \wedge S^{m\sigma}_t)[m+m\sigma]) \\
&\cong H^{0}_{C_2Nis}(X, C_*A_{tr,C_2}(S^{m-k}_t \wedge S^{(m-l)\sigma}_t)[(m-i)+(m-j)\sigma]) \\
&\cong H^{a}_{C_2Nis}(X, \underline{A}(V)[p\sigma]).
\end{aligned}$$

The first two isomorphisms are immediate. The third follows from [Hov01, Theorem 8.10] and the standard adjunction relating $\mathrm{H}_{\bullet,C_2}(k)$ and $\mathrm{SH}_{C_2}(k)$, the fourth from Theorem A.6, the remaining isomorphisms follow from (3.1) and equivariant cancellation [HVØ15, Theorem 9.8].

□

Remark 3.5. Under the isomorphisms above, the product structures arising from the pairing of spectra $\mathbf{M}\underline{A} \wedge \mathbf{M}\underline{A} \to \mathbf{M}\underline{A}$ agrees with that arising from the pairing of complexes $\underline{A}(V) \otimes \underline{A}(W) \to \underline{A}(V \oplus W)$.

3.3. Basic properties.

We record some of the basic properties of Bredon motivic cohomology.

Cancellation. Let $V = s + t\sigma$ be a virtual representation. We defined Bredon motivic cohomology via a representing spectrum in $\mathrm{SH}_{C_2}(k)$. Immediate from this definition we have natural isomorphisms

$$\widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(T^V \wedge \mathsf{E}, \underline{A}) \cong \widetilde{H}^{a-2s+(p-2t)\sigma,b-s+(q-t)\sigma}_{C_2}(\mathsf{E}, \underline{A}). \tag{3.6}$$

Equivariant transfers. Recall we write $D^-(C_2\,\mathrm{Cor}_k)$ for the derived category of bounded above chain complexes of equivariant Nisnevich sheaves with equivariant transfers on $C_2\mathrm{Sm}/k$.

Proposition 3.7. *Let X be a smooth C_2-scheme over k and $K_\bullet$ a cochain complex of presheaves with equivariant transfers. Then there is a natural isomorphism $\mathrm{Ext}^n_{D^-(C_2\,\mathrm{Cor}_k)}(\mathbb{Z}_{tr,C_2}(X), K_\bullet) \cong H^n_{C_2Nis}(X, K_\bullet)$.*

Proof. The argument is as in [Voe00, Proposition 3.1.9], using that smooth C_2-schemes have finite equivariant Nisnevich cohomological dimension [HVØ15, Corollary 3.9] together with [HVØ15, Theorem 4.15]. □

Corollary 3.8. *If $K_\bullet$ is a cochain complex of sheaves with equivariant transfers then $H^n_{C_2Nis}(-, K_\bullet)$ is a presheaf with equivariant transfers.*

Theorem 3.4 shows that for any a, b, p, q, $H^{a+p\sigma,b+q\sigma}_{C_2}(X, \underline{A})$ is naturally identified with equivariant Nisnevich hypercohomology groups with coefficients in a (bounded above) cochain complex of sheaves with equivariant transfers. It is therefore a presheaf with equivariant transfers.

Mayer-Vietoris sequences. Since Bredon motivic cohomology is a representable theory, associated to an equivariant distinguished square (2.2) is a Mayer-Vietoris long exact sequence

$$\begin{aligned}\cdots \to H^{a+p\sigma,b+q\sigma}_{C_2}(X,\underline{A}) \to & H^{a+p\sigma,b+q\sigma}_{C_2}(U,\underline{A}) \oplus H^{a+p\sigma,b+q\sigma}_{C_2}(Y,\underline{A}) \\ & \to H^{a+p\sigma,b+q\sigma}_{C_2}(W,\underline{A}) \to H^{a+1+p\sigma,b+q\sigma}_{C_2}(X,\underline{A}) \to \cdots .\end{aligned}$$

Ring structure. When A is a commutative ring, then $\mathbf{M}\underline{A}$ is a commutative ring spectrum. We thus have a cup product pairing for smooth C_2-varieties

$$H^{a+p\sigma,b+q\sigma}_{C_2}(X,\underline{A}) \times H^{c+s\sigma,d+t\sigma}_{C_2}(X,\underline{A}) \xrightarrow{\cup} H^{(a+c)+(p+s)\sigma,(b+d)+(q+t)\sigma}_{C_2}(X,\underline{A})$$

given by the usual formula

$$x \cup y := \Delta^*(x \boxtimes y),$$

where $\Delta : X \to X \times X$ is the diagonal and $\boxtimes$ is the external product. The cup product makes $H^{*,*}_{C_2}(X,\underline{A})$ into a $\mathbb{Z}^4$-graded ring. Since it is a representable theory, it is a module over $\pi_0^{C_2}(\mathbf{1}) := \mathrm{End}_{\mathrm{SH}_{C_2}(k)}(\mathbf{1})$ and in fact an algebra over this ring.

Consider the following endomorphisms of $\mathbf{1}$ in $\mathrm{SH}_{C_2}(k)$

$$\begin{aligned}\epsilon &= \Sigma^{-2}_{S^1_t}\left(S^1_t \wedge S^1_t \xrightarrow{\text{twist}} S^1_t \wedge S^1_t\right), \\ \epsilon' &= \Sigma^{-2}_{S^\sigma_t}\left(S^\sigma_t \wedge S^\sigma_t \xrightarrow{\text{twist}} S^\sigma_t \wedge S^\sigma_t\right), \\ u &= \Sigma^{-2}_{S^\sigma}\left(S^\sigma \wedge S^\sigma \xrightarrow{\text{twist}} S^\sigma \wedge S^\sigma\right).\end{aligned} \tag{3.9}$$

We also write $\epsilon, \epsilon', u \in H^{0,0}_{C_2}(k,\underline{A})$ for the respective elements in cohomology, induced by these endomorphisms.

Proposition 3.10. *Let $x \in H^{a+p\sigma,b+q\sigma}_{C_2}(X,\underline{A})$ and $y \in H^{c+s\sigma,d+t\sigma}_{C_2}(X,\underline{A})$. Then*

$$x \cup y = (-1)^{ac}(u)^{ps}(\epsilon)^{bd}(\epsilon')^{qt}(y \cup x).$$

Proof. This follows by the same argument as [Dug14, Proposition 6.13] (see also Remark 6.14 of loc. cit.). The point is to carefully analyze the endomorphism of $S^{\alpha_1+\alpha_2,\beta_1+\beta_2}$ arising from the twist $S^{\alpha_1,\beta_1} \wedge S^{\alpha_2,\beta_2} \to S^{\alpha_2,\beta_2} \wedge S^{\alpha_1,\beta_1}$, where $(\alpha_1,\beta_1) = (a+p\sigma, b+q\sigma)$ and $(\alpha_2,\beta_2) = (c+s\sigma, d+t\sigma)$. □

Remark 3.11. We will see in Proposition 3.24 below that $\epsilon = 1$, $\epsilon' = 1$, and $u = -1$ in $H^{\star,\star}_{C_2}(k,\underline{A})$ and therefore $x \cup y = (-1)^{ac+ps}(y \cup x)$.

Topological realization. Let $k = \mathbb{C}$. The functor $C_2\mathrm{Sm}/\mathbb{C} \to C_2\mathrm{Top}$, $X \mapsto X(\mathbb{C})$ extends to a functor $\mathrm{Re}_{\mathbb{C}} : \mathrm{SH}_{C_2}(\mathbb{C}) \to \mathrm{SH}_{C_2}$ from the stable equivariant motivic homotopy category over $\mathbb{C}$ to the classical stable equivariant homotopy category, see Section A.5. For a topological space M with C_2-action we write $H^{a+p\sigma}_{C_2}(M,\underline{A})$ for the topological Bredon cohomology theory, and $\widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(\mathcal{X},\underline{A})$ for the reduced Bredon cohomology of a pointed motivic C_2-space $\mathcal{X}$. By Theorem A.31, the Betti realization $\mathrm{Re}_{\mathbb{C}}(\mathbf{M}\underline{A})$ represents topological Bredon cohomology with $\underline{A}$-coefficients. This yields the comparison functor

$$\mathrm{Re}_{\mathbb{C}} : H^{a+p\sigma,b+q\sigma}_{C_2}(X,\underline{A}) \to H^{a+b\sigma}_{C_2}(X(\mathbb{C}),\underline{A}).$$

Moreover, this yields a ring homomorphism since $\mathrm{Re}_{\mathbb{C}} : \mathrm{SH}_{C_2}(\mathbb{C}) \to \mathrm{SH}_{C_2}$ is a symmetric monoidal functor, see Theorem A.16.

Change of groups. Consider the adjunction $C_2 \times - : \mathrm{Sm}/k \rightleftarrows C_2\mathrm{Sm}/k : (-)^e$, where X^e is the underlying smooth scheme of a smooth scheme with C_2-action. This extends to an adjunction on pointed motivic spaces and by Proposition A.11 it stabilizes to yield the adjunction

$$C_{2+} \wedge - : \mathrm{SH}(k) \rightleftarrows \mathrm{SH}_{C_2}(k) : (-)^e$$

on stable homotopy categories. The functor $(-)^e$ should be thought of as forgetting the action. Similarly, the adjunction $(-)^{triv} : \mathrm{Sm}/k \rightleftarrows C_2\mathrm{Sm}/k : (-)^{C_2}$ induces an adjunction on stable homotopy categories

$$(-)^{triv} : \mathrm{SH}(k) \rightleftarrows \mathrm{SH}_{C_2}(k) : (-)^{C_2}.$$

Here $(-)^{triv} : \mathrm{Sm}/k \to C_2\mathrm{Sm}/k$ sends a smooth scheme X to the C_2-scheme consisting of X with the trivial action. If E is a motivic spectrum then $C_{2+} \wedge \mathsf{E}$ agrees with $C_{2+} \wedge (\mathsf{E})^{triv}$ (the latter being the smash product of two equivariant motivic spectra). Typically we simply write E again for the spectrum E^{triv}.

Write $H^{*,*}_{\mathcal{M}}(-, A)$ for motivic cohomology theory and $\mathbf{M}A$ for the representing motivic spectrum. There is a natural map

$$\phi : \widetilde{H}_{C_2}^{a+p\sigma,b+q\sigma}(\mathcal{X}, \underline{A}) \to \widetilde{H}_{\mathcal{M}}^{a+p,b+q}(\mathcal{X}^e, A) \tag{3.12}$$

obtained from $(-)^e : [\mathcal{X}, S^{a+b\sigma,b+q\sigma} \wedge \mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)} \to [\mathcal{X}^e, (S^{a+b\sigma,b+q\sigma} \wedge \mathbf{M}\underline{A})^e]_{\mathrm{SH}(k)}$ and the facts that $(S^{a+b\sigma,b+q\sigma} \wedge \mathbf{M}\underline{A})^e = S^{a+b,b+q} \wedge (\mathbf{M}\underline{A})^e$ and $(\mathbf{M}\underline{A})^e$ is a T^2-spectrum representing motivic cohomology. When $k = \mathbb{C}$, by Proposition A.17 and Theorem A.31 we have a commutative square

$$\begin{tikzcd}
\widetilde{H}_{C_2}^{a+p\sigma,b+q\sigma}(\mathcal{X}, \underline{A}) \arrow[r, "\phi"] \arrow[d] & \widetilde{H}_{\mathcal{M}}^{a+p,b+q}(\mathcal{X}^e, A) \arrow[d] \\
\widetilde{H}_{C_2}^{a+p\sigma}(\mathrm{Re}_{\mathbb{C}}(\mathcal{X}), \underline{A}) \arrow[r, "\phi"] & \widetilde{H}_{sing}^{a+p}(\mathrm{Re}_{\mathbb{C}}(\mathcal{X})^e, A).
\end{tikzcd} \tag{3.13}$$

Proposition 3.14. *Let $\mathcal{X}$ be a pointed motivic C_2-space. For integers a, b, p, q, there is a natural isomorphism*

$$\widetilde{H}_{C_2}^{a+p\sigma,b+q\sigma}(C_{2+} \wedge \mathcal{X}, \underline{A}) \xrightarrow{\cong} \widetilde{H}_{\mathcal{M}}^{a+p,b+q}(\mathcal{X}^e, A).$$

Moreover, when $k = \mathbb{C}$, there is a commutative square

$$\begin{tikzcd}
\widetilde{H}_{C_2}^{a+p\sigma,b+q\sigma}(C_{2+} \wedge \mathcal{X}, \underline{A}) \arrow[r, "\cong"] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & \widetilde{H}_{\mathcal{M}}^{a+p,b+q}(\mathcal{X}^e, A) \arrow[d, "\mathrm{Re}_{\mathbb{C}}"] \\
\widetilde{H}_{C_2}^{a+p\sigma}(C_{2+} \wedge \mathrm{Re}_{\mathbb{C}}(\mathcal{X}), \underline{A}) \arrow[r, "\cong"] & \widetilde{H}_{sing}^{a+p}(\mathrm{Re}_{\mathbb{C}}(\mathcal{X})^e, A).
\end{tikzcd}$$

Proof. By Lemma A.9, the adjoint of the inclusion $\mathcal{X}^e \to (C_{2+} \wedge \mathcal{X})^e = \mathcal{X}^e \coprod \mathcal{X}^e$ as the first summand, is an isomorphism $i : C_{2+} \wedge \mathcal{X}^e \cong C_{2+} \wedge \mathcal{X}$ in the equivariant motivic homotopy category. Together with the adjunction isomorphism and the observation that $(\mathbf{M}\underline{A})^e$ represents motivic cohomology, we obtain isomorphisms

$$\begin{aligned}
[C_{2+} \wedge \mathcal{X}, \Sigma^{a+p\sigma,b+q\sigma}\mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)} &\cong [C_{2+} \wedge (\mathcal{X})^e, \Sigma^{a+p\sigma,b+q\sigma}\mathbf{M}\underline{A}]_{\mathrm{SH}_{C_2}(k)} \\
&\cong [\mathcal{X}^e, \Sigma^{a+p\sigma,b+q\sigma}\mathbf{M}A]_{\mathrm{SH}(k)},
\end{aligned}$$

yielding the first part of the proposition.

We have $\mathrm{Re}_{\mathbb{C}}(\mathcal{X})^e \cong \mathrm{Re}_{\mathbb{C}}(\mathcal{X}^e)$ and $C_{2+} \wedge \mathrm{Re}_{\mathbb{C}}(\mathcal{X}) \cong \mathrm{Re}_{\mathbb{C}}(C_{2+} \wedge \mathcal{X})$ by Proposition A.15. The map $\mathrm{Re}_{\mathbb{C}}(i)$ is an isomorphism $C_{2+} \wedge \mathrm{Re}_{\mathbb{C}}(\mathcal{X})^e \cong C_{2+} \wedge \mathrm{Re}_{\mathbb{C}}(\mathcal{X})$. Together with the isomorphism $\mathrm{Re}_{\mathbb{C}}(\mathbf{M}\underline{A}) \cong \mathrm{H}\underline{A}$ in Theorem A.31 this implies the second part of the proposition. ☐

Proposition 3.15. *Let X be a smooth k-scheme considered as a C_2-scheme with trivial action. For integers m, n, the map (3.12) is an isomorphism*

$$\phi : H^{m,n}_{C_2}(X, \underline{A}) \xrightarrow{\cong} H^{m,n}_{\mathcal{M}}(X, A).$$

Proof. The functor $(-)^e : C_2\mathrm{Sm}/k \to \mathrm{Sm}/k$ takes equivariant Nisnevich covers to Nisnevich covers and so induces a morphism of sites $(\mathrm{Sm}/k)_{Nis} \to (C_2\mathrm{Sm}/k)_{C_2Nis}$. This induces a map on cohomology $\phi : H^m_{C_2Nis}(X, \underline{A}) \to H^m_{Nis}(X, A(n))$. Here $A(n)$ is the Nisnevich sheafification of $C_*A_{tr}(T^{\wedge n})[-2n]$, i.e., it is the "usual" weight-n motivic complex. Under the isomorphism of Theorem 3.4 this map ϕ is identified with the map ϕ in the statement of the proposition.

There is also a morphism of sites $t : (C_2\mathrm{Sm}/k)_{C_2Nis} \to (\mathrm{Sm}/k)_{Nis}$, induced by $(-)^{triv} : \mathrm{Sm}/k \to C_2\mathrm{Sm}/k$. The functor t_* is exact and $t_*A(n) = \underline{A}(n)$, so we have an induced isomorphism $\psi : H^m_{Nis}(X, A(n)) \cong H^m_{C_2Nis}(X, \underline{A}(n))$; this is an inverse to ϕ. ☐

Proposition 3.16. *Let X be a smooth quasi-projective C_2-scheme over k. Suppose that X has free action. For integers m, n, there are isomorphisms*

$$\begin{tikzcd} H^{m,n}_{\mathcal{M}}(X/C_2, A) & H^{m,n}_{C_2}(X/C_2, \underline{A}) \arrow[l, "\cong", "\phi"'] \arrow[r, "\cong", "\pi^*"'] & H^{m,n}_{C_2}(X, \underline{A}), \end{tikzcd}$$

where $\pi : X \to X/C_2$ is the quotient. Moreover, when $k = \mathbb{C}$, there are commutative squares

$$\begin{tikzcd} H^{m,n}_{\mathcal{M}}(X/C_2, A) \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & H^{m,n}_{C_2}(X/C_2, \underline{A}) \arrow[l, "\cong", "\phi"'] \arrow[r, "\cong", "\pi^*"'] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & H^{m,n}_{C_2}(X, \underline{A}) \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] \\ H^m_{sing}(X(\mathbb{C})/C_2, A) & H^m_{C_2}(X(\mathbb{C})/C_2, \underline{A}) \arrow[l, "\cong", "\phi"'] \arrow[r, "\cong", "\pi^*"'] & H^m_{C_2}(X(\mathbb{C}), \underline{A}). \end{tikzcd}$$

Proof. The first arrow is an isomorphism by Proposition 3.15. By Theorem 3.4, the second map is identified with the map between equivariant Nisnevich hypercohomology groups $\pi^* : H^m_{C_2Nis}(X/C_2, \underline{A}(n)) \to H^m_{C_2Nis}(X, \underline{A}(n))$. Since X/C_2 has trivial action we have the isomorphism

$$H^m_{C_2Nis}(X/C_2, \underline{A}(n)) \cong H^m_{Nis}(X/C_2, A(n)).$$

Under this identification, the isomorphism

$$H^m_{Nis}(X/C_2, A(n)) \cong H^m_{C_2Nis}(X, \underline{A}(n))$$

of [HVØ15, Lemma 3.19] is $\pi^* : H^m_{C_2Nis}(X/C_2, \underline{A}(n)) \to H^m_{C_2Nis}(X, \underline{A}(n))$.

When $k = \mathbb{C}$, we have that $X(\mathbb{C})/C_2 = (X/C_2)(\mathbb{C})$. The commutativity of the first square is a specialization of (3.13). The commutativity of the second is immediate. ☐

We embed $C_2 \subseteq \mathbb{A}(\sigma)$ via $C_2 = \{\pm 1\}$. By the equivariant homotopical purity theorem [HKØ15, Theorem 7.6], there is an equivariant motivic weak equivalence

$$C_{2+} \wedge T^{\sigma} \simeq \mathbb{P}(\sigma \oplus 1)/\mathbb{P}(\sigma \oplus 1) \setminus C_2.$$

Since $\mathbb{P}(\sigma\oplus 1)\setminus\mathbb{A}(\sigma)\subseteq\mathbb{P}(\sigma\oplus 1)\setminus C_2$ and $T^\sigma\simeq\mathbb{P}(\sigma\oplus 1)/\mathbb{P}(\sigma\oplus 1)\setminus\mathbb{A}(\sigma)$ we have a map $t':T^\sigma\to C_{2+}\wedge T^\sigma$, and hence a stable map $t:S^0\to C_{2+}$.

Remark 3.17. Presumably topological Bredon cohomology is a presheaf with equivariant transfers (in the sense of [HVØ15, §4]), but establishing this would require a lengthy digression. For this reason we use the map τ in the following proposition rather than transfer maps coming from the theory of presheaves with equivariant transfers.

Proposition 3.18. *Let E be a motivic C_2-spectrum over $\mathbb{C}$. Write $\pi : C_{2+}\wedge\mathsf{E}\to\mathsf{E}$ for the projection. Then the diagram below commutes and $\tau^*\pi^*=2$,*

$$\begin{tikzcd}
\widetilde{H}^{\star,\star}_{C_2}(\mathsf{E},\underline{A}) \arrow[r, "\pi^*"] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & \widetilde{H}^{\star,\star}_{C_2}(C_{2+}\wedge\mathsf{E},\underline{A}) \arrow[r, "\tau^*"] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & \widetilde{H}^{\star,\star}_{C_2}(\mathsf{E},\underline{A}) \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] \\
\widetilde{H}^{\star}_{C_2}(\mathrm{Re}_{\mathbb{C}}(\mathsf{E}),\underline{A}) \arrow[r, "\pi^*"] & \widetilde{H}^{\star}_{C_2}(C_{2+}\wedge\mathrm{Re}_{\mathbb{C}}(\mathsf{E}),\underline{A}) \arrow[r, "\tau^*"] & \widetilde{H}^{\star}_{C_2}(\mathrm{Re}_{\mathbb{C}}(\mathsf{E}),\underline{A}).
\end{tikzcd}$$

Proof. The commutativity of the diagram is immediate. It suffices to treat the case $\mathsf{E}=S^0$ and to see that $\tau^*(1)=2$. The topological realization of τ is the Spanier-Whitehead dual of the projection $C_{2+}\to S^0$. In particular $\tau^*(1)=2$ in $H^0_{C_2}(\mathrm{pt.},\underline{A})$. It remains to show that $\tau^*(1)=2\in H^{0,0}_{C_2}(k,\underline{A})=A$. This follows from the commutative diagram

$$\begin{tikzcd}
A=\widetilde{H}^{0,0}_{C_2}(C_{2+},\underline{A}) \arrow[r, "\tau^*"] \arrow[d, "\cong"] & \widetilde{H}^{0,0}_{C_2}(S^0,\underline{A})=A \arrow[d, "\cong"] \\
A=\widetilde{H}^{0}_{\mathcal{M}}(S^0\vee S^0,\underline{A}) \arrow[r, "\tau^*"] & \widetilde{H}^{0}_{\mathcal{M}}(S^0,\underline{A})=A
\end{tikzcd}$$

and that the bottom arrow sends 1 to 2. □

3.4. **Thom isomorphisms.** Let R be a C_2-equivariant motivic commutative ring spectrum, X a smooth C_2-scheme over k, and $E\to X$ a C_2-equivariant vector bundle.

Definition 3.19. An R-*Thom class* (or simply *Thom class*, when R is understood) for E is a class $u\in\mathsf{R}^{\star,\star}_{C_2}(\mathrm{Th}(E))$ with the property that for any equivariant map $f:Y\to X$ of smooth C_2-schemes over k, the composition

$$\mathsf{R}^{\star,\star}_{C_2}(Y_+)\xrightarrow{\mathrm{id}\otimes f^*u}\mathsf{R}^{\star,\star}_{C_2}(Y_+)\otimes\mathsf{R}^{\star,\star}_{C_2}(\mathrm{Th}(f^*E))\xrightarrow{\Delta^*}\mathsf{R}^{\star,\star}_{C_2}(\mathrm{Th}(f^*E))$$

is an isomorphism. Here, $\Delta:\mathrm{Th}(f^*E)\to Y_+\wedge\mathrm{Th}(f^*E)$ is the Thom diagonal.

Proposition 3.20. *Let $V=a+b\sigma$. There are classes $u_V\in\widetilde{H}^{2a+2b\sigma,a+b\sigma}_{C_2}(T^V,\underline{A})$ such that*

$$\widetilde{H}^{\star,\star}_{C_2}(X_+,\underline{A})\xrightarrow{-\cup(1_X\times u_V)}\widetilde{H}^{\star+2a+2b\sigma,\star+a+b\sigma}_{C_2}(X_+\wedge T^V,\underline{A})$$

is an isomorphism, for any C_2-variety X. Moreover, if ϕ is an automorphism of the C_2-equivariant vector bundle $X\times\mathbb{A}(V)\to X$, then

$$\phi^*(1_X\times u_V)=1_X\times u_V. \tag{3.21}$$

Proof. By [HVØ15, Theorem 9.7] there are $\lambda_1 \in \widetilde{H}^{1,1}_{C_2}(S^1_t, \underline{A})$ and $\lambda_\sigma \in \widetilde{H}^{\sigma,\sigma}_{C_2}(S^\sigma_t, \underline{A})$ such that for any smooth C_2-variety X, multiplication with these elements induces isomorphisms

$$-\cup(1_X \times \lambda_1) : \widetilde{H}^{\star,\star}_{C_2}(X, \underline{A}) \cong \widetilde{H}^{\star+1,\star+1}_{C_2}(X_+ \wedge S^1_t, \underline{A}), \text{ and}$$
$$-\cup(1_X \times \lambda_\sigma) : \widetilde{H}^{\star,\star}_{C_2}(X, \underline{A}) \cong \widetilde{H}^{\star+\sigma,\star+\sigma}_{C_2}(X_+ \wedge S^\sigma_t, \underline{A}).$$

Define $u_1 = \Sigma^{1,0}\lambda_1$ and $u_\sigma = \Sigma^{\sigma,0}\lambda_\sigma$. For a representation $V = a + b\sigma$ define $u_V = (u_1)^a(u_\sigma)^b$. This element satisfies the first condition.

It remains to check that (3.21) holds for any equivariant bundle automorphism ϕ of $X \times \mathbb{A}(V)$. We will proceed by induction on the dimension of V.

We first consider the case $V = \sigma$. Write $\alpha = a + p\sigma$, $\beta = b + q\sigma$ and write

$$H^{\alpha,\beta}_{C_2}(X, \underline{A})_{(-\sigma)} = \operatorname{coker}\left(H^{\alpha,\beta}_{C_2}(X \times \mathbb{A}(\sigma), \underline{A}) \xrightarrow{i^*} H^{\alpha,\beta}_{C_2}(X \times (\mathbb{A}(\sigma) \setminus \{0\}), \underline{A})\right).$$

Since $H^{\alpha,\beta}_{C_2}(-, \underline{A})$ is a presheaf with equivariant transfers (see Section 3.3), if X is affine then by [HVØ15, Proposition 8.3] the map i^* has a section and we have a natural splitting

$$H^{\alpha,\beta}_{C_2}(X \times (\mathbb{A}(\sigma) \setminus \{0\}), \underline{A}) = H^{\alpha,\beta}_{C_2}(X \times \mathbb{A}(\sigma), \underline{A}) \oplus H^{\alpha,\beta}_{C_2}(X, \underline{A})_{(-\sigma)}. \tag{3.22}$$

Consider the cofiber sequence $(\mathbb{A}(\sigma) \setminus \{0\})_+ \to \mathbb{A}(\sigma)_+ \to T^\sigma$. The induced long exact sequences break into (split) short exact sequences

$$0 \to H^{\alpha,\beta}_{C_2}(\mathbb{A}(\sigma), \underline{A}) \xrightarrow{i^*} H^{\alpha,\beta}_{C_2}(\mathbb{A}(\sigma) \setminus \{0\}, \underline{A}) \xrightarrow{\delta} \widetilde{H}^{\alpha+1,\beta}_{C_2}(T^\sigma, \underline{A}) \to 0.$$

The element u_σ lifts to an element $u'_\sigma \in H^{2\sigma-1,\sigma}_{C_2}(\mathbb{A}(\sigma) \setminus \{0\}, \underline{A})$. We choose u'_σ so that under the splitting (3.22), we have $u'_\sigma \in H^{2\sigma-1,\sigma}_{C_2}(k, \underline{A})_{(-\sigma)}$. Let ϕ be an automorphism of the equivariant vector bundle $X \times \mathbb{A}(\sigma)$ over X. By naturality, to show that $\phi^*(1_X \times u_\sigma) = 1_X \times u_\sigma$ for a smooth affine C_2-variety X, we are reduced to showing

$$\phi_0^*(1_X \times u'_\sigma) = (1_X \times u'_\sigma) + \beta, \tag{3.23}$$

where β is some element in $\ker(\delta)$ and ϕ_0 is the restriction of ϕ to $X \times \mathbb{A}(\sigma) \setminus \{0\}$. For any X (not necessarily affine), the group of equivariant linear automorphisms of $X \times \mathbb{A}(\sigma)$ over X is $(\mathcal{O}^*_X)^{C_2}$, i.e., an equivariant automorphism is given by multiplication with an invariant unit. An invariant unit is specified (uniquely) by an equivariant map $X \to \mathbb{G}_m$. By naturality, to verify the relation (3.21) for $1_X \times u_\sigma$ and any automorphism ϕ of $X \times \mathbb{A}(\sigma)$, it suffices to verify it for $X = \mathbb{G}_m$ and ϕ the automorphism of $\mathbb{G}_m \times \mathbb{A}(\sigma) \setminus \{0\} \to \mathbb{G}_m$ given by multiplication with the canonical unit t of $\mathbb{G}_m$. In this case, ϕ_0 is the map $\langle pr_1, \mu\rangle$, where pr_1 is the projection to the first factor and $\mu : \mathbb{G}_m \times (\mathbb{A}(\sigma) \setminus \{0\}) \to \mathbb{A}(\sigma) \setminus \{0\}$ is the multiplication. We have that $\phi_0^*(1_{\mathbb{G}_m} \times u'_\sigma) = 1 \cup \mu^*(u'_\sigma) = \mu^*(u'_\sigma)$.

From the naturality of the decomposition (3.22) we see that μ^* restricts to a map

$$\mu^* : H^{2\sigma-1,\sigma}(k, \underline{A})_{(-\sigma)} \to H^{2\sigma-1,\sigma}(\mathbb{G}_m, \underline{A})_{(-\sigma)}.$$

Moreover, this map has a splitting induced by $e : k \to \mathbb{G}_m$, the inclusion at $1 \in \mathbb{G}_m$.

It follows that $\mu^*(1_{\mathbb{G}_m} \times u'_\sigma) = 1_{\mathbb{G}_m} \times u'_\sigma + \gamma$, where $\gamma \in \ker(e^*)$. An easy diagram chase shows that $\delta(\ker(e^*)) = 0$, and so (3.23) holds. It follows that (3.21) holds for $1_X \times u_\sigma$ for any X and any automorphism of $X \times \mathbb{A}(\sigma)$. A similar (but easier) argument establishes (3.21) for $1_X \times u_1$ as well. This takes care of the case when V has dimension one.

Now we proceed by induction on the dimension of V. Let $V = a + b\sigma$ be an n-dimensional representation. We may assume that X is a point of the equivariant Nisnevich topology. Recall that for a finite group G, the points of the equivariant Nisnevich topology are of the form $G \times_H \operatorname{Spec}(R)$ where $H \subseteq G$ is a subgroup and R is an essentially smooth Henselian local k-algebra with H-action, see [HVØ15, Theorem 3.14]. For $G = C_2$, there are two possibilities, $X = C_2 \times \operatorname{Spec}(R)$ or $X = \operatorname{Spec}(R)$, where R is a smooth local ring with C_2-action. In the first case the claim follows from Proposition 3.14 and that the u_V are nonequivariant Thom classes. Now we consider the case that $X = \operatorname{Spec}(R)$, where R is a smooth local ring with C_2-action. The equivariant automorphisms of $X \times \mathbb{A}(V)$ are $\operatorname{Aut}_R(V_R)^{C_2}$. If A is a matrix with entries in R, write A^σ for the matrix obtained by applying the involution σ to its entries. Under the identification $\operatorname{End}_R(V_R) = \operatorname{Mat}_{n\times n}(R)$ the C_2-action is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} A^\sigma & -(B^\sigma) \\ -(C^\sigma) & D^\sigma \end{bmatrix}.$$

Here, A is an $a \times a$-matrix and B is an $b \times b$-matrix, and $a + b = n$. It follows that this matrix is in $\operatorname{Mat}_{n\times n}(R)^{C_2}$ if and only if A and D have coefficients in R^{C_2} and σ acts by -1 on the coefficients of B, C.

Let $x \in R$ and for $i \neq j$ write $E_{ij}(x)$ for the elementary matrix corresponding to adding x times row j to row i, i.e., it has x in position (i, j) and is the same as the identity matrix in all other entries. Then $E_{ij}(x)$ is in $\operatorname{GL}_n(R)^{C_2}$ if

(i) either $i, j \leq a$ or $a < i, j$ and $x \in R$ is invariant, or
(ii) $i \leq a$, $j > a$ or $i > a$, $j \leq a$ and $\sigma x = -x$.

The matrix $E_{ij}(x)$ is equivariantly homotopic to the identity via the equivariant $\mathbb{A}^1$-homotopy $t \mapsto E_{ij}(tx)$.

Let T_{ij} be the elementary matrix corresponding to switching the ith and jth rows. If $i, j \leq a$ then T_{ij} is in $\operatorname{Mat}_{n\times n}(R)^{C_2}$. Moreover, in this case there is an algebraic map $\mathbb{A}^1 \to \operatorname{GL}_a(R^{C_2}) \subseteq \operatorname{GL}_n(R)^{C_2}$ joining T_{ij} and the diagonal matrix $\langle -1, 1 \ldots, 1\rangle$. Thus there is an equivariant $\mathbb{A}^1$-homotopy joining T_{ij} and $\langle -1, 1, \ldots, 1\rangle$ in $\operatorname{GL}_{a+b}(R)$. Similarly, T_{ij} is equivariantly $\mathbb{A}^1$-homotopic to the identity for $i, j > a$.

Let $M = (m_{ij})$ be an invertible matrix in $\operatorname{Mat}_{n\times n}(R)^{C_2}$. We show that there is an $\mathbb{A}^1$-homotopy

$$M \simeq_{\mathbb{A}^1} \begin{bmatrix} u & 0 \\ 0 & M' \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} M' & 0 \\ 0 & u \end{bmatrix}.$$

Here, $u \in R$ is an invariant unit and M' is an invertible $(n-1) \times (n-1)$-matrix in $\operatorname{Mat}_{(n-1)\times(n-1)}(R)^{C_2}$. First we assume the entry m_{11} of M is a unit. In this case, the claim follows by multiplying with the elementary matrices $E_{i1}(-m_{i1}/m_{11})$, $i > 1$ on the left and with $E_{1j}(-m_{1j}/m_{11})$, $j > 1$ on the right. If m_{11} is not a unit but some m_{ij} is a unit for $i, j \leq a$, then by multiplying by T_{1i} on the left and T_{j1} on the right, we are reduced to the previous case. If all entries m_{ij} of A (i.e., $i, j \leq a$) are non-units, we can repeat the previous arguments for the elements of D, if at least one of its entries is a unit. The only remaining case is when all entries of A and of D are in the maximal ideal of R. For $t \in \mathbb{A}^1$ we write M_t for the matrix which agrees with M in all positions except $(M_t)_{11} = t + m_{11}$. This gives a map $\mathbb{A}^1 \to \operatorname{Mat}_{n\times n}(R)^{C_2}$. Write M'_t for the $a \times a$-matrix consisting of $(M_t)_{ij}$ for

$i, j \leq a$. The reduction of M_t modulo the maximal ideal of R is

$$\overline{M_t} = \begin{bmatrix} \overline{M_t'} & \overline{B} \\ \overline{C} & 0 \end{bmatrix}.$$

Since $M = M_0$ is invertible so is $\overline{M}_t$. It follows that M_t is invertible for all t. Now M_1 is equivariantly $\mathbb{A}^1$-homotopic to M and the previous case applies to M_1, so we are done. □

Recall (3.9) that we have elements $\epsilon, \epsilon', u \in H^{0,0}_{C_2}(k, \underline{A})$ which determine the commutativity properties of the ring $H^{\star,\star}_{C_2}(X, \underline{A})$. These elements are $\epsilon = \Sigma_T^{-2}\tau_T^*(\Sigma_T^2 1)$, $\epsilon' = \Sigma_{T^\sigma}^{-2}\tau_{T^\sigma}^*(\Sigma_{T^\sigma}^2 1)$, and $u = \Sigma_{S^\sigma}^{-2}\tau_{S^\sigma}^*(\Sigma_{S^\sigma}^2 1)$, where $\tau_{\mathsf{E}} : \mathsf{E} \wedge \mathsf{E} \to \mathsf{E} \wedge \mathsf{E}$ is the twist endomorphism.

Proposition 3.24. *In $H^{0,0}_{C_2}(k, \underline{A})$, $\epsilon = 1$, $\epsilon' = 1$, and $u = -1$. In particular, if $x \in H^{a+p\sigma, b+q\sigma}_{C_2}(X, \underline{A})$ and $y \in H^{c+s\sigma, d+t\sigma}_{C_2}(X, \underline{A})$, then*

$$x \cup y = (-1)^{ac+ps}(y \cup x).$$

Proof. By Proposition 3.20, $H^{4,2}_{C_2}(k, \underline{A})$ is a free $H^{0,0}_{C_2}(k, \underline{A})$-module with basis u_2. The map τ_T is induced by the twist automorphism of the vector bundle $\mathbb{A}^2 \to \operatorname{Spec}(k)$ and so by Proposition 3.20, $\tau_T^*(u_2) = u_2$. In particular, $\tau_T^* = \mathrm{id}$ and it follows that $\epsilon = \Sigma_T^{-2}\tau_T^*(\Sigma_T^2 1) = 1$, as claimed. The argument for ϵ' is entirely similar.

Now we compute u. Recall from Section 3.1 the complex $\mathbb{Z}_{top}(\sigma)$ of presheaves with transfers. Under the identification of Theorem 3.4 the element u corresponds to the twist map $\mathbb{Z}_{top}(\sigma) \otimes^{tr,\mathbb{L}} \mathbb{Z}_{top}(\sigma) \to \mathbb{Z}_{top}(\sigma) \otimes^{tr,\mathbb{L}} \mathbb{Z}_{top}(\sigma)$ in $D^-(C_2 \operatorname{Cor}_k)$. The twist map $C \otimes D \to D \otimes C$ is given componentwise by $(-1)^{pq}$ times the twist $C^p \otimes D^q \to D^q \otimes C^p$. The complex $\mathbb{Z}_{top}(\sigma) \otimes^{tr,\mathbb{L}} \mathbb{Z}_{top}(\sigma)$ is

$$\mathbb{Z}_{tr,C_2}(C_2 \times C_2) \xrightarrow{\langle q_2, -q_1 \rangle} \mathbb{Z}_{tr,C_2}(C_2) \oplus \mathbb{Z}_{tr,C_2}(C_2) \xrightarrow{p_* \oplus p_*} \mathbb{Z},$$

where $\mathbb{Z}$ is in degree 0, $p : C_2 \to \operatorname{Spec}(k)$ is the projection, and $q_i : C_2 \times C_2 \to C_2$ is the projection to the ith factor. A chain homotopy between the twist map and -1 is given by $\{s_i\}$, $s_i = 0$, $i \neq 0, -1$, and $s_0 = \langle 0, p^t \rangle$, $s_{-1} = \Delta' \oplus \Delta'$. Here $\Delta' : C_2 \to C_2 \times C_2$ is given by $e \mapsto e \times \sigma$, $\sigma \mapsto \sigma \times e$. □

Definition 3.25. Let V be a C_2-representation and $E \to X$ a C_2-equivariant vector bundle. Say that E is *type* V if every point $x \in X$ is contained in an invariant open neighborhood $U \subseteq X$ such that $E|_U$ is isomorphic, as a C_2-equivariant vector bundle, to the product bundle $U \times \mathbb{A}(V) \to U$.

Theorem 3.26. *Let X be a smooth C_2-scheme over k, $V = a + b\sigma$, and $E \to X$ a C_2-equivariant vector bundle of type V. Then there are Thom classes*

$$\operatorname{th}(E) \in \widetilde{H}^{2a+2b\sigma, a+b\sigma}_{C_2}(\operatorname{Th}(\mathrm{E}), \underline{A}).$$

Proof. By assumption, there is a cover $X = U_1 \cup \cdots \cup U_n$ by open invariant subschemes such that $E|_{U_i} \cong U_i \times \mathbb{A}(V)$. Proceeding by induction on n, we can use the Mayer-Vietoris long exact sequence to patch the elements $1_{U_i} \times u_V$ constructed in the previous proposition. The condition of (3.21) guarantees that they patch together. □

In the following, A denotes an abelian group.

Corollary 3.27. *Let $i : Z \hookrightarrow X$ be a closed immersion of smooth C_2-schemes over k, with open complement $j : U \hookrightarrow X$ and normal bundle $\mathcal{N}_i$. Suppose that $Z = \coprod Z_r$, with each Z_r invariant, and $\mathcal{N}_i|_{Z_r}$ is of type $a_r + b_r\sigma$. Then there is a Gysin long exact sequence*

$$\cdots \to \oplus_r H^{\star - 2\alpha_r, \star - \alpha_r}_{C_2}(Z_r, \underline{A}) \to H^{\star,\star}_{C_2}(X, \underline{A}) \xrightarrow{j^*} H^{\star,\star}_{C_2}(U, \underline{A}) \to \cdots .$$

Proof. By equivariant homotopical purity [HKØ15, Theorem 7.6], we have a cofiber sequence of motivic C_2-spaces

$$U \to X \to \mathrm{Th}(\mathcal{N}_i). \tag{3.28}$$

This induces a long exact sequence

$$\cdots \to \widetilde{H}^{\star,\star}_{C_2}(\mathrm{Th}(\mathcal{N}_i), \underline{A}) \to H^{\star,\star}_{C_2}(X, \underline{A}) \xrightarrow{j^*} H^{\star,\star}_{C_2}(U, \underline{A}) \to \widetilde{H}^{\star+1,\star}_{C_2}(\mathrm{Th}(\mathcal{N}_i), \underline{A}) \to \cdots .$$

Note that $\mathrm{Th}(\mathcal{N}_i) = \vee_r \mathrm{Th}(\mathcal{N}_i|_{Z_r})$. Applying the previous theorem to each $\mathrm{Th}(\mathcal{N}_i|_{Z_\alpha})$ identifies the long exact sequence induced by (3.28) with the desired Gysin sequence. ☐

Remark 3.29. An important case is the following. Let X be a smooth C_2-scheme and Z a connected component of the fixed point subscheme $X^{C_2} \subseteq X$. Then the fibers of the normal bundle of $Z \subseteq X$ are of type $\mathrm{codim}_X(Z)\sigma$. Indeed, by Luna's slice theorem [Lun73, p. 97], for any $z \in Z$ we have $(T_z X)^{C_2} = T_z(X^{C_2}) = T_z(Z)$.

When $k = \mathbb{C}$, the same construction as in the proof of Theorem 3.26 applies to topological Bredon cohomology. Moreover, this construction is compatible with the Betti realization functor $\mathrm{Re}_{\mathbb{C}} : \widetilde{H}^{\star,\star}_{C_2}(\mathrm{Th}(E), \underline{A}) \to \widetilde{H}^{\star}_{C_2}(\mathrm{Th}(E)(\mathbb{C}), \underline{A})$.

Proposition 3.30. *Let $k = \mathbb{C}$ and $E \to X$ be a C_2-equivariant vector bundle and suppose* $\mathrm{th}(E)$ *is a Thom class. Then* $\mathrm{Re}_{\mathbb{C}}(\mathrm{th}(E)) \in \widetilde{H}^{\star}_{C_2}(\mathrm{Th}(E)(\mathbb{C}), \underline{A})$ *is a Thom class.*

Proof. To show that $\mathrm{Re}_{\mathbb{C}}(\mathrm{th}(E))$ is a Thom class it suffices to show that $i^*(\mathrm{th}(E))$ is a generator of the free $H^*(C_2/H, \underline{A})$-module $\widetilde{H}^*(\mathrm{Th}(i^*E), \underline{A})$, where $H \subseteq C_2$ is a subgroup and $i : C_2/H \to X(\mathbb{C})$ is an equivariant map [May96, XVI.9]. Write $V = a + b\sigma$, where $i^*E = C_2/H \times \mathbb{A}(V)$. Then, $i^*(\mathrm{th}(E)) = a(\Sigma_{T^V} 1)$ in $\widetilde{H}^{2a+2b\sigma, a+b\sigma}_{C_2}(Th(i^*E), \underline{A})$, for some $a \in H^{0,0}_{C_2}(\mathrm{Spec}(\mathbb{C}), \underline{A}) = A$. It follows that $i^*\mathrm{Re}_{\mathbb{C}}(\mathrm{th}(E)) = a(\Sigma_{S^{V(\mathbb{C})}} 1)$, which is a generator of $\widetilde{H}^{2a+2b\sigma}_{C_2}(\mathrm{Th}(i^*E(\mathbb{C})), \underline{A})$, and so $\mathrm{Re}_{\mathbb{C}}(\mathrm{th}(E))$ is a Thom class. ☐

4. BREDON COHOMOLOGY AND EQUIVARIANT HIGHER CHOW GROUPS

In [HVØ15, Theorem 5.19] we constructed a natural comparison map between the Bredon motivic cohomology groups and Edidin-Graham's equivariant higher Chow groups. In this section we elaborate on the comparison between these two constructions of equivariant motivic cohomology. Throughout, A denotes an abelian group.

Proposition 4.1. *Let X be a smooth quasi-projective C_2-scheme over k. There is a natural isomorphism*

$$CH^b_{C_2}(X, 2b - a, A) \cong H^{a,b}_{C_2}(X \times \mathbf{E}C_2, \underline{A}).$$

Proof. By definition, $CH^b_{C_2}(X, 2b-a, A) = CH^b(X \times_{C_2} (\mathbb{A}(n\sigma) \setminus \{0\}), 2b-a, A)$ for n sufficiently large, see [EG98, p. 599, 605]. In particular, the value of this latter group is constant for $n \gg 0$. Write $U_n = \mathbb{A}(n\sigma) \setminus \{0\}$. Using the isomorphism between higher Chow groups and motivic cohomology [Voe02, Corollary 2] together with Proposition 3.16 we obtain the natural isomorphisms

$$\begin{aligned} CH^b_{C_2}(X, 2b-a, A) &\cong \lim_n H^{a,b}_{\mathcal{M}}(X \times_{C_2} U_n, A) \\ &\cong \lim_n H^{a,b}_{C_2}(X \times U_n, \underline{A}) \\ &\cong H^{a,b}_{C_2}(X \times \mathbf{E}C_2, \underline{A}). \end{aligned}$$

For the last isomorphism we have used the Milnor exact sequence

$$0 \to \lim^1_n H^{a,b}_{C_2}(X \times U_n) \to H^{a,b}_{C_2}(X \times \mathbf{E}C_2) \to \lim_n H^{a,b}_{C_2}(X \times U_n) \to 0,$$

and the fact that the $\lim^1$-term vanishes. □

The groups $H^{\star,\star}_{C_2}(X \times \mathbf{E}C_2, \underline{A})$ define the Borel motivic cohomology of X. In light of the identification above, we view $H^{\star,\star}_{C_2}(X \times \mathbf{E}C_2, \underline{A})$ as a generalized version of equivariant higher Chow groups (in which the grading is by representations instead of just integers). By the motivic isotropy separation cofiber sequence (2.7), the projection map $\mathbf{E}C_{2+} \to S^0$ induces the comparison map between the Bredon motivic cohomology and the Borel motivic cohomology theories. Their difference is measured by $\widetilde{\mathbf{E}}C_2$.

Lemma 4.2. *Let X be a smooth, quasi-projective C_2-scheme. Suppose that either (i) X has free action, (ii) $b < 0$, or (iii) $a \leq 1$ and A is finite, then*

$$\widetilde{H}^{a+p\sigma, b+q\sigma}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) = 0.$$

Proof. By Proposition 5.9 below, we have isomorphisms

$$\widetilde{H}^{a+p\sigma, b+q\sigma}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \cong \widetilde{H}^{a,b}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}).$$

First we consider the case when X has free action. Then X/G is smooth and applying Proposition 3.16, the map $H^{a,b}_{C_2}(X, \underline{A}) \to H^{a,b}_{C_2}(X \times \mathbf{E}C_2, \underline{A})$ becomes identified with the natural isomorphism

$$H^{a,b}_{\mathcal{M}}(X/G, A) \xrightarrow{\cong} H^{a,b}_{\mathcal{M}}(X \times^G \mathbf{E}C_2, A)$$

for all a, b. This establishes the vanishing in case (i).

Now consider the case when X has trivial action. In this case we have

$$H^{a,b}_{C_2}(X \times \mathbf{E}C_2, \underline{A}) = H^{a,b}_{\mathcal{M}}(X \times \mathbf{B}C_2, A).$$

The projection map $X \times \mathbf{B}C_2 \to X$ affords a section and the long exact sequence associated to the motivic isotropy sequence breaks up into short exact sequences

$$0 \to H^{a,b}_{C_2}(X, \underline{A}) \to H^{a,b}_{\mathcal{M}}(X \times \mathbf{B}C_2, A) \to \widetilde{H}^{a+1,b}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \to 0.$$

The two left groups vanish whenever $b < 0$ and so does the third, which establishes case (ii). The space $\mathbf{B}C_2$ is the complement of the zero section of the line bundle $\mathcal{O}(-2)$ on $\mathbb{P}^\infty$, see e.g., [Voe03b, Lemma 6.4]. It thus fits into a cofiber sequence

$$\mathbf{B}C_2 \to \mathbb{P}^\infty \to \mathrm{Th}(\mathcal{O}(-2))$$

of motivic spaces. Now we suppose that A is finite. We have, as a consequence of the Bloch-Kato conjectures, that $H^{a,b}_{\mathcal{M}}(X, A) = 0$ for $a < 0$. Combined with the sequence in motivic cohomology resulting from the above cofiber sequence, the

Thom isomorphism, and the projective bundle theorem, we find an isomorphism $H^{a,b}_{\mathcal{M}}(X,A) \to H^{a,b}_{\mathcal{M}}(X \times \mathbf{B}C_2, A)$ for $a \leq 0$ and A finite.

Now consider an arbitrary smooth C_2-scheme X. The fixed points $X^{C_2} \subseteq X$ are smooth and its open complement $U = X \setminus X^{C_2}$ has free action. The previous paragraphs applied to X^{C_2} and U together with the Gysin sequence of Corollary 3.27 yields the result. □

Theorem 4.3. *Let X be a smooth quasi-projective C_2-scheme over k. Then the natural map*

$$H^{a+p\sigma,b+q\sigma}_{C_2}(X, \underline{A}) \to H^{a+p\sigma,b+q\sigma}_{C_2}(X \times \mathbf{E}C_2, \underline{A})$$

is an isomorphism if either (i) X has free action, (ii) $b < 0$, or (iii) and A is finite $a \leq 0$. In case (iii) the map is injective if $a = 1$.

Proof. This follows immediately from the previous lemma and the motivic isotropy separation cofiber sequence (2.7). □

Remark 4.4. Combining the identification of Proposition 4.1 and the periodicity of Corollary 5.8 we find that there is a natural map

$$\begin{aligned} H^{a+p\sigma,b+q\sigma}_{C_2}(X, \underline{\mathbb{Z}/2}) &\to H^{a+p,b+q}_{\mathcal{M}}(X \times^G \mathbf{E}C_2, \mathbb{Z}/2) \\ &\cong CH^{b+q}(X, 2(b+q) - a - p, \mathbb{Z}/2) \end{aligned}$$

which is an isomorphism if (i) X has free action, (ii) $b < 0$, or (iii) $a \leq 0$. Moreover in case (iii) the map is an injection for $a = 1$.

5. Periodicity and Borel motivic cohomology

In this section we show that the ring $H^{\star,\star}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$ is periodic with periods $(\sigma - 1, 0)$ and $(\sigma - 1, \sigma - 1)$. It follows that the groups $H^{\star,\star}_{C_2}(X \times \mathbf{E}C_2, \underline{\mathbb{Z}/2})$ are also periodic. These form a generalized "geometric" Borel motivic cohomology theory. The integer graded portion of these groups is isomorphic to equivariant higher Chow groups, see Section 4. This periodicity of the generalized Borel motivic cohomology will play an important role in our comparison theorem between motivic and topological Bredon cohomology.

We remind the reader that our convention is that $(*,*)$ stands for an integer bigrading and $(\star,\star)$ stands for the bigrading determined by representations, see (1.4). We begin with the cohomology of C_2.

Lemma 5.1. *Let A be an abelian group. There is an $H^{*,*}_{\mathcal{M}}(k,A)$-algebra isomorphism*

$$H^{\star,\star}_{C_2}(C_2, \underline{A}) \cong H^{*,*}_{\mathcal{M}}(k,A)[s^{\pm 1}, t^{\pm 1}],$$

where $s \in H^{\sigma-1,0}_{C_2}(C_2, \underline{A})$ and $t \in H^{\sigma-1,\sigma-1}_{C_2}(C_2, \underline{A})$.

Proof. Write $(\alpha, \beta) = (a + p\sigma, b + q\sigma)$ and $|a + p\sigma| = a + p$. We have isomorphisms

$$\begin{aligned} H^{\star,\star}_{C_2}(C_2, \underline{A}) \xrightarrow{\cong} \widetilde{H}^{\star+\alpha,\star+\beta}_{C_2}(S^{\alpha,\beta} \wedge C_{2+}, \underline{A}) &\xrightarrow{\cong} \widetilde{H}^{\star+\alpha,\star+\beta}_{C_2}(S^{|\alpha|,|\beta|} \wedge C_{2+}, \underline{A}) \\ &\xleftarrow{\cong} H^{\star+\alpha-|\alpha|,\star+\beta-|\beta|}_{C_2}(C_2, \underline{A}); \end{aligned}$$

the first and last isomorphisms are instances of the suspension isomorphism, and the middle isomorphism follows from Lemma A.9. This is an isomorphism of free one-dimensional $H^{\star,\star}_{C_2}(C_2, \underline{A})$-modules and thus is given by multiplication with an invertible element $x_{\alpha,\beta}\in H^{\alpha-|\alpha|,\beta-|\beta|}_{C_2}(C_2,\underline{A})$. Taking $s=x_{\sigma-1,0}$ and $t=x_{\sigma-1,\sigma-1}$, we get an $H^{*,*}_{\mathcal{M}}(k,A)$-algebra map

$$H^{*,*}_{\mathcal{M}}(k,A)[s^{\pm1},t^{\pm1}]\to H^{\star,\star}_{C_2}(C_2,\underline{A}),$$

which is an isomorphism. □

Lemma 5.2. *The map* $C_{2+}\to S^0$ *induces isomorphisms*

(i) $H^{\sigma-1,0}_{C_2}(k,\underline{\mathbb{Z}/2})\xrightarrow{\cong}H^{\sigma-1,0}_{C_2}(C_2,\underline{\mathbb{Z}/2})$,
(ii) $H^{\sigma-1,\sigma-1}_{C_2}(k,\underline{\mathbb{Z}/2})\xrightarrow{\cong}H^{\sigma-1,\sigma-1}_{C_2}(C_2,\underline{\mathbb{Z}/2})$.

Proof. Consider the exact sequence induced by the cofiber sequence (2.4),

$$0\to H^{\sigma-1,0}_{C_2}(k,\underline{\mathbb{Z}/2})\to H^{\sigma-1,0}_{C_2}(C_2,\underline{\mathbb{Z}/2})\xrightarrow{\psi}H^{\sigma,0}_{C_2}(S^\sigma,\underline{\mathbb{Z}/2})\to H^{\sigma,0}_{C_2}(k,\underline{\mathbb{Z}/2})\to 0.$$

Under the identifications $H^{\sigma-1,0}_{C_2}(C_2,\underline{\mathbb{Z}/2})\cong\mathbb{Z}/2\cong H^{\sigma,0}_{C_2}(S^\sigma,\underline{\mathbb{Z}/2})$, coming from Proposition 3.14, Proposition 3.15, and (3.6), we claim the homomorphism ψ is multiplication by 2. This follows by considering the commutative diagram, where the vertical maps are the forgetful maps

$$\begin{tikzcd}
& \widetilde{H}^{\sigma,0}_{C_2}(S^1\wedge C_{2+}) \arrow[r, "\psi"] \arrow[dl, "\cong"'] \arrow[d] & \widetilde{H}^{\sigma,0}_{C_{2+}}(S^\sigma) \arrow[d, hook] \\
\widetilde{H}^{1,0}_{\mathcal{M}}(S^1) \arrow[r] & \widetilde{H}^{1,0}_{\mathcal{M}}(S^1\vee S^1) \arrow[r] & \widetilde{H}^{1,0}_{\mathcal{M}}(S^1).
\end{tikzcd}$$

The diagonal map is the standard adjunction. We have that $\widetilde{H}^{0,0}_{C_2}(S^\sigma)=0$ and so the left vertical is seen to be injective from the exact sequence (arising from the cofiber sequence $C_{2+}\wedge S^\sigma\to S^\sigma\to S^{2\sigma}$),

$$\widetilde{H}^{0,0}_{C_2}(S^\sigma)\to\widetilde{H}^{\sigma,0}_{C_2}(S^\sigma)\to\widetilde{H}^{\sigma,0}_{C_2}(C_{2+}\wedge S^\sigma).$$

The lower horizontal composition is 2, which establishes the claim and thus establishes (i).

To establish (ii), consider now the exact sequence

$$\widetilde{H}^{0,\sigma}_{C_2}(S^{1,1},\underline{\mathbb{Z}/2})\to\widetilde{H}^{\sigma,\sigma}_{C_2}(S^{1,1},\underline{\mathbb{Z}/2})\to\widetilde{H}^{\sigma,\sigma}_{C_2}(S^{1,1}\wedge C_{2+},\underline{\mathbb{Z}/2})\xrightarrow{\psi}\widetilde{H}^{1,\sigma}_{C_2}(S^{1,1},\underline{\mathbb{Z}/2}).$$

It follows from [HVØ15, Corollary 5.15] that $\widetilde{H}^{0,\sigma}_{C_2}(S^{1,1},\underline{\mathbb{Z}/2})\cong\widetilde{H}^{0,1}_{C_2}(S^{1,1},\underline{\mathbb{Z}/2})=0$, and $\widetilde{H}^{1,\sigma}_{C_2}(S^{1,1},\underline{\mathbb{Z}/2})\cong\mathbb{Z}/2\cong\widetilde{H}^{1,1}_{C_2}(S^{1,1},\underline{\mathbb{Z}/2})$. Under this identification, we claim that ψ is multiplication by 2. This follows similarly as in the previous paragraph by considering the commutative diagram

$$\begin{tikzcd}
& \widetilde{H}^{1+\sigma,\sigma}_{C_2}(C_{2+}\wedge S^1\wedge S^{1,1}) \arrow[r, "\psi"] \arrow[dl, "\cong"'] \arrow[d] & \widetilde{H}^{1+\sigma,\sigma}_{C_{2+}}(S^\sigma\wedge S^{1,1}) \arrow[d, hook] \\
\widetilde{H}^{2,1}_{\mathcal{M}}(T) \arrow[r] & \widetilde{H}^{2,1}_{\mathcal{M}}(T\vee T) \arrow[r] & \widetilde{H}^{2,1}_{\mathcal{M}}(T).
\end{tikzcd}$$

Since the lower horizontal composition is multiplication by 2, (ii) follows once we see that the right vertical map is injective. We have an exact sequence

$$\widetilde{H}^{1+\sigma,\sigma}_{C_2}(S^{2\sigma}\wedge S^{1,1})\to\widetilde{H}^{1+\sigma,\sigma}_{C_2}(S^\sigma\wedge S^{1,1})\to\widetilde{H}^{1+\sigma,\sigma}_{C_2}(C_{2+}\wedge S^1\wedge S^{1,1}),$$

and so it suffices to see that $\widetilde{H}^{1,\sigma}_{C_2}(S^\sigma \wedge S^{1,1}) = 0$. This follows from the exact sequence

$$\widetilde{H}^{1,\sigma}_{C_2}(C_{2+} \wedge S^1 \wedge S^{1,1}) \to \widetilde{H}^{1,\sigma}_{C_2}(S^\sigma \wedge S^{1,1}) \to \widetilde{H}^{1,\sigma}_{C_2}(S^{1,1}) \to \widetilde{H}^{1,\sigma}_{C_2}(C_{2+} \wedge S^{1,1})$$

since the left hand group is zero and by [HVØ15, Corollary 5.17] the rightmost map is induced by the inclusion $(\mathcal{O}^*_{\mathbb{G}^\sigma_m})^{C_2} \subseteq \mathcal{O}^*_{\mathbb{G}_m}$ and therefore an injection. ☐

Lemma 5.3. *Let A be an abelian group. There are isomorphisms*

$$H^{0,0}_{C_2}(\mathbf{E}C_2, \underline{A}) \cong \lim_n H^{0,0}_{C_2}(\mathbb{A}(n\sigma) \setminus \{0\}, \underline{A}) \cong A.$$

Proof. We have $H^{0,0}_{C_2}(\mathbb{A}(n\sigma) \setminus \{0\}, \underline{A}) \cong H^{0,0}_{\mathcal{M}}((\mathbb{A}(n\sigma) \setminus \{0\})/C_2, A) = A$ by Proposition 3.16 and by [Voe03b, Proposition 6.1] the maps

$$H^{0,0}_{\mathcal{M}}((\mathbb{A}(n\sigma) \setminus \{0\})/C_2, A) \to H^{0,0}_{\mathcal{M}}((\mathbb{A}((n+1)\sigma) \setminus \{0\})/C_2, A)$$

are isomorphisms. ☐

Write

$$s \in H^{\sigma-1,0}_{C_2}(k, \underline{\mathbb{Z}/2}) \quad \text{and} \quad t \in H^{\sigma-1,\sigma-1}_{C_2}(k, \underline{\mathbb{Z}/2})$$

for the elements obtained from Lemma 5.1 and Lemma 5.2. We write as well s, t for the corresponding images in $H^{\star,\star}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$.

Theorem 5.4. *The elements $s \in H^{\sigma-1,0}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$ and $t \in H^{\sigma-1,\sigma-1}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$ are invertible.*

Proof. Consider the equivariant embedding $i_n : C_2 \subseteq \mathbb{A}(n\sigma) \setminus \{0\}$ given by including at $\{\pm 1\}$. We show that i_n induces isomorphisms

(i) $i_n^* : H^{1-\sigma,0}_{C_2}(\mathbb{A}(n\sigma) - \{0\}, \underline{\mathbb{Z}/2}) \xrightarrow{\cong} H^{1-\sigma,0}_{C_2}(C_2, \underline{\mathbb{Z}/2})$,

(ii) $i_n^* : H^{1-\sigma,1-\sigma}_{C_2}(\mathbb{A}(n\sigma) \setminus \{0\}, \underline{\mathbb{Z}/2}) \xrightarrow{\cong} H^{1-\sigma,1-\sigma}_{C_2}(C_2, \underline{\mathbb{Z}/2})$.

This will imply the theorem as follows. For each n, the elements s^{-1}, t^{-1} in $H^{\star,\star}_{C_2}(C_2, \underline{\mathbb{Z}/2})$ lift uniquely to elements u_n, v_n in $H^{\star,\star}_{C_2}(\mathbb{A}(n\sigma) \setminus \{0\}, \underline{\mathbb{Z}/2})$. The uniqueness of these lifts implies that $\{u_n\}$ and $\{v_n\}$ determine elements (u_n), (v_n) in $\lim_n H^{\star,\star}_{C_2}(\mathbb{A}(n\sigma) \setminus \{0\}, \underline{\mathbb{Z}/2})$. These in turn lift to elements $\overline{u}$, $\overline{v}$ in $H^{\star,\star}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$. We now have that $s \cup \overline{u}$ and $t \cup \overline{v}$ in $H^{0,0}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2}) = \mathbb{Z}/2$ are equal to 1 since they map to $1 \in H^{0,0}_{C_2}(C_2, \underline{\mathbb{Z}/2}) = \mathbb{Z}/2$.

For typographical simplicity we will suppress the coefficients $\mathbb{Z}/2$ of the cohomology groups from the notation. We will also write $U_n := \mathbb{A}(n\sigma) \setminus 0$. Observe that we have a map $U_n/C_2 \to U_n/\mathbb{G}_m = \mathbb{P}^{n-1}$. We have an isomorphism $U_n/C_2 \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-2) \setminus \mathbb{P}^{n-1}$ and therefore a cofiber sequence of motivic spaces

$$U_n/C_2 \to \mathbb{P}^{n-1} \to \mathrm{Th}(\mathcal{O}(-1)). \tag{5.5}$$

To establish (i) we use the commutative square

$$\begin{tikzcd}
\widetilde{H}^{1,0}_{C_2}(S^1 \wedge C_{2+} \wedge (U_n)_+) \arrow[r, "i_n^*"] \arrow[d] & \widetilde{H}^{1,0}_{C_2}(S^1 \wedge C_{2+} \wedge C_{2+}) \arrow[d] \\
\widetilde{H}^{1,0}_{C_2}(S^\sigma \wedge (U_n)_+) \arrow[r, "i_n^*"] & \widetilde{H}^{1,0}_{C_2}(S^\sigma \wedge C_{2+})
\end{tikzcd}$$

where the vertical arrows are induced by the cofiber sequence (2.4). Using Proposition 3.16 we have $H^{1,0}_{C_2}(U_n) = H^{1,0}_{\mathcal{M}}(U_n/C_2) = 0$ which implies that the left hand vertical arrow is surjective. In particular we see that the $\mathbb{Z}/2$-module $\widetilde{H}^{1,0}_{C_2}(S^\sigma \wedge (U_n)_+)$ has rank at most one. Using Proposition 3.14, the composition of the top horizontal and and right vertical arrow is identified with the composition

$$\widetilde{H}^{1,0}_{\mathcal{M}}(S^1 \wedge (\mathbb{A}^n \setminus \{0\})_+) \to \widetilde{H}^{1,0}_{\mathcal{M}}(S^1 \wedge S^0_+) \to \widetilde{H}^{1,0}_{\mathcal{M}}(S^1).$$

This composition is surjective and so $\widetilde{H}^{1,0}_{C_2}(S^\sigma \wedge (U_n)_+) \to \widetilde{H}^{1,0}_{C_2}(S^\sigma \wedge C_{2\,+})$ is also surjective. But this implies that it is an isomorphism.

We now establish (ii). Consider the commutative diagram with exact columns which we obtain from the cofiber sequence (2.4),

$$\begin{tikzcd}
\widetilde{H}^{1,1}_{C_2}(S^{2\sigma,\sigma} \wedge (U_n)_+) \arrow[r] \arrow[d] & \widetilde{H}^{1,1}_{C_2}(S^{2\sigma,\sigma} \wedge C_{2\,+}) \arrow[d] \\
\widetilde{H}^{\sigma+1,1}_{C_2}(S^{2\sigma,\sigma} \wedge (U_n)_+) \arrow[r, "f"] \arrow[d] & \widetilde{H}^{\sigma+1,1}_{C_2}(S^{2\sigma,\sigma} \wedge C_{2\,+}) \arrow[d] \\
\widetilde{H}^{2,1}_{\mathcal{M}}(S^{2,1} \wedge (\mathbb{A}^n \setminus \{0\})_+) \arrow[r, hook] \arrow[d] & \widetilde{H}^{2,1}_{\mathcal{M}}(S^{2,1} \wedge S^0_+) \arrow[d] \\
\widetilde{H}^{2,1}_{C_2}(S^{2\sigma,\sigma} \wedge (U_n)_+) \arrow[r] \arrow[d, two heads, "g"'] & \widetilde{H}^{2,1}_{C_2}(S^{2\sigma,\sigma} \wedge C_{2\,+}) \arrow[d] \\
\widetilde{H}^{\sigma+2,1}_{C_2}(S^{2\sigma,\sigma} \wedge (U_n)_+) \arrow[r] & \widetilde{H}^{\sigma+2,1}_{C_2}(S^{2\sigma,\sigma} \wedge C_{2\,+}).
\end{tikzcd}$$

To show that f is an isomorphism we show that it is an injection whose domain is a rank one $\mathbb{Z}/2$-module. We make use of the cofiber sequence (2.5) to compute some ranks of the $\mathbb{Z}/2$-modules on the left hand side. By [HVØ15, Proposition 8.3], there is an equivariant finite correspondence $\lambda : \mathbb{A}(\sigma) \times U_n \to U_1 \times U_n$ whose composition with the inclusion $U_1 \times U_n \subseteq \mathbb{A}(\sigma) \times U_n$ is the identity. This implies that the long exact sequences induced by (2.5) break into short exact sequences

$$0 \to \widetilde{H}^{a,1}_{C_2}((U_n)_+) \to \widetilde{H}^{a,1}_{C_2}((U_1 \times U_n)_+) \to \widetilde{H}^{a+1,1}_{C_2}(S^{2\sigma,\sigma} \wedge (U_n)_+) \to 0. \tag{5.6}$$

The inclusion $U_1 \times U_n \subseteq U_{n+1}$ induces a map $\widetilde{H}^{a,1}_{C_2}((U_{n+1})_+) \to \widetilde{H}^{a,1}_{C_2}((U_1 \times U_n)_+)$ which is identified with the map $H^{a,1}_{\mathcal{M}}(U_{n+1}/C_2) \to H^{a,1}_{\mathcal{M}}((U_1 \times U_n)/C_2)$. The closed complement of $(U_1 \times U_n)/C_2$ in U_{n+1}/C_2 is smooth, and so using the Gysin sequence we find that this map is an isomorphism for $a = 0$. It follows that $\widetilde{H}^{1,1}_{C_2}(S^{2\sigma,\sigma} \wedge (U_n)_+) = 0$. This implies that f is injective.

It remains to see that the rank of $\widetilde{H}^{\sigma+1,1}_{C_2}(S^{2\sigma,\sigma} \wedge (U_n)_+)$ equals one. To prove this we show that g is an isomorphism. Since g is surjective, it suffices to see that the domain and codomain of g both have the same rank. We show they are both of rank one. Consider the Gysin exact sequence, where the displayed ranks follow easily using the exact sequence obtained from the cofiber sequence (5.5),

$$\begin{aligned}0 \to \overset{\text{rank } 2}{H^{1,1}_{\mathcal{M}}(U_n/C_2)} \to H^{1,1}_{\mathcal{M}}((U_1 \times U_n)/C_2) \to \overset{\text{rank } 1}{H^{0,0}_{\mathcal{M}}(U_n/C_2)} \\ \to \underset{\text{rank } 1}{H^{2,1}_{\mathcal{M}}(U_{n+1}/C_2)} \to H^{2,1}_{\mathcal{M}}((U_1 \times U_n)/C_2) \to 0.\end{aligned}$$

Since $H^{2,1}_{\mathcal{M}}((U_1 \times U_n)/C_2)$ contains a rank one submodule (namely $H^{2,1}_{\mathcal{M}}((U_n)/C_2)$) it has rank one. We conclude that $H^{2,1}_{\mathcal{M}}((U_1 \times U_n)/C_2)$ has rank three. Using (5.6) we see that $\widetilde{H}^{2,1}_{C_2}(S^{2\sigma,\sigma} \wedge (U_n)_+)$ has rank one. From the exact sequences

$$0 = H^{2,1}_{\mathcal{M}}(U_1 \times U_n) \to H^{2,1}_{C_2}(U_1 \times U_n) \to H^{\sigma+2,1}_{C_2}(U_1 \times U_2) \to 0$$

and

$$0 = H^{2,1}_{\mathcal{M}}(U_n) \to H^{2,1}_{C_2}(U_n) \to H^{\sigma+2,1}_{C_2}(U_2) \to 0,$$

and (5.6), we find that $\widetilde{H}^{\sigma+2,1}_{C_2}(S^{2\sigma,\sigma} \wedge (U_n)_+)$ has rank one. It follows that g is an isomorphism, and so we conclude that f is an isomorphism, as desired. □

Remark 5.7. Topologically, there is an isomorphism

$$H^{\star}_{C_2}(EC_2, \underline{\mathbb{Z}/2}) \cong H^*_{sing}(BC_2, \mathbb{Z}/2)[u^{\pm 1}],$$

where u has degree $\sigma - 1$, see [Car99, Lemma 27]. The computation of the previous theorem implies there is an inclusion

$$H^{\star,\star}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2}) \supseteq H^{*,*}_{\mathcal{M}}(\mathbf{B}C_2, \mathbb{Z}/2)[s^{\pm 1}, t^{\pm 1}].$$

When $k = \mathbb{C}$, note the $\mathrm{Re}_{\mathbb{C}}(s) = \mathrm{Re}_{\mathbb{C}}(t) = u$.

Corollary 5.8. *Multiplication by* $s^{d-b}t^{-d} \in H^{b-b\sigma,d-d\sigma}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$ *induces a natural isomorphism*

$$\widetilde{H}^{a+b\sigma,c+d\sigma}_{C_2}(\mathsf{E} \wedge (\mathbf{E}C_2)_+, \underline{\mathbb{Z}/2}) \cong \widetilde{H}^{a+b,c+d}_{C_2}(\mathsf{E} \wedge (\mathbf{E}C_2)_+, \underline{\mathbb{Z}/2})$$

of $H^{\star,\star}_{C_2}(\mathbf{E}C_2, \underline{\mathbb{Z}/2})$*-modules for any motivic* C_2*-spectrum* E.

We note as well that Proposition 2.9 immediately implies that the cohomology of any $X_+ \wedge \widetilde{\mathbf{E}}C_2$ is periodic in the following sense.

Proposition 5.9. *Let* X *be a smooth* C_2*-scheme over* k *and* A *be a commutative ring. For all integers a,b,p,q, there are* $H^{\star,\star}_{C_2}(X, \underline{A})$*-module isomorphisms*

$$\widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \cong \widetilde{H}^{a+(p-1)\sigma,b+q\sigma}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A})$$

and

$$\widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \cong \widetilde{H}^{a+(p-2)\sigma,b+(q-1)\sigma}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}).$$

6. COMPARING MOTIVIC AND TOPOLOGICAL BREDON COHOMOLOGY OVER $\mathbb{C}$

Let X be a smooth variety over a field. The Beilinson-Lichtenbaum conjecture [SV00, Conjecture 6.8] is the assertion that the map

$$H^{p,q}_{\mathcal{M}}(X, \mathbb{Z}/n) \to H^{p}_{et}(X, \mu_n^{\otimes q}) \tag{6.1}$$

is an isomorphism when $p \leq q$ and is an injection when $p = q + 1$. The validity of this conjecture is a consequence of the Milnor and Bloch-Kato conjectures [Voe03a, Voe11] together with [SV00, Theorem 7.4]. Now if X is a complex variety this can be rephrased using singular cohomology; topological realization

$$H^{p,q}_{\mathcal{M}}(X,\mathbb{Z}/n)\to H^p_{sing}(X(\mathbb{C}),\mathbb{Z}/n) \text{ is } \begin{cases}\text{an isomorphism} & \text{if } p\le q,\\ \text{a monomorphism} & \text{if } p=q+1.\end{cases} \tag{6.2}$$

In this section, we establish a C_2-equivariant generalization of the Beilinson-Lichtenbaum conjecture for smooth complex varieties X with involution. We begin with a consideration of the Borel part of the Bredon cohomologies.

Proposition 6.3. *Let X be a smooth complex C_2-variety and A a finite abelian group. Then*

$$\mathrm{Re}_{\mathbb{C}}: H^{a+p\sigma,b+q\sigma}_{C_2}(X\times \mathbf{E}C_2,\underline{A})\to H^{a+p\sigma}_{C_2}(X(\mathbb{C})\times \mathbf{E}C_2(\mathbb{C}),\underline{A}),$$

is an isomorphism if $a+p\le b+q$ and an injection if $a+p=b+q+1$.

Proof. We first assume that X is quasi-projective. It suffices to assume that $A=\mathbb{Z}/p^i$ where p is a prime. Suppose that 2 is invertible in A. Consider the commutative diagram coming from Proposition 3.18, where the coefficient group A has been suppressed

$$\begin{tikzcd}
H^{a+p\sigma,b+q\sigma}_{C_2}(X\times \mathbf{E}C_2) \arrow[r, hook, "\pi^*"] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & H^{a+p,b+q}_{\mathcal{M}}(X\times \mathbf{E}C_2) \arrow[r, two heads, "\tau^*"] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & H^{a+p\sigma,b+q\sigma}_{C_2}(X\times \mathbf{E}C_2) \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] \\
H^{a+p\sigma}_{C_2}((X\times \mathbf{E}C_2)(\mathbb{C})) \arrow[r, hook, "\pi^*"] & H^{a+p}_{sing}((X\times \mathbf{E}C_2)(\mathbb{C})) \arrow[r, two heads, "\tau^*"] & H^{a+p\sigma}_{C_2}((X\times \mathbf{E}C_2)(\mathbb{C})).
\end{tikzcd}$$

The horizontal compositions are multiplication by 2 and so are isomorphisms. By the Beilinson-Lichtenbaum conjecture (6.2), the middle map is an isomorphism for $a+p\le b+q$ and an injection for $a+p=b+q+1$. The same is thus true of the outer vertical arrows.

It remains to consider the case $A=\mathbb{Z}/2^i$. By comparing the exact sequences arising from the short exact sequence $0\to\mathbb{Z}/2^{i-1}\to\mathbb{Z}/2^i\to\mathbb{Z}/2\to 0$ and induction, we are reduced to the case $A=\mathbb{Z}/2$.

We now assume that $A=\mathbb{Z}/2$ (and continue suppressing the coefficients as needed). Using the periodicity from Corollary 5.8 we may replace $(a+p\sigma,b+q\sigma)$ by $(a+p,b+q)$ (and we write simply (a,b) for this new integer bidegree). Consider the comparison of Milnor exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \lim^1_n H^{a-1,b}_{C_2}(U_n) \arrow[r] \arrow[d] & H^{a,b}_{C_2}(X\times \mathbf{E}C_2) \arrow[r] \arrow[d] & \lim_n H^{a,b}_{C_2}(U_n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \lim^1_n H^{a-1,b}_{C_2}(U_n(\mathbb{C})) \arrow[r] & H^{a,b}_{C_2}(X(\mathbb{C})\times \mathbf{E}C_2(\mathbb{C})) \arrow[r] & \lim_n H^{a,b}_{C_2}(U_n(\mathbb{C})) \arrow[r] & 0
\end{tikzcd}$$

where $U_n:=X\times(\mathbb{A}(n\sigma)\setminus\{0\})$. Since U_n is a smooth quasi-projective variety with free action, the quotient U_n/C_2 exists and is a smooth quasi-projective variety. The map $H^{a,b}_{C_2}(U_n)\to H^a_{C_2}(U_n(\mathbb{C}))$ is identified with the map $H^a_{\mathcal{M}}(U_n/C_2)\to H^a_{sing}(U_n(\mathbb{C})/C_2)$ by Proposition 3.16. Therefore, using [Voe03a, Corollary 6.9(2)], the map (i) is an isomorphism for $a+p\le b+q$ and an injection for $a+p=b+q+1$ by applying the Beilinson-Lichtenbaum conjecture (6.1) to the left and right hand vertical maps of the above comparison of Milnor exact sequences.

To deduce the proposition for a general smooth X from the quasi-projective case, we use that X is locally affine in the equivariant Nisnevich topology, see Remark 2.3.

There are several ways to turn this observation into a formal argument; we proceed directly as follows. Suppose that we have a cartesian square of smooth C_2-complex varieties

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d] & Y \arrow[d, "\phi"] \\
U \arrow[r, hook] & X,
\end{tikzcd}$$

where Y is quasi-projective, ϕ is equivariant étale, U is an invariant open, and the restriction $\phi|_{Y\setminus W}$ has an equivariant section. Then, if the proposition is true for U it is also true for X. Indeed, this square leads to a distinguished equivariant Nisnevich square, via standard techniques,

$$\begin{tikzcd}
W' \arrow[r, hook] \arrow[d] & Y' \arrow[d] \\
U \arrow[r, hook] & X,
\end{tikzcd}$$

where Y' is open in Y; in particular, it is quasi-projective. Comparing the resulting Mayer-Vietoris long exact sequences then shows that under these assumptions, the proposition holds for X.

Now we let $A \subseteq X$ be a dense invariant affine open and Y any quasi-projective equivariant Nisnevich cover of X. Let $\emptyset = Z_{n+1} \subseteq Z_n \subseteq \cdots Z_1 \subseteq Z_0 := X \setminus A$ be an equivariant splitting sequence for $Y|_{Z_0}$. Set $X_i = X \setminus Z_i$ and $Y_i = Y|_{X_i}$. The cartesian square

$$\begin{tikzcd}
Y_i \arrow[r, hook] \arrow[d] & Y_{i+1} \arrow[d] \\
X_i \arrow[r, hook] & X_{i+1}
\end{tikzcd}$$

satisfies the conditions of the previous paragraph, and so proceeding by induction we find that the proposition holds for each X_i. □

Next we consider the isotropic part of the Bredon cohomologies.

Proposition 6.4. *Let X be a smooth complex C_2-variety and A a finite abelian group. Then*

$$\mathrm{Re}_{\mathbb{C}} : \widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \to \widetilde{H}^{a+p\sigma}_{C_2}(X(\mathbb{C})_+ \wedge \widetilde{\mathbf{E}}C_2(\mathbb{C}), \underline{A}),$$

is an isomorphism if $a \le b$ and an injection if $a = b+1$.

Proof. First we consider the special case when X has trivial action. By the periodicities supplied by Proposition 5.9 and the corresponding ones in topological Bredon cohomology, we may assume that $p = q = 0$.

Since X has trivial action, the map $H^{a,b}_{C_2}(X, \underline{A}) \to H^a_{C_2}(X(\mathbb{C}), \underline{A})$ is naturally isomorphic to the map $H^{a,b}_{\mathcal{M}}(X, A) \to H^a_{sing}(X(\mathbb{C}), A)$, by Proposition 3.15. In particular, by the Beilinson-Lichtenbaum conjecture, it is an isomorphism if $a \le b$ and an injection if $a = b+1$. The map $H^{a,b}_{C_2}(X\times\mathbf{E}C_2, \underline{A}) \to H^a_{C_2}(X(\mathbb{C})\times\mathbf{E}C_2(\mathbb{C}), \underline{A})$ is an isomorphism for $a \le b$ and an injection for $a = b+1$, by the previous proposition. The proposition thus follows for X with trivial action by comparing the long exact sequences associated to the motivic isotropy cofiber sequence

$$X_+ \wedge \mathbf{E}C_{2+} \to X_+ \to X_+ \wedge \widetilde{\mathbf{E}}C_2. \tag{6.5}$$

We now treat the general case. The fixed point scheme X^{C_2} is smooth and so by the equivariant homotopical purity theorem [HKØ15, Theorem 7.6], we have a cofiber sequence

$$X \setminus X^{C_2} \to X \to \mathrm{Th}(\mathcal{N}),$$

where $\mathcal{N}$ is the normal bundle of the inclusion $X^{C_2} \subseteq X$. By Lemma 4.2, the map $X \to \mathrm{Th}(\mathcal{N})$ induces an isomorphism

$$\widetilde{H}^{\star,\star}_{C_2}(\mathrm{Th}(\mathcal{N}) \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \xrightarrow{\cong} \widetilde{H}^{\star,\star}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \tag{6.6}$$

and similarly for the topological Bredon cohomology. Note that X^{C_2} is a disjoint union $X^{C_2} = \coprod_r Z_r$ of connected smooth varieties with trivial action. We may apply Theorem 3.26 and Proposition 3.30, together with Remark 3.29 to each $\mathcal{N}|_{Z_r}$ in order to obtain the commutative square

$$\begin{tikzcd}
\widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(\mathrm{Th}(\mathcal{N}) \wedge \widetilde{\mathbf{E}}C_2) \arrow[r, "\cong"] \arrow[d] & \bigoplus^r \widetilde{H}^{a+(p-2p_r)\sigma,\, b+(q-p_r)\sigma}_{C_2}(Z_{r+} \wedge \widetilde{\mathbf{E}}C_2) \arrow[d] \\
\widetilde{H}^{a+p\sigma}_{C_2}(\mathrm{Th}(\mathcal{N}(\mathbb{C})) \wedge \widetilde{\mathbf{E}}C_2(\mathbb{C})) \arrow[r, "\cong"] & \bigoplus^r \widetilde{H}^{a+(p-2p_r)\sigma}_{C_2}(Z_r(\mathbb{C})_+ \wedge \widetilde{\mathbf{E}}C_2(\mathbb{C})),
\end{tikzcd}$$

where $p_r = \mathrm{codim}_X(Z_r)$. The proposition holds for the right hand map and thus it holds for the left hand map as well. Applying the isomorphism (6.6) yields the conclusion of the proposition. □

Combining the previous two results now implies our equivariant generalization of the Beilinson-Lichtenbaum conjectures over $\mathbb{C}$.

Theorem 6.7. *Let X be a smooth complex C_2-variety and A a finite abelian group. The comparison map*

$$\mathrm{Re}_{\mathbb{C}} : H^{a+p\sigma,b+q\sigma}_{C_2}(X, \underline{A}) \to H^{a+p\sigma}_{C_2}(X(\mathbb{C}), \underline{A})$$

is

(i) *an isomorphism if both $a + p \leq b + q$ and $a \leq b$,*
(ii) *an injection if both $a + p \leq b + q + 1$ and $a \leq b + 1$.*

Proof. This follows by comparing the long exact sequences induced by the motivic isotropy cofiber sequence $X_+ \wedge \mathbf{E}C_{2+} \to X_+ \to X_+ \wedge \widetilde{\mathbf{E}}C_2$ together with Proposition 6.3 and Proposition 6.4. □

Notice that in the case $p = q = 0$ and X complex variety with trivial $\mathbb{Z}/2-$action the above theorem reduces to the usual Beilinson-Lichtenbaum conjecture for complex varieties (see (6.2)).

7. Comparing motivic and topological Bredon cohomology over $\mathbb{R}$

Let X be a smooth real variety and write $\Sigma_2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The space $X(\mathbb{C})$ has an Σ_2-action. This extends to the topological realization functor $\mathrm{Re}_{\mathbb{C},\Sigma_2} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}_{\Sigma_2}$, see [HO14, Proposition 4.8]. We have $\mathrm{Re}_{\mathbb{C},\Sigma_2}(\mathbf{M}\underline{A}) = \mathrm{H}\underline{A}$ [HO14, Theorem 4.17] and thus a comparison map relating motivic cohomology and Bredon cohomology. By [HV12, Corollary 5.11], the Beilinson-Lichtenbaum conjecture (6.1) for real varieties can be reinterpreted as the statement that the map

$$H^{a,b}_{\mathcal{M}}(X, A) \to H^{a-b+b\sigma}_{\Sigma_2}(X(\mathbb{C}), \underline{A}) \text{ is } \begin{cases} \text{an isomorphism} & \text{if } a \leq b, \\ \text{a monomorphism} & \text{if } a = b + 1. \end{cases} \tag{7.1}$$

We now consider a smooth real variety X with an C_2-action. The space of complex points $X(\mathbb{C})$ has two involutions, one coming from complex conjugation and the other coming from action on the variety X. To avoid confusing these actions, we write C_2 for the group acting algebraically on the real variety X while we write $\Sigma_2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ for the Galois group and its action is by complex conjugation. Thus $X(\mathbb{C})$ has an $C_2 \times \Sigma_2$-action. This extends to the topological realization functor, see Theorem A.19,

$$\mathrm{Re}_{\mathbb{C},\,\Sigma_2} : \mathrm{SH}_{C_2}(\mathbb{R}) \to \mathrm{SH}_{C_2\times\Sigma_2}.$$

Write τ_1 (resp. τ_2) for the nontrivial element of C_2 (resp. of Σ). Write σ for the $C_2 \times \Sigma_2$-representation which is defined by letting τ_1 act by -1 and τ_2 by the identity. Write ϵ for the representation which is defined by letting τ_1 act by the identity and τ_2 by -1. The four representations of the Klein group $C_2 \times \Sigma_2$ are 1, σ, ϵ, and $\sigma \otimes \epsilon$.

The effect of the topological realization functor on spheres is as follows.

Lemma 7.2. *We have* $\mathrm{Re}_{\mathbb{C},\,\Sigma_2}(S^1) \simeq S^1$, $\mathrm{Re}_{\mathbb{C},\,\Sigma_2}(S^\sigma) \simeq S^\sigma$, $\mathrm{Re}_{\mathbb{C},\,\Sigma_2}(S^1_t) \simeq S^\epsilon$, *and* $\mathrm{Re}_{\mathbb{C},\,\Sigma_2}(S^\sigma_t) \simeq S^{\sigma\otimes\epsilon}$. *Thus*

$$\mathrm{Re}_{\mathbb{C},\,\Sigma_2}(S^{a+p\sigma,b+q\sigma}) \simeq S^{(a-b)+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon}.$$

Proof. The first relation is obvious. The second follows from the cofiber sequence $C_{2+} \to S^0 \to S^\sigma$. We have that $S^1_t(\mathbb{C})$ and $S^\sigma_t(\mathbb{C})$ are equivariantly homotopic to the unit circle in $\mathbb{C}^*$. In the first case, $\tau_1(z) = z$ and $\tau_2(z) = \overline{z}$, and in the second case $\tau_1(z) = 1/z$ and $\tau_2(z) = \overline{z}$. The displayed equalities follow immediately from these formulae. □

Since $\mathrm{Re}_{\mathbb{C},\Sigma_2}(\mathbf{M}\underline{A}) \cong \mathrm{H}\underline{A}$ in $\mathrm{SH}_{C_2\times\Sigma_2}$, see Theorem A.31, there exists a comparison map

$$\mathrm{Re}_{\mathbb{C},\Sigma_2} : H^{a+p\sigma,b+q\sigma}_{C_2}(X,\underline{A}) \to H^{a-b+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon}_{C_2\times\Sigma_2}(X(\mathbb{C}),\underline{A}). \tag{7.3}$$

Observe that $\mathrm{Re}_{\mathbb{C},\Sigma_2}(\mathbb{A}(\sigma)) = \sigma + \sigma\otimes\epsilon$. Applying $\mathrm{Re}_{\mathbb{C},\Sigma_2}$ to the motivic isotropy cofiber sequence (2.7) yields the cofiber sequence of $C_2 \times \Sigma_2$-spaces

$$\mathrm{E}_{\Sigma_2}C_{2+} \to S^0 \to \widetilde{\mathrm{E}}_{\Sigma_2}C_2.$$

Here $\mathrm{E}_{\Sigma_2}C_2$ is the Σ_2-equivariant universal C_2-space, see [May96, VII.1] or [HHR09, Definition B.108]. We have $\widetilde{\mathrm{E}}_{\Sigma_2}C_2 = \mathrm{colim}_n\, S^{n\sigma\otimes\epsilon} \wedge S^{n\sigma}$, which immediately shows the following.

Proposition 7.4. *The unit maps* $S^0 \to S^\sigma$, $S^0 \to S^{\sigma\otimes\epsilon}$ *induce* $C_2\times\Sigma_2$*-equivariant homotopy equivalences* $\widetilde{\mathrm{E}}_{\Sigma_2}C_2 \simeq \widetilde{\mathrm{E}}_{\Sigma_2}C_2 \wedge S^\sigma$ *and* $\widetilde{\mathrm{E}}_{\Sigma_2}C_2 \simeq \widetilde{\mathrm{E}}_{\Sigma_2}C_2 \wedge S^{\sigma\otimes\epsilon}$. *In particular,* $\widetilde{H}^\star_{C_2\times\Sigma_2}(\mathsf{E} \wedge \widetilde{\mathrm{E}}_{\Sigma_2}C_2, \underline{A})$ *is* σ*-periodic as well as* $\sigma\otimes\epsilon$*-periodic for any* $C_2 \times \Sigma_2$*-equivariant spectrum* E.

Since (7.3) is a ring map, Theorem 5.4 implies there are invertible elements $s, t \in H^\star_{C_2\times\Sigma_2}(\mathrm{E}_{\Sigma_2}C_2, \underline{\mathbb{Z}/2})$, the degree of s is $\sigma - 1$ and the degree of t is $\sigma\otimes\epsilon - \epsilon$. This immediately implies the following.

Proposition 7.5. *Multiplication by* $s^{-b}t^{-d} \in H^{b-b\sigma+d\epsilon-d\sigma\otimes\epsilon}_{C_2\times\Sigma_2}(\mathrm{E}_{\Sigma_2}C_2, \underline{\mathbb{Z}/2})$ *induces a natural isomorphism*

$$\widetilde{H}^{a+b\sigma+c\epsilon+d\sigma\otimes\epsilon}_{C_2\times\Sigma_2}(\mathsf{E} \wedge (\mathrm{E}_{\Sigma_2}C_2)_+, \underline{\mathbb{Z}/2}) \cong \widetilde{H}^{a+b+(c+d)\epsilon}_{C_2\times\Sigma_2}(\mathsf{E} \wedge (\mathrm{E}_{\Sigma_2}C_2)_+, \underline{\mathbb{Z}/2})$$

of $H^\star_{C_2\times\Sigma_2}(\mathrm{E}_{C_2}\Sigma_2, \underline{\mathbb{Z}/2})$*-modules for any* $C_2 \times \Sigma_2$*-spectrum* E.

We proceed, as in the previous section, towards our equivariant version of the Beilinson-Lichtenbaum conjecture over the reals by establishing the following case.

Proposition 7.6. *Let X be a smooth real variety with C_2-action and A a finite abelian group. Then the map*

$$H_{C_2}^{a+p\sigma,b+q\sigma}(X\times \mathbf{E}C_2,\underline{A}) \to H_{C_2\times\Sigma_2}^{a-b+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon}(X(\mathbb{C})\times \mathrm{E}_{\Sigma_2}C_2,\underline{A})$$

is an isomorphism for every $a+p\leq b+q$ and a monomorphism for $a+p=b+q+1$.

Proof. It suffices to consider the case of a quasi-projective X by the same argument as in the second half of Proposition 6.3. It also suffices to consider the case $A=\mathbb{Z}/2$ by the same argument as in Proposition 6.3, using an obvious variant of Proposition 3.18.

Using Corollary 5.8 and Proposition 7.5, we can replace $(a+p\sigma, b+q\sigma)$ by $(a+p,b+q)$ and $a-b+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon$ by $(a+p-b-q)+(b+q)\sigma\otimes\epsilon$. For simplicity, we reindex, replacing $a+p$ by a and $b+q$ by b (i.e., $p=0$ and $q=0$). Considering the comparison of Milnor exact sequences, as in Proposition 6.3, we need to see that the map

$$H^{a,b}_{C_2}(U_n,\underline{\mathbb{Z}/2}) \to H^{a+p-b-q,b+q}_{C_2\times C_2}(U_n(\mathbb{C}),\underline{\mathbb{Z}/2}) \tag{7.7}$$

is an isomorphism for $a\leq b$ and a monomorphism for $a=b+1$. Here $U_n := X\times(\mathbb{A}(n\sigma)\setminus\{0\})$.

If Y is a $C_2\times\Sigma_2$-space, then we have $H^{s+t\epsilon}_{C_2\times\Sigma_2}(Y,\underline{A})\cong H^{s+t\sigma}_{\Sigma_2}(Y/C_2,\underline{A})$. Moreover, if $Y=X(\mathbb{C})$, where X is a real variety with free C_2-action, arguing as in Proposition 3.16, we have a commutative square

$$\begin{tikzcd}
H^{m,n}_{C_2}(X,\underline{A}) \arrow[r, "\cong"] \arrow[d, "\mathrm{Re}_{\mathbb{C},\,\Sigma_2}"'] & H^{m,n}_{\mathcal{M}}(X/C_2,A) \arrow[d, "\mathrm{Re}_{\mathbb{C},\,\Sigma_2}"] \\
H^{m-n+n\epsilon}_{C_2\times\Sigma_2}(X(\mathbb{C}),\underline{A}) \arrow[r, "\cong"] & H^{m-n+n\sigma}_{\Sigma_2}(X(\mathbb{C})/C_2,\underline{A}).
\end{tikzcd}$$

In particular the map (7.7) is identified with the map

$$H^{a,b}_{\mathcal{M}}(U_n/C_2,\mathbb{Z}/2)\to H^{a-b+b\sigma}_{C_2}((U_n/C_2)(\mathbb{C}),\underline{\mathbb{Z}/2}).$$

Therefore, the map (7.7) is an isomorphism for $a\leq b$ and an injection for $a=b+1$ by applying the Beilinson-Lichtenbaum conjecture for a real variety (7.1).

□

We also have the following:

Proposition 7.8. *Let X be a smooth real variety with C_2-action and A a finite abelian group. The map*

$$\widetilde{H}^{a+p\sigma,b+q\sigma}_{C_2}(X_+\wedge\widetilde{\mathbf{E}}C_2,\underline{A})\to \widetilde{H}^{a-b+b\epsilon}_{C_2\times\Sigma_2}(X(\mathbb{C})_+\wedge\widetilde{\mathrm{E}}_{\Sigma_2}C_2,\underline{A})$$

is an isomorphism for $a\leq b$ and a monomorphism for $a=b+1$.

Proof. In the case of a free C_2-action on X the above map is an isomorphism for all the indexes because the groups are zero. Consider the comparison of long exact sequences in cohomology associated to the motivic isotropy cofiber sequence

$$X_+\wedge\mathbf{E}C_2\to X_+\to X_+\wedge\widetilde{\mathbf{E}}C_2.$$

Suppose that X has trivial action. Using Proposition 5.9 and Proposition 7.4, we may assume that $p = 0$ and $q = 0$. Consider the map

$$\widetilde{H}^{a,b}_{C_2}(X_+ \wedge \widetilde{\mathbf{E}}C_2, \underline{A}) \to \widetilde{H}^{a-b+b\epsilon}_{C_2\times\Sigma_2}(X(\mathbb{C})_+ \wedge \widetilde{\mathrm{E}}_{\Sigma_2}C_2, \underline{A}).$$

Because X has a trivial action, we have that the middle map in the long exact sequence given by the motivic isotropy cofiber sequence is identified with

$$H^{a,b}_{\mathcal{M}}(X, \underline{A}) \to H^{a-b,b}_{C_2}(X(\mathbb{C}), \underline{A}).$$

This is an isomorphism if $a \leq b$ and a monomorphism if $a = b+1$ by the Beilinson-Lichtenbaum conjecture (7.1).

The other significant map in the diagram is

$$\widetilde{H}^{a,b}_{C_2}(X \times \mathbf{E}C_2, \underline{A}) \to \widetilde{H}^{a-b+b\epsilon}_{C_2\times\Sigma_2}(X(\mathbb{C}) \times E_{\Sigma_2}C_2, \underline{A})$$

which according to Proposition 7.6 is an isomorphism for any $a \leq b$ and monomorphism for any $a = b+1$. This confirms the isomorphism in the statement when the action is trivial.

The isomorphism in the general case follows as in Proposition 6.4 by considering the equivariant cofiber sequence for the real variety X

$$X \setminus X^{C_2} \to X \to Th(\mathcal{N}),$$

where $\mathcal{N}$ is the normal bundle of the inclusion $X^{C_2} \subset X$. □

Theorem 7.9. *Let X be a smooth real variety with C_2-action and A an abelian group. Then the map*

$$H^{a+p\sigma,b+q\sigma}_{C_2}(X, \underline{A}) \to H^{a-b+(p-q)\sigma+b\epsilon+q\sigma\otimes\epsilon}_{C_2\times\Sigma_2}(X(\mathbb{C}), \underline{A})$$

is

(i) *an isomorphism if both $a+p \leq b+q$ and $a \leq b$,*
(ii) *an injection if both $a+p \leq b+q+1$ and $a \leq b+1$.*

Proof. This follows by comparing the long exact sequences induced by the motivic isotropy cofiber sequence $X_+ \wedge \mathbf{E}C_{2+} \to X_+ \to X_+ \wedge \widetilde{\mathbf{E}}C_2$ together with Proposition 7.6 and Proposition 7.8. □

When $p = q = 0$ and X a smooth real variety with trivial C_2-action the above theorem reduces to the version of the Beilinson-Lichtenbaum conjecture for a real variety (see 7.1) established in [HV12]. (Of course this was used to prove the theorem.)

Appendix A. Equivariant motivic homotopy theory

Unstable equivariant motivic homotopy theory was first considered by Voevodsky [Del09]. A stable version was considered by Hu-Kriz-Ormsby [HKO11] as part of their work on the completion problem in Hermitian K-theory. General foundations and model structures are constructed in [HKØ15] and representability results for equivariant algebraic K-theory are also established. A general framework for stable equivariant motivic homotopy theory emphasizing the six functor formalism is introduced in [Hoy15]. Alternate versions of a homotopy theory for smooth G-schemes are studied in [Her13] and [CJ14]; however, theories of interest, such as equivariant algebraic K-theory, are not representable in the homotopy categories constructed there.

The main results in this paper rely on a Betti or topological realization functor, which is constructed in this appendix, relating equivariant motivic homotopy and classical equivariant homotopy theory. We begin by giving a brief but self-contained construction of a model for the unstable and stable equivariant motivic homotopy categories, which is geared towards the construction of the Betti realization functor. We also record the details of the construction of several comparison functors between equivariant and nonequivariant homotopy categories in this setting. Finally in the last sections of this appendix we verify that the topological realization of the Bredon motivic cohomology spectrum is the topological Bredon cohomology spectrum.

A.1. **Unstable equivariant motivic homotopy theory.** To keep exposition streamlined, we restrict attention to the case of a finite group G over a field k. A *motivic G-space* over k is defined to be a presheaf of simplicial sets on $G\mathrm{Sm}/k$. We write $G\mathrm{Spc}(k)$ and $G\mathrm{Spc}_\bullet(k)$ respectively for the categories of motivic G-spaces and pointed motivic G-spaces over k. We are primarily interested in the stable equivariant motivic homotopy category, and so we only treat the unstable model structure for pointed motivic G-spaces. The category $G\mathrm{Spc}_\bullet(k)$ is a symmetric monoidal category via the pointwise smash product $(F \wedge G)(X) := F(X) \wedge G(X)$ of motivic G-spaces.

We make use of an equivariant version of the "closed flasque model structure" introduced in [PPR09], a variation on the flasque model structure of [Isa05]. This model structure is particularly well-suited for topological realization.

Let $\mathcal{Z} = \{Z_i \hookrightarrow X\}$ be a finite collection of closed immersions ($\emptyset \to X$ is allowed) in $G\mathrm{Sm}/k$. Define

$$\cup\mathcal{Z} := \mathrm{coeq}(\coprod_{r,r'} Z_r \times_X Z_{r'} \rightrightarrows \coprod_r Z_r),$$

where the coequalizer is computed in $\mathrm{Spc}(k)$ (i.e., $\cup Z$ is the categorical union of the $Z_r \subseteq X$). Write $i_{\mathcal{Z}} : \cup\mathcal{Z} \to X$ for the resulting monomorphism.

The pushout product $i \,\square\, j$ of two maps $i : A \to X$ and $j : B \to Y$ is defined to be the map $A \wedge Y \coprod_{A\wedge B} X \wedge B \to X \wedge Y$. Define two sets of maps:

(1) I^c is the set of maps of the form $(i_{\mathcal{Z}})_+ \,\square\, g_+$, where $\mathcal{Z}$ is a finite set of closed equivariant immersions in $G\mathrm{Sm}/k$ and $g : \partial\Delta^n \to \Delta^n$, $n \geq 0$ is the standard inclusion.
(2) J^c is the set of maps of the form $(i_{\mathcal{Z}})_+ \,\square\, g_+$, where $\mathcal{Z}$ is a finite set of closed equivariant immersions in $G\mathrm{Sm}/k$ and $g : \Lambda^{n,k} \to \Delta^n$, $n \geq 1$, $0 \leq i \leq n$ is the standard inclusion.

Definition A.1. Let $f : F \to G$ be a map of pointed motivic G-spaces.

(1) Say that f is a *schemewise weak equivalence* provided $f : F(U) \to G(U)$ is a weak equivalence of simplicial sets for all U in $G\mathrm{Sm}/k$.
(2) Say that f is a *closed flasque fibration* if it has the right lifting property with respect to J^c.
(3) Say that f is a *closed flasque cofibration* if it has the left lifting property with respect to acyclic closed schemewise fibrations.

The schemewise weak equivalences, closed flasque cofibrations, and closed flasque fibrations define the *global closed flasque model structure* on $G\mathrm{Spc}_\bullet(k)$, see e.g., [HKØ15, Theorem 3.8].

Proposition A.2. *The global closed flasque model structure is a simplicial, proper, cellular, monoidal model structure on* $G\mathrm{Spc}_\bullet(k)$*. The sets* I^c *and* J^c *are respectively generating cofibrations and acyclic cofibrations.*

For a distinguished equivariant Nisnevich square Q as in (2.2), write P_Q for the pushout of $A \leftarrow B \rightarrow Y$ in $G\mathrm{Spc}(k)$.

Definition A.3.

1. The *closed flasque (equivariant Nisnevich) local model structure* on $G\mathrm{Spc}_\bullet(k)$ is the left Bousfield localization of the global closed flasque model structure at the set of maps $\{(P_Q)_+ \to X_+\}$, where Q ranges over the set of distinguished equivariant Nisnevich squares. The associated homotopy category is denoted $\mathrm{H}^{Nis}_{G,\bullet}(k)$.
2. The *closed flasque motivic model structure* on $G\mathrm{Spc}_\bullet(k)$ is the left Bousfield localization of the closed flasque local model structure at the set of projections $(X \times \mathbb{A}^1)_+ \to X_+$ for all X in $G\mathrm{Sm}/k$. For brevity, we refer to the weak equivalences and fibrations of this model structure as *motivic weak equivalences* and *motivic fibrations*. The associated homotopy category is the unstable equivariant motivic homotopy category and is denoted $\mathrm{H}_{G,\bullet}(k)$.

Theorem A.4. *The closed flasque local and closed flasque motivic model structures are simplicial, proper, cellular, monoidal model structures on* $G\mathrm{Spc}_\bullet(k)$*. The identity functor from the projective model structures to the closed flasque model structures is a left Quillen equivalence. In particular, the homotopy category* $\mathrm{H}_{G,\bullet}(k)$ *coincides with the one defined in* [HKØ15] *(and hence also with the one defined by Voevodsky in* [Del09]*).*

Proof. The global closed flasque model structure is left proper (in fact proper) as well as cellular and so by [Hir03, Theorem 4.1.1] the left Bousfield localization of this model structure at a set of maps exists (and is again left proper and cellular). This implies that the above model structures exist, are simplicial, left proper, and cellular. Right properness follows from the fact that the local projective and motivic model structures are proper, see [HKØ15, Theorem 4.3].

Every projective cofibration is a closed flasque cofibration and the weak equivalences in the global model structures coincide so that the identity is a Quillen equivalence between the global model structures, and hence a Quillen equivalence on the localized model structures.

To show that these are symmetric monoidal model structures we need to check that if f and g are cofibrations, then the pushout product of $f \,\square\, g$ is also a cofibration and that it is a weak equivalence if either f or g is an acyclic cofibration. It suffices to assume that f and g are generating cofibrations. Note that taking smash products preserve motivic weak equivalences since every object is injective cofibrant. The pushout product of maps of the form $(i_{\mathcal{Z}})_+ \to X_+$ are again of the same form. The pushout product axiom in simplicial sets therefore implies that the pushout product of closed flasque cofibrations is again a cofibration. If one of f or g is a weak equivalence, so is the pushout product because the smash product preserves equivariant motivic equivalences. □

Remark A.5. It follows from the definitions that F is motivic fibrant on $G\mathrm{Sm}/k$ if and only if the following three conditions hold.

1. F is fibrant in the global closed flasque model structure.

(2) F is *equivariant Nisnevich excisive*, i.e., for any distinguished square in $G\mathrm{Sm}/k$ the square

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(Y) \arrow[d] \\ F(A) \arrow[r] & F(B) \end{tikzcd}$$

is homotopy cartesian.

(3) F is $\mathbb{A}^1$-*invariant*, i.e., $F(X \times \mathbb{A}^1) \to F(X)$ is a weak equivalence for all X.

A.2. **Hypercohomology and motivic homotopy.** Let $\mathcal{F}$ be a presheaf of simplicial abelian groups on $G\mathrm{Sm}/k$. Write $\mathcal{N}\mathcal{F}$ for the associated presheaf of normalized cochain complexes. Forgetting the group structure, we view $\mathcal{F}$ as a pointed motivic G-space, with 0 as basepoint. There is a natural isomorphism (see e.g., [MV99, Proposition 2.1.26]) of homotopy classes of maps in $\mathrm{H}^{Nis}_{G,\bullet}(k)$ and sheaf cohomology groups

$$[S^n \wedge X_+, \mathcal{F}]_{\mathrm{H}^{Nis}_{G,\bullet}(k)} \cong H^{-n}_{GNis}(X, (\mathcal{N}\mathcal{F})_{GNis}).$$

Moreover, if $\mathcal{F}$ is equivariant Nisnevich excisive then both of these groups agree with the homotopy group $\pi_n\mathcal{F}(X)$.

In [HVØ15] we introduced an equivariant generalization of Voevodsky's machinery of presheaves with transfers. A presheaf with equivariant transfers is an additive presheaf on the category $G\,\mathrm{Cor}_k$, which has the same objects as $G\mathrm{Sm}/k$ and whose maps are given by $\mathrm{Cor}_k(X,Y)^G$. From the viewpoint of motivic homotopy theory, a fundamental feature of the transfer structure is that it allows one to construct a "small" motivic fibrant replacement.

Write $\mathcal{L}_{GNis}$ for a fibrant replacement functor in the equivariant Nisnevich local model structure of Theorem A.4. If F is a presheaf of abelian groups recall that $C_*F(X)$ is the simplicial abelian group $F(X \times \Delta^\bullet_k)$. By the exponent $\exp(G)$ of a group G we mean the least common multiple of the orders of elements of the group.

Theorem A.6. *Suppose that G is abelian group, $|G|$ is coprime to $\mathrm{char}(k)$, and k contains a primitive $\exp(G)$th-root of unity. Let $\mathcal{F}$ be a presheaf with equivariant transfers. Then $\mathcal{F} \to \mathcal{L}_{GNis}C_*\mathcal{F}$ is a motivic fibrant replacement functor. In particular, we have a natural isomorphism*

$$[S^n \wedge X_+, \mathcal{F}]_{\mathrm{H}_{G,\bullet}(k)} \cong H^{-n}_{GNis}(X, \mathcal{N}C_*\mathcal{F}).$$

Proof. By definition $\mathcal{L}C_*\mathcal{F}$ is closed flasque fibrant and equivariant Nisnevich excisive. It remains to see that $\mathcal{L}_{GNis}C_*\mathcal{F}$ is $\mathbb{A}^1$-invariant. We have natural isomorphisms

$$\pi_n\mathcal{L}_{GNis}C_*\mathcal{F}(X) \cong [S^n \wedge X_+, C_*\mathcal{F}]_{\mathrm{H}^{Nis}_{G,\bullet}(k)} \cong H^{-n}_{GNis}(X, \mathcal{N}C_*\mathcal{F}).$$

The $\mathbb{A}^1$-invariance of this presheaf follows from [HVØ15, Theorem 8.12] together with an application of the spectral sequence

$$H^p_{GNis}(X, \mathcal{H}^q) \Longrightarrow H^{p+q}_{GNis}(X, \mathcal{N}C_*\mathcal{F}),$$

where $\mathcal{H}^q$ is the sheafification of the presheaf $U \mapsto H^q\mathcal{N}C_*\mathcal{F}(U)$. ☐

A.3. **Adjunctions of motivic spaces.** Next we record some motivic analogues of familiar adjunctions in topology relating G-spaces and ordinary spaces. We begin with the adjunctions

$$i := (-)^{triv} : \mathrm{Sm}/k \rightleftarrows G\mathrm{Sm}/k : (-)^G =: \phi,$$
$$\epsilon := G \times - : \mathrm{Sm}/k \rightleftarrows G\mathrm{Sm}/k : (-)^e =: \rho.$$

Here X^{triv} is the scheme X considered as a G-scheme with trivial action, X^G is the fixed points scheme (which is smooth by Luna's Slice Theorem [Lun73, p. 97]), and $(-)^e$ simply forgets the action. We obtain several adjoint pairs of functors on based motivic spaces

$$i^* : \mathrm{Spc}_\bullet(k) \rightleftarrows G\mathrm{Spc}_\bullet(k) : i_*,$$
$$i_* = \phi^* : G\mathrm{Spc}_\bullet(k) \rightleftarrows \mathrm{Spc}_\bullet(k) : \phi_*,$$
$$\epsilon^* : \mathrm{Spc}_\bullet(k) \rightleftarrows G\mathrm{Spc}_\bullet(k) : \epsilon_*,$$
$$\epsilon_* = \rho^* : G\mathrm{Spc}_\bullet(k) \rightleftarrows \mathrm{Spc}_\bullet(k) : \rho_*.$$

Proposition A.7. *Each of the pairs* (i^*, i_*), (ϕ^*, ϕ_*), (ϵ^*, ϵ_*), *and* (ρ^*, ρ_*) *are Quillen adjoints on the closed flasque motivic model structures.*

Proof. Each of the functors i, ϕ, ϵ, and ρ commute with fiber products and preserve closed immersions. It follows that i^*, ϕ^*, ϵ^*, and ρ^* preserve generating cofibrations and generating acyclic cofibrations for the global closed flasque model structure. These are thus left Quillen adjoints on the global closed flasque model structure.

Both i and ϵ send distinguished Nisnevich squares to distinguished equivariant Nisnevich squares. The functor ρ sends distinguished equivariant Nisnevich squares to distinguished Nisnevich squares and by [Her13, Corollary 3.2.6], ϕ does as well. Moreover, i, ϕ, ϵ, and ρ all send maps of the form $X \times \mathbb{A}^1 \to X$ to maps of the same form. It follows by the universal property of Bousfield localization that the functors i^*, ϕ^*, ϵ^*, and ρ^* are also left Quillen functors on the closed flasque motivic model structure.

□

We may now define motivic analogues of the classical change of groups functors.

Definition A.8. Define the

1. *trivial action* functor by $(-)^{triv} := \mathbb{L}i^*$,
2. *G-fixed points* functor by $(-)^G := \mathbb{R}i_*$,
3. *induced motivic G-space* functor by $G_+ \wedge - := \mathbb{L}\epsilon^*$,
4. *coinduced motivic G-space* functor by $F(G_+, -) := \mathbb{R}\rho_*$,
5. *underlying motivic space* functor by $(-)^e := \mathbb{R}\epsilon_*$.

Note that $i_* = \phi^*$ and $\epsilon_* = \rho^*$ are both Quillen left and Quillen right adjoints. We thus have natural motivic equivalences $\mathbb{R}i_* = (-)^G \simeq \mathbb{L}i_*$ and $\mathbb{R}\epsilon_* = (-)^e \simeq \mathbb{L}\epsilon_*$.

In summary, we have adjunctions

$$(-)^{triv} : \mathrm{H}_\bullet(k) \rightleftarrows \mathrm{H}_{G,\bullet}(k) : (-)^G,$$
$$G_+ \wedge - : \mathrm{H}_\bullet(k) \rightleftarrows \mathrm{H}_{G,\bullet}(k) : (-)^e,$$
$$(-)^e : \mathrm{H}_{G,\bullet}(k) \rightleftarrows \mathrm{H}_\bullet(k) : F(G_+, -).$$

We note that $(G_+ \wedge X)^e \simeq \coprod_{|G|} X$. Indeed, we have $\epsilon_*\epsilon^*(X) = \coprod_{|G|} X$ since this formula holds for smooth schemes and both sides commute with colimits. If X^{cof} is a cofibrant replacement for X, then we have the weak equivalences $(G_+ \wedge X)^e = \mathbb{R}\epsilon_*\mathbb{L}\epsilon^*(X) \simeq \mathbb{L}\epsilon_*\mathbb{L}\epsilon^*(X) \simeq \epsilon_*\epsilon^*(X^{cof}) = \coprod_{|G|} X^{cof}$, which establishes the claim. The inclusion $X^e \to (G_+ \wedge X)^e$ at the summand corresponding to $e \in G$ induces the map $i : G_+ \wedge X^e \to G_+ \wedge X$.

Lemma A.9. *The map* $i : G_+ \wedge X^e \xrightarrow{\cong} G_+ \wedge X$ *is an isomorphism in* $\mathrm{H}_{G,\bullet}(k)$.

Proof. We have that $G_+ \wedge (-)^e \cong \mathbb{L}\epsilon^*\mathbb{L}\epsilon_*(-) = \mathbb{L}\epsilon^*\mathbb{L}\rho^*(-)$. Note that these commute with homotopy colimits, as does $G_+ \wedge -$. It thus suffices to assume that X is representable. If Y is a G-scheme, then the map $G \times Y^e \to G \times Y$ given by $(g, y) \mapsto (g, gy)$ yields the desired equivariant isomorphism. $\square$

A.4. **Stable equivariant motivic homotopy theory.** Stable model structures on motivic spectra are constructed in [Jar00], yield the stable motivic homotopy category. A stable equivariant motivic homotopy category was first constructed in [HKO11]. In this paper we will work with model structures for equivariant motivic spectra by using Hovey's machinery [Hov01, Section 8].

Let V be a representation. The associated motivic representation sphere is the quotient $T^V := \mathbb{P}(V \oplus 1)/\mathbb{P}(V)$, where the quotient is taken in the category of presheaves. It is naturally an object of $G\mathrm{Spc}_\bullet(k)$. We also write $T^{nV} := (T^V)^{\wedge n}$. Since $\mathbb{P}(V) \to \mathbb{P}(V \oplus 1)$ is a closed flasque cofibration, T^V is closed flasque cofibrant motivic G-space. We write ρ_G for the regular representation.

Let K be a pointed motivic G-space. We write $\mathbf{hom}(K, -)$ for the right adjoint of $K \wedge -$. In particular, $\Sigma_{T^{\rho_G}} F = T^{\rho_G} \wedge F$ and $\Omega_{T^{\rho_G}} F = \mathbf{hom}(T^{\rho_G}, F)$.

Definition A.10. A symmetric K-spectrum is a sequence $E = (E_0, E_1, \dots)$ consisting of objects E_n of $G\mathrm{Spc}_\bullet(k)$ together with the following data

(i) a Σ_n-action on E_n,
(ii) Σ_n-equivariant maps $\sigma_n : E_n \wedge K \to E_{n+1}$,

where the structure maps σ_n are required to satisfy the condition that the composites $E_n \wedge K^{\wedge p} \to E_{n+p}$ are $\Sigma_n \times \Sigma_p$-equivariant for all $n, p \geq 0$.

A map $E \to F$ of symmetric K-spectra is a collection of Σ_n-equivariant maps $E_n \to F_n$ which are compatible with the structure maps. Write $G\mathrm{Spt}^{\Sigma}_K(k)$ for the category of symmetric K-spectra in $G\mathrm{Spc}_\bullet(k)$. Now suppose that K is a closed flasque cofibrant based motivic G-space. The category $G\mathrm{Spc}_\bullet(k)$ equipped with the closed flasque motivic model structure is a left proper, cellular, simplicial, symmetric monoidal model category, and so we can use [Hov01, Definition 8.7] to define a stable model structure on the category of symmetric K-spectra. It is again a symmetric monoidal model category by [Hov01, Theorem 8.11]. If K' is another closed flasque cofibrant based motivic G-space, we write $G\mathrm{Spt}^{\Sigma}_{K,K'}(k)$ for the category of symmetric (K, K')-bispectra (i.e., symmetric K'-spectra in $G\mathrm{Spt}^{\Sigma}_K(k)$). This is again equipped with the stable model structure of [Hov01, Section 8].

A symmetric K-spectrum $\mathcal{E}$ is fibrant if and only if it is an Ω_K-spectrum, i.e., E_i is motivic fibrant and $\mathcal{E}_i \to \Omega_K E_{i+1}$ is a motivic weak equivalence for all i. By Theorem 3.4 the Bredon motivic cohomology spectrum $\mathbf{M}\underline{A}$ is an $\Omega_{T^{\rho_{C_2}}}$-spectrum.

The adjunctions of Section A.3 stabilize.

Proposition A.11. *The adjoint pairs of Section A.3 induce adjoint pairs*

$$\begin{aligned} (-)^{triv} : \mathrm{SH}(k) &\rightleftarrows \mathrm{SH}_G(k) : (-)^G, \\ G_+ \wedge - : \mathrm{SH}(k) &\rightleftarrows \mathrm{SH}_G(k) : (-)^e, \\ (-)^e : \mathrm{SH}_G(k) &\rightleftarrows \mathrm{SH}(k) : F(G_+, -). \end{aligned}$$

The functors $(-)^{triv}$*, and* $(-)^e$ *are symmetric monoidal and* $(-)^G$ *is lax monoidal.*

Proof. First we note that we have an equivalence, as tensor triangulated categories

$$\mathrm{SH}_G(k) \simeq \mathrm{Ho}(G\mathrm{Spt}^{\Sigma}_{T,T^{\rho_G}}(k))$$

where the category $G\mathrm{Spt}^{\Sigma}_{T,T^{\rho_G}}(k)$ of symmetric (T, T^{ρ_G})-bispectra equipped with the stable model structure as above. Indeed, the model category $G\mathrm{Spt}^{\Sigma}_{T,T^{\rho_G}}(k)$ is isomorphic to $G\mathrm{Spt}^{\Sigma}_{T^{\rho_G},T}(k)$ and the endofunctor $-\wedge T$ on $G\mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(k)$ is a Quillen equivalence, since $T \wedge T^{\tilde{\rho}_G} = T^{\rho_G}$ (where $\tilde{\rho}_G$ is the reduced regular representation). By [Hov01, Theorem 9.1], the stabilization functor (which is symmetric monoidal) $G\mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(k) \to G\mathrm{Spt}^{\Sigma}_{T^{\rho_G},T}(k)$ is therefore a left Quillen equivalence.

The monoidal functor $i^* : \mathrm{Spc}_\bullet(k) \to G\mathrm{Spc}_\bullet(k)$ makes $G\mathrm{Spc}_\bullet(k)$ into a $\mathrm{Spc}_\bullet(k)$-model category in the sense of [Hov99, Definition 4.2.18] and the Quillen functor i^* is a $\mathrm{Spc}_\bullet(k)$-module functor (see [Hov99, Definition 4.1.7]). By [Hov01, Theorem 9.3], the Quillen pairs (i^*, i_*) extends to a Quillen pair on spectra. Combined with the stablization adjunction, we have the composition of Quillen functors

$$\begin{tikzcd} \mathrm{Spt}^{\Sigma}_{T}(k) \arrow[r, shift left, "i^*"] & G\mathrm{Spt}^{\Sigma}_{T}(k) \arrow[l, shift left, "i_*"] \arrow[r, shift left, "\Sigma^{\infty}_{T^{\rho_G}}"] & G\mathrm{Spt}^{\Sigma}_{T,T^{\rho_G}}(k). \arrow[l, shift left, "\Omega^{\infty}_{T^{\rho_G}}"] \end{tikzcd}$$

The pair $((-)^{triv}, (-)^G)$ is the induced adjunction on homotopy categories, i.e., $(-)^{triv} = \mathbb{L}(\Sigma^{\infty}_{T^{\rho_G}} \circ i^*)$ and $(-)^G = \mathbb{R}(i_* \circ \Omega^{\infty}_{T^{\rho_G}})$.

The other two adjunctions are attained as follows. We use $\mathrm{Spt}^{\Sigma}_{T^{|G|}}(k)$ as our model for $\mathrm{SH}(k)$. Note for X in $\mathrm{Spc}_\bullet(k)$ and Y in $G\mathrm{Spc}_\bullet(k)$ we have a natural isomorphism $\epsilon^*(X \wedge \rho^*(Y)) \cong \epsilon^*(X) \wedge Y$. Indeed, this holds when X is in $\mathrm{Sm}k$ and Y is in $G\mathrm{Sm}/k$ and both sides commute with colimits. Since $\rho^*(T^{\rho_G}) = T^{|G|}$, we have $\epsilon^*(X \wedge T^{|G|}) = \epsilon^*(X) \wedge T^{\rho_G}$. We also have $\rho^*(Y \wedge T^{\rho_G}) = \rho^*(Y) \wedge T^{|G|}$. By [HO14, Lemma 4.1], the adjunctions (ϵ^*, ϵ_*) and (ρ^*, ρ_*) extend to Quillen adjunctions on stable model structures

$$\begin{aligned} \epsilon^* : \mathrm{Spt}^{\Sigma}_{T^{|G|}}(k) &\rightleftarrows G\mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(k) : \epsilon_*, \\ \rho^* : G\mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(k) &\rightleftarrows \mathrm{Spt}^{\Sigma}_{T^{|G|}}(k) : \rho_*. \end{aligned}$$

We have $\epsilon_* = \rho^* : G\mathrm{Spc}_\bullet(k) \to \mathrm{Spc}_\bullet(k)$. The prolongations to functors on spectra are levelwise isomorphic and it is straightforward to verify that they are in fact isomorphic as spectra. In other words, $\epsilon_* = \rho^* : G\mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(k) \to \mathrm{Spt}^{\Sigma}_{T^{|G|}}(k)$. In particular, $\mathbb{R}\epsilon_* \simeq \mathbb{L}\rho^*$. The adjunctions $(G_+ \wedge -, (-)^e)$ and $((-)^e, F(G_+, -))$ on the homotopy categories are thus obtained as $G_+ \wedge - = \mathbb{L}\epsilon^*$, $F(G_+, -) = \mathbb{R}\rho_*$, and $(-)^e = \mathbb{L}\rho^*$.

Since i^* and ρ^* are symmetric monoidal functors, so are $(-)^{triv}$ and $(-)^e$. Since $(-)^G$ is right adjoint to a symmetric monoidal functor, it is lax monoidal. □

In [Her13] a different homotopy category of equivariant motivic spectra is constructed. He uses a different topology, the fixed point Nisnevich topology, see loc. cit. for details. We obtain a model structure Quillen equivalent to his by starting with the global flasque model structure on $G\mathrm{Spc}_\bullet(k)$ and then localizing at the fixed point Nisnevich equivalences and $\mathbb{A}^1$-equivalences; write $G\mathrm{Spc}_\bullet(k)^{fp}$ for this model structure. Starting with this model structure on motivic G-spaces and imposing the corresponding stable model structure on symmetric T^{ρ_G}-spectra using [Hov01] as above results in a model structure which we denote by $G\mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(k)^{fp}$.

Lemma A.12. *The model categories $G\mathrm{Spc}_\bullet(k)^{fp}$ and $G\mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(k)^{fp}$ are Quillen equivalent respectively to the model structures on motivic G-spaces and on symmetric T^{ρ_G}-spectra introduced in* [Hov01].

Proof. In loc. cit., Herrmann starts with the global injective model structure on presheaves of simplicial sets on $G\mathrm{Sm}/k$ rather than the global closed flasque model structure. The identity is a left Quillen equivalence from the global closed flasque model structure to the global injective model structure. The lemma thus follows from standard facts about Bousfield localization. □

A.5. **Topological realization over $\mathbb{C}$ (unstable).** If X is a complex variety, we consider $X(\mathbb{C})$ as a topological space with the Euclidean topology. If X has a G-action, then $X(\mathbb{C})$ also has a G-action and $X \mapsto X(\mathbb{C})$ defines a functor $G\mathrm{Sm}/\mathbb{C} \to G\mathrm{Top}$. The topological realization functor $\mathrm{Re}_{\mathbb{C}} : G\mathrm{Spc}_\bullet(\mathbb{C}) \to G\mathrm{Top}_\bullet$ is defined by the Kan extension

$$\mathrm{Re}_{\mathbb{C}}(F) = \operatorname*{colim}_{(X\times\Delta^n)_+\to F} (X(\mathbb{C}) \times \Delta^n_{top})_+$$

where Δ^n_{top} is the standard topological n-simplex, considered with trivial action. It has a right adjoint $K \mapsto \mathcal{S}\mathrm{ing}_{\mathbb{C}}(K)$, defined by $\mathcal{S}\mathrm{ing}_{\mathbb{C}}(K)(X) = \underline{\mathrm{Hom}}_G(X(\mathbb{C}), K)$ (where $\underline{\mathrm{Hom}}_G(-,-)$ is the simplicial set of continuous equivariant maps). Equip $G\mathrm{Top}_\bullet$ with the model structure where a map $X \to Y$ is a weak equivalence or a fibration if $X^H \to Y^H$ is a weak equivalence or a fibration for all subgroups of G, see e.g., [MM02, Theorem III.1.8]. The resulting homotopy category $\mathrm{H}_{G,\bullet}$ is the classical unstable equivariant homotopy category. For the corresponding model structure on G-simplicial sets, see e.g., [DRØ03, §9.2].

Proposition A.13. *The adjoint pair*

$$\mathrm{Re}_{\mathbb{C}} : G\mathrm{Spc}_\bullet(\mathbb{C}) \rightleftarrows G\mathrm{Top}_\bullet : \mathcal{S}\mathrm{ing}_{\mathbb{C}}$$

is a Quillen adjunction and $\mathrm{Re}_{\mathbb{C}}$ *is a symmetric monoidal functor. Additionally,* $\mathrm{Re}_{\mathbb{C}}$ *is a left Quillen functor on* $G\mathrm{Spc}_\bullet(\mathbb{C})^{fp}$.

Proof. The argument is similar to that given in [PPR09, Theorem A.23]. First we show that $\mathrm{Re}_{\mathbb{C}}$ is a left adjoint on the closed flasque model structure. For any finite collection of closed immersions $\{Z_i \to X\}$ in $G\mathrm{Sm}/\mathbb{C}$, $\mathrm{Re}_{\mathbb{C}}((i_{\mathcal{Z}})_+)$ is the inclusion of an equivariant subcomplex, and hence an equivariant cofibration. It follows that $\mathrm{Re}_{\mathbb{C}}$ sends I^c to cofibrations in $G\mathrm{Top}_\bullet$ and J^c to weak homotopy equivalences in $G\mathrm{Top}_\bullet$. This implies that $\mathcal{S}\mathrm{ing}_{\mathbb{C}}$ preserves trivial fibrations as well as fibrations between fibrant objects. By Dugger's lemma [Dug01, Corollary A2], $\mathrm{Re}_{\mathbb{C}}$ is thus a left Quillen functor on the closed flasque global model structure.

Next we claim that $\mathrm{Re}_{\mathbb{C}}$ sends fixed point Nisnevich distinguished squares to homotopy pushouts. By [Her13, Corollary 2.13], an equivariant Nisnevich square is also a fixed point Nisnevich square. A square in $G\mathrm{Top}_\bullet$ is a homotopy pushout if and only if it is so on all fixed points. Since $X^H(\mathbb{C}) = X(\mathbb{C})^H$, it suffices to show that the square obtained by taking complex points of a distinguished Nisnevich square is a homotopy pushout square in topological spaces. This follows from [DI04, Theorem 5.2], establishing the claim. As $\mathbb{A}^1(\mathbb{C})$ is equivariantly contractible, $\mathrm{Re}_{\mathbb{C}}(X_+) \to \mathrm{Re}_{\mathbb{C}}((X \times \mathbb{A}^1)_+)$ is an equivariant homotopy equivalence. It follows by the universal property of Bousfield localization that $\mathrm{Re}_{\mathbb{C}}$ is a left Quillen functor on the closed flasque motivic model structure as well as on $G\mathrm{Spc}_\bullet(\mathbb{C})^{fp}$.

That $\mathrm{Re}_{\mathbb{C}}$ is symmetric monoidal is a simple consequence of the fact that there is a natural equivariant homeomorphism $(X \times Y)(\mathbb{C}) \cong X(\mathbb{C}) \times Y(\mathbb{C})$. □

Remark A.14. That $\mathrm{Re}_{\mathbb{C}}$ is a left Quillen functor on $\mathrm{Spc}_\bullet(\mathbb{C})^{fp}$ has the following useful consequence. If $X \to Y$ is a map of motivic G-spaces such that $X^H \to Y^H$ is a motivic weak equivalence for all subgroups H of G, then the induced map $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(X) \to \mathbb{L}\mathrm{Re}_{\mathbb{C}}(Y)$ is an equivariant weak equivalence.

Proposition A.15. *The squares of left adjoints commutes up to natural isomorphisms*

$$\begin{tikzcd}
\mathrm{H}_\bullet(\mathbb{C}) \arrow[r, "G_+\wedge -"] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & \mathrm{H}_{G,\bullet}(\mathbb{C}) \arrow[d, "\mathrm{Re}_{\mathbb{C}}"] & \mathrm{H}_\bullet(\mathbb{C}) \arrow[r, "(-)^{triv}"] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & \mathrm{H}_{G,\bullet}(\mathbb{C}) \arrow[d, "\mathrm{Re}_{\mathbb{C}}"] & \mathrm{H}_{G,\bullet}(\mathbb{C}) \arrow[r, "(-)^{e}"] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & \mathrm{H}_\bullet(\mathbb{C}) \arrow[d, "\mathrm{Re}_{\mathbb{C}}"] \\
\mathrm{H}_\bullet \arrow[r, "G_+\wedge -"] & \mathrm{H}_{G,\bullet}, & \mathrm{H}_\bullet \arrow[r, "(-)^{triv}"] & \mathrm{H}_{G,\bullet}, & \mathrm{H}_{G,\bullet} \arrow[r, "(-)^{e}"] & \mathrm{H}_\bullet.
\end{tikzcd}$$

Proof. Using the notation from Section A.3, we have a natural isomorphism $\mathrm{Re}_{\mathbb{C}} \circ \epsilon^*(-) \cong G_+ \wedge \mathrm{Re}_{\mathbb{C}}(-)$ of functors $\mathrm{Spc}_\bullet(\mathbb{C}) \to G\mathrm{Top}$. Indeed, this is clear on representable motivic spaces and since these functors commute with colimits, this suffices. Similarly, we have natural isomorphisms $\mathrm{Re}_{\mathbb{C}} \circ i^*(-) \cong (\mathrm{Re}_{\mathbb{C}}(-))^{triv}$ and $\mathrm{Re}_{\mathbb{C}} \circ \rho^*(-) \cong (\mathrm{Re}_{\mathbb{C}}(-))^e$. These isomorphisms imply the isomorphisms of derived functors on the homotopy categories. □

A.6. **Topological realization over $\mathbb{C}$ (stable).** Now we turn our attention to a stable realization functor. For any real orthogonal representation V there is a representation sphere $S^V = V_+$, where $(-)_+$ denotes the one-point compactification. Since $\mathbb{C}[G] = \mathbb{R}[G] \oplus \mathbb{R}[G]$ as real representations, we have $\mathrm{Re}_{\mathbb{C}}(T^{\rho_G}) = S^{2\rho_G}$.

If E is a motivic G-spectrum, define the topological $S^{2\rho_G}$-spectrum $\mathrm{Re}_{\mathbb{C}}\mathsf{E}$ by $(\mathrm{Re}_{\mathbb{C}}\mathsf{E})_i = \mathrm{Re}_{\mathbb{C}}\mathsf{E}_i$ with structure maps

$$\mathrm{Re}_{\mathbb{C}}\mathsf{E}_i \wedge S^{2\rho_G} = \mathrm{Re}_{\mathbb{C}}(\mathsf{E}_i \wedge T^{\rho_G}) \to \mathrm{Re}_{\mathbb{C}}\mathsf{E}_{i+1}.$$

The functor $\mathcal{S}\mathrm{ing}_{\mathbb{C}}$ extends as well to a functor on $S^{2\rho_G}$-spectra. In fact, we obtain an adjoint pair of functors

$$\mathrm{Re}_{\mathbb{C}} : \mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(\mathbb{C}) \rightleftarrows \mathrm{Spt}^{\Sigma}_{S^{2\rho_G}}(G\mathrm{Top}_\bullet) : \mathcal{S}\mathrm{ing}_{\mathbb{C}}.$$

Theorem A.16. *The adjoint pairs*

$$\mathrm{Re}_{\mathbb{C}} : \mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(\mathbb{C}) \rightleftarrows \mathrm{Spt}^{\Sigma}_{S^{2\rho_G}}(G\mathrm{Top}_\bullet) : \mathcal{S}\mathrm{ing}_{\mathbb{C}},$$
$$\mathrm{Re}_{\mathbb{C}} : \mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(\mathbb{C})^{fp} \rightleftarrows \mathrm{Spt}^{\Sigma}_{S^{2\rho_G}}(G\mathrm{Top}_\bullet) : \mathcal{S}\mathrm{ing}_{\mathbb{C}}$$

are both Quillen adjoint pairs. Moreover, $\mathrm{Re}_{\mathbb{C}}$ *is symmetric monoidal.*

Proof. This is straightforward from Proposition A.13, cf. [PPR09, Theorem A.45]. □

Proposition A.17. *The squares of left adjoints commute up to natural isomorphisms*

$$\begin{tikzcd}
\mathrm{SH}(\mathbb{C}) \arrow[r, "G_+\wedge -"] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & \mathrm{SH}_G(\mathbb{C}) \arrow[d, "\mathrm{Re}_{\mathbb{C}}"] & \mathrm{SH}(\mathbb{C}) \arrow[r, "(-)^{triv}"] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & \mathrm{SH}_G(\mathbb{C}) \arrow[d, "\mathrm{Re}_{\mathbb{C}}"] & \mathrm{SH}_G(\mathbb{C}) \arrow[r, "(-)^e"] \arrow[d, "\mathrm{Re}_{\mathbb{C}}"'] & \mathrm{SH}(\mathbb{C}) \arrow[d, "\mathrm{Re}_{\mathbb{C}}"] \\
\mathrm{SH} \arrow[r, "G_+\wedge -"] & \mathrm{SH}_G, & \mathrm{SH} \arrow[r, "(-)^{triv}"] & \mathrm{SH}_G, & \mathrm{SH}_G \arrow[r, "(-)^e"] & \mathrm{SH}.
\end{tikzcd}$$

Proof. Straightforward using the unstable result in Proposition A.15. ☐

A.7. **Topological realization over** $\mathbb{R}$. Write $\Sigma_2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. If X is a real variety with G-action, then $X(\mathbb{C})$ is a $(G\times\Sigma_2)$-space, where the Σ_2-action is via complex conjugation. We thus have a functor $G\mathrm{Sm}/\mathbb{R} \to (G\times\Sigma_2)\mathrm{Top}$ which induces the topological realization functor $\mathrm{Re}_{\mathbb{C},\Sigma_2} : G\mathrm{Spc}_\bullet(\mathbb{C}) \to (G\times\Sigma_2)\mathrm{Top}_\bullet$, defined by the Kan extension

$$\mathrm{Re}_{\mathbb{C},\Sigma_2}(F) = \operatorname*{colim}_{(X\times\Delta^n)_+\to F} (X(\mathbb{C})\times\Delta^n_{top})_+.$$

Its right adjoint is defined by $\mathcal{S}\mathrm{ing}_{\mathbb{C},\Sigma_2}(K)(X) = \underline{\mathrm{Hom}}_{G\times\Sigma_2}(X(\mathbb{C}),K)$, where K is a $G\times\Sigma_2$-space and X is a smooth real G-variety. Equip $(G\times\Sigma_2)\mathrm{Top}_\bullet$ with the model structure where a map $X\to Y$ is a weak equivalence or a fibration if $X^H\to Y^H$ is a weak equivalence or a fibration for all subgroups of G, see e.g., [MM02, Theorem III.1.8]. The resulting homotopy category $\mathrm{H}_{\bullet,G\times\Sigma_2}$ is the classical unstable equivariant homotopy category.

Proposition A.18. *The adjoint pair*

$$\mathrm{Re}_{\mathbb{C},\Sigma_2} : G\mathrm{Spc}_\bullet(\mathbb{R}) \rightleftarrows (G\times\Sigma_2)\mathrm{Top}_\bullet : \mathcal{S}\mathrm{ing}_{\mathbb{C},\Sigma_2}$$

is a Quillen adjunction and $\mathrm{Re}_{\mathbb{C},\Sigma_2}$ *is a symmetric monoidal functor. Additionally,* $\mathrm{Re}_{\mathbb{C},\Sigma_2}$ *is a left Quillen functor on* $G\mathrm{Spc}_\bullet(\mathbb{R})^{fp}$.

Proof. The argument is similar to Proposition A.13. ☐

Now we turn our attention to a stable realization functor. Since $\mathbb{C}[G]=\mathbb{R}[G\times\Sigma_2]$ as real representations, we have $\mathrm{Re}_{\mathbb{C}}(T^{\rho_G}) = S^{\rho_{G\times\Sigma_2}}$.

If E is a motivic G-spectrum, define the topological $S^{\rho_{G\times\Sigma_2}}$-spectrum $\mathrm{Re}_{\mathbb{C},\Sigma_2}\mathsf{E}$ by $(\mathrm{Re}_{\mathbb{C},\Sigma_2}\mathsf{E})_i = \mathrm{Re}_{\mathbb{C},\Sigma_2}\mathsf{E}_i$ with structure maps

$$\mathrm{Re}_{\mathbb{C},\Sigma_2}\mathsf{E}_i \wedge \mathrm{Re}_{\mathbb{C},\Sigma_2}(S^{\rho_{G\times\Sigma_2}}) = \mathrm{Re}_{\mathbb{C},\Sigma_2}(\mathsf{E}_i\wedge T^{\rho_G}) \to \mathrm{Re}_{\mathbb{C},\Sigma_2}\mathsf{E}_{i+1}.$$

The functor $\mathcal{S}\mathrm{ing}_{\mathbb{C}}$ extends as well to a functor on $S^{\rho_{G\times\Sigma_2}}$-spectra, adjoint to $\mathrm{Re}_{\mathbb{C},\Sigma_2}$.

Theorem A.19. *The adjoint pairs*

$$\mathrm{Re}_{\mathbb{C},\Sigma_2} : \mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(\mathbb{C}) \rightleftarrows \mathrm{Spt}^{\Sigma}_{S^{\rho_{G\times\Sigma_2}}}((G\times\Sigma_2)\mathrm{Top}_\bullet) : \mathcal{S}\mathrm{ing}_{\mathbb{C},\Sigma_2},$$
$$\mathrm{Re}_{\mathbb{C},\Sigma_2} : \mathrm{Spt}^{\Sigma}_{T^{\rho_G}}(\mathbb{C})^{fp} \rightleftarrows \mathrm{Spt}^{\Sigma}_{S^{\rho_{G\times\Sigma_2}}}((G\times\Sigma_2)\mathrm{Top}_\bullet) : \mathcal{S}\mathrm{ing}_{\mathbb{C},\Sigma_2}$$

are both Quillen adjoint pairs Moreover, $\mathrm{Re}_{\mathbb{C},\Sigma_2}$ *is symmetric monoidal.*

Proof. This is straightforward from Proposition A.18, cf. [PPR09, Theorem A.45]. ☐

A.8. **Symmetric powers.** In order to analyze the topological realization of the Bredon motivic cohomology spectrum, we need to make precise that it is represented by symmetric powers. In this subsection we introduce and analyze symmetric powers for motivic G-spaces. This discussion parallels the treatment of symmetric powers in [Del09] and [Voe10a], and we mostly follow [Lev14, Appendix]. Throughout this subsection, we assume that $\mathrm{char}(k) = 0$.

Write $G\mathrm{Sm}'/k$ for the category of smooth, quasi-projective G-schemes over k and $G\mathrm{Spc}'(k)$ and $G\mathrm{Spc}'_\bullet(k)$ for the corresponding categories of simplicial presheaves and pointed simplicial presheaves. These can also be given a motivic model structure as above. The inclusion $\phi : G\mathrm{Sm}'/k \subseteq G\mathrm{Sm}/k$ induces an adjoint pair $\phi^* : G\mathrm{Spc}'(k) \rightleftarrows G\mathrm{Spc}(k) : \phi_*$ and similarly for based motivic spaces.

Lemma A.20. *The adjunction $\phi^* : G\mathrm{Spc}'(k) \rightleftarrows G\mathrm{Spc}(k) : \phi_*$ is a Quillen equivalence on motivic model structures. Similarly for based motivic G-spaces.*

Proof. This follows easily from the fact that smooth G-schemes are locally affine in the equivariant Nisnevich topology, see Remark 2.3. ☐

Write $G\mathrm{Sch}'/k$ for the category of reduced, quasi-projective G-schemes of finite type over k. Consider the functor $(-)^{\times n} : G\mathrm{Sch}'/k \to (\Sigma_n \times G)\mathrm{Sch}'/k$ which sends a G-scheme X to the n-fold product $X^{\times n}$, where G acts diagonally and Σ_n acts by permuting the factors. The composition of $(-)^{\times n}$ and the Yoneda embedding gives us a functor $G\mathrm{Sch}'/k \to \mathrm{sPre}((\Sigma_n \times G)\mathrm{Sch}'/k)$ and we define $\Pi^{(n)}(-)$ to be its left Kan extension, yielding the functor

$$\Pi^{(n)} : \mathrm{sPre}(G\mathrm{Sch}'/k) \to \mathrm{sPre}((\Sigma_n \times G)\mathrm{Sch}'/k),$$

and similarly for pointed motivic spaces.

Let $N \trianglelefteq K$ be a normal subgroup of a group K and write $\Gamma = K/N$ for the quotient group. If X is a quasi-projective K-scheme over k then a quotient scheme X/N exists and the K-action on X induces a Γ-action on the scheme X/N. We write $q_{K,N} : K\mathrm{Sch}'/k \to \Gamma\mathrm{Sch}/k$ for the quotient functor, i.e., $q_{K,N}(X) = X/N$. The functor $q_{K,N}$ induces an adjoint pair of functors

$$(q_{K,N})^* : \mathrm{sPre}(K\mathrm{Sch}'/k) \rightleftarrows \mathrm{sPre}(\Gamma\mathrm{Sch}/k) : (q_{K,N})_*$$

and similarly for based presheaves. For $\mathcal{X}$ in $K\mathrm{Spc}'(k)$, define $\mathcal{X}/N := (q_{K,N})^*(\mathcal{X})$. If $\mathcal{X}$ is represented by a quasi-projective G-scheme X, then $\mathcal{X}/N$ is represented by X/N.

Define the *nth symmetric product* $\mathrm{Sym}^n : \mathrm{sPre}(G\mathrm{Sch}'/k) \to \mathrm{sPre}(G\mathrm{Sch}'/k)$ by

$$\mathrm{Sym}^n(\mathcal{X}) := \Pi^{(n)}(\mathcal{X})/\Sigma_n.$$

The inclusion of categories $i : G\mathrm{Sm}'/k \subseteq G\mathrm{Sch}'/k$ induces an adjoint pair of functors

$$i^* : G\mathrm{Spc}'(k) \rightleftarrows \mathrm{sPre}(G\mathrm{Sch}'/k) : i_*$$

and similarly for pointed motivic spaces. Note that i_* preserves colimits, as these are computed sectionwise. If $\mathcal{X}$ is an object of $\mathrm{sPre}(G\mathrm{Sch}'/k)$, when no confusion should arise we again write $\mathrm{Sym}^n(\mathcal{X})$ for $i_*(\mathrm{Sym}^n(\mathcal{X}))$ in $G\mathrm{Spc}'(k)$ and also for $\phi^* i_*(\mathrm{Sym}^n(k))$ in $G\mathrm{Spc}(k)$.

If $\mathcal{X}$ is an object of $\mathrm{sPre}_\bullet(G\mathrm{Sch}'/k)$, then $\mathrm{Sym}^n(\mathcal{X})$ has a canonical basepoint and we write $\mathrm{Sym}^n_\bullet(\mathcal{X})$ for the corresponding pointed motivic space in $G\mathrm{Spc}'_\bullet(k)$.

Remark A.21. Our discussion of equivariant symmetric products here roughly follows the nonequivariant treatment in [Lev14]. To avoid confusion it should be noted that our definition is slightly different than the one given there. The difference is that we use $\Pi^{(n)}(\mathcal{X})$ (which is a presheaf of simplicial sets, defined via the Kan extension of the n-fold product functor on schemes) instead of $\mathcal{X}^{\times n}$ (the n-fold product of presheaves viewed as a presheaf of Σ_n-simplicial sets). Of course the resulting symmetric products agree in the cases under consideration (namely the quotient of a scheme by a closed subscheme).

We have addition and multiplication maps,

$$\begin{aligned} &\mathrm{Sym}^m_\bullet(\mathcal{X}) \times \mathrm{Sym}^n_\bullet(\mathcal{X}) \xrightarrow{\sigma_{m,n}} \mathrm{Sym}^{m+n}_\bullet(\mathcal{X}) \text{ and} \\ &\mathrm{Sym}^m_\bullet(\mathcal{X}) \wedge \mathrm{Sym}^n_\bullet(\mathcal{Y}) \xrightarrow{\mu_{m,n}} \mathrm{Sym}^{mn}_\bullet(\mathcal{X} \wedge \mathcal{Y}), \end{aligned}$$

obtained as follows. By definition $(\Pi^{(m)}\mathcal{X})/\Sigma_m = \mathrm{colim}_{X\to\mathcal{X}}(X^{\times m})/\Sigma_m$. The map $\sigma_{m,n}$ is induced by the obvious map $(X^{\times m})/\Sigma_m \times (X^{\times n})/\Sigma_n \to (X^{\times(m+n)})/\Sigma_{m+n}$. Similarly, $\mu_{m,n}$ is induced by the map $X^{\times m}/\Sigma_m \times Y^n/\Sigma_n \to (X \times Y)^{mn}/\Sigma_{mn}$. Stabilization maps $st_{n+1} := \sigma_{n,1}(-,*) : \mathrm{Sym}^n_\bullet(\mathcal{X}) \to \mathrm{Sym}^{n+1}_\bullet(\mathcal{X})$ are thus obtained by adding the basepoint. Define

$$\mathrm{Sym}^\infty_\bullet(\mathcal{X}) := \underset{n}{\mathrm{colim}}(\mathcal{X} \xrightarrow{st_2} \mathrm{Sym}^2_\bullet(\mathcal{X}) \xrightarrow{st_3} \mathrm{Sym}^3_\bullet(\mathcal{X}) \xrightarrow{st_4} \cdots)$$

Let (X, s) be a pointed quasi-projective G-scheme and let $A \subseteq X$ be a closed reduced invariant subscheme which contains the basepoint s. Write $(X, A)^{(n)} \subseteq X^{\times n}$ for the $\Sigma_n \times G$-orbit of $A \times X^{\times n-1}$, i.e., it is the subscheme consisting of tuples $(x_1, \ldots, x_n)$ such that at least one x_i is in A. Write $\mathrm{Sym}^n_\bullet(X, A) := (X, A)^{(n)}/\Sigma_n$. Write $\overline{\pi}_{1,n-1} : A \times \mathrm{Sym}^{n-1}_\bullet(X) \to \mathrm{Sym}^{n-1}_\bullet(X/A)$ for the composition of the projection $A \times \mathrm{Sym}^{n-1}_\bullet(X) \to \mathrm{Sym}^{n-1}_\bullet(X)$ followed by the quotient map $\mathrm{Sym}^{n-1}_\bullet(X) \to \mathrm{Sym}^{n-1}_\bullet(X/A)$. Restricting the addition map $\sigma'_{1,n-1} : A \times \mathrm{Sym}^{n-1}_\bullet(X) \to \mathrm{Sym}^n_\bullet(X)$, we have a finite equivariant map. Its image is a closed subset which we equip with the reduced induced scheme structure and this subscheme agrees with the subscheme $\mathrm{Sym}^n_\bullet(X, A) \subseteq \mathrm{Sym}^n_\bullet(X)$.

Proposition A.22. *There is a pushout square in* $G\mathrm{Spc}_\bullet(k)$

(A.23)
$$\begin{tikzcd} \mathrm{Sym}^n_\bullet(X, A) \arrow[r] \arrow[d, "\pi_n"'] & \mathrm{Sym}^n_\bullet(X) \arrow[d] \\ \mathrm{Sym}^{n-1}_\bullet(X/A) \arrow[r, "st_n"] & \mathrm{Sym}^n_\bullet(X/A) \end{tikzcd}$$

such that $\pi_n \circ \sigma'_{1,n-1} = \overline{\pi}_{1,n-1}$. *Moreover, this is a homotopy pushout square.*

Proof. The last statement that this is a homotopy pushout will follow by showing it is a pushout square, since $\mathrm{Sym}^n_\bullet(X, A) \to \mathrm{Sym}^n_\bullet(X)$ is a monomorphism, which is a cofibration for the global injective model structure.

It suffices to establish the pushout square in $G\mathrm{Spc}'_\bullet(k)$. Note that we have $A^{\times n} \subseteq (X, A)^{(n)}$ and write $\Pi^{(n)}(X/A, *) = (X, A)^{(n)}/A^{\times n}$ for the quotient in

$\mathrm{sPre}_\bullet((\Sigma_n \times G)\mathrm{Sch}'/k)$. We thus have pushout squares in $\mathrm{sPre}_\bullet((\Sigma_n \times G)\mathrm{Sch}'/k)$

$$\begin{tikzcd}
A^{\times n} \arrow[r, hook] \arrow[d] & (X,A)^{(n)} \arrow[r, hook] \arrow[d] & X^{\times n} \arrow[d] \\
* \arrow[r] & \Pi^{(n)}(X/A,*) \arrow[r, hook] & \Pi^{(n)}(X/A).
\end{tikzcd}$$

We write $\mathrm{Sym}^n_\bullet(X/A,*) = \Pi^{(n)}(X/A,*)/\Sigma_n$. Applying the functors $(q_{\Sigma_n\times G,\Sigma_n})^*$ and i_* to the previous pushout squares, we obtain the pushout squares in $G\mathrm{Spc}'_\bullet(k)$,

$$\begin{tikzcd}
\mathrm{Sym}^n_\bullet(A) \arrow[r] \arrow[d] & \mathrm{Sym}^n_\bullet(X,A) \arrow[r, hook, "i"] \arrow[d, "\tilde{\pi}_n"] & \mathrm{Sym}^n_\bullet(X) \arrow[d] \\
* \arrow[r] & \mathrm{Sym}^n_\bullet(X/A,*) \arrow[r, hook, "j"] & \mathrm{Sym}^n_\bullet(X/A).
\end{tikzcd}$$

It remains to construct an isomorphism $\mathrm{Sym}^n(X/A,*) \cong \mathrm{Sym}^{n-1}_\bullet(X/A)$ in $G\mathrm{Spc}'_\bullet(k)$ and show that j is identified with st_n and $\tilde{\pi}_n \circ \sigma'_{1,n-1}$ is identified with $\overline{\pi}_{1,n-1}$.

Addition of a basepoint gives a commutative diagram of maps in $G\mathrm{Spc}'_\bullet(k)$

$$\begin{tikzcd}
\mathrm{Sym}^{n-1}_\bullet(A) \arrow[r] \arrow[d, "\sigma_{n,1}(-{,}*)"] & \mathrm{Sym}^{n-1}_\bullet(X) \arrow[r] \arrow[d, "\sigma_{n,1}(-{,}*)"] & \mathrm{Sym}^{n-1}_\bullet(X/A) \arrow[d, "\psi"] \\
\mathrm{Sym}^n_\bullet(A) \arrow[r] & \mathrm{Sym}^n_\bullet(X,A) \arrow[r] & \mathrm{Sym}^n_\bullet(X/A,*),
\end{tikzcd}$$

where ψ is the map induced by $\sigma_{n,1}(-,*)$. We now proceed to define an inverse, ϕ, to ψ in $G\mathrm{Spc}_\bullet(k)$. Let $f : Y \to \mathrm{Sym}^n_\bullet(X/A,*)$ be a map in $G\mathrm{Spc}'_\bullet(k)$, where Y is a smooth quasi-projective G-scheme over k. It lifts to a map $f' : Y \to \mathrm{Sym}^n_\bullet(X,A)$. Write $Z := (X,A)^{(n)} \times_{\mathrm{Sym}^n_\bullet(X,A)} Y$, which we consider as a $\Sigma_n \times G$-scheme, and write $g : Z \to (X,Z)^{(n)}$ for the resulting map. The map $Z/\Sigma_n \to Y$ is a G-equivariant finite morphism and for any field K, $(Z/\Sigma_n)(K) \to Y(K)$ is a bijection. This implies that it is a homeomorphism, thus Z/Σ_n is irreducible and since it is reduced, it is integral. Since $\mathrm{char}(k) = 0$, it follows that $Z/\Sigma_n \to Y$ is birational, see e.g., [Hum75, Chapter 1 §4.6, Theorem]. Zariski's Main Theorem implies that it is an isomorphism. Now let $n : Z^N \to Z$ be the normalization. The $\Sigma_n \times G$-action lifts uniquely to an action on Z^N. Since $Z^N/\Sigma_n \to Z/\Sigma_n$ is finite and birational, it is an isomorphism by Zariski's Main Theorem. We may replace Z by Z^N and g by gn and assume that Z is normal. In particular, Z is a disjoint union of its irreducible components. The map $g : Z \to (X,A)^{(n)}$ is given by a tuple $g = (g_1, \dots, g_n)$ of maps $g_i : Z \to X$. Let Z_1 be an irreducible component. Its image under g is contained in an irreducible component of $(X,A)^{(n)}$ and so there is some i such that $g_i(Z_1) \subseteq A$. Since Σ_n acts on $(X,A)^{(n)}$ by permuting the factors, we can choose Z_1 so that $g_n(Z_1) \subseteq A$. Let $\Sigma_{n-1} \subseteq \Sigma_n$ be the embedding as the subgroup of permutations of $\{1, \dots, n\}$ which fix n. Let $\overline{Z}$ be the Σ_{n-1}-orbit of Z_1. The map $\Sigma_n \times_{\Sigma_{n-1}} \overline{Z} \to Z$ is an isomorphism and so we have isomorphisms $\overline{Z}/\Sigma_{n-1} \cong Z/\Sigma_n \cong Y$. Consider the Σ_{n-1}-equivariant map $(g_1, \dots, g_{n-1}) : \overline{Z} \to X^{\times n-1}$. Passing to quotients we have a map $Y \to \mathrm{Sym}^{n-1}_\bullet(X)$ and hence a map $\phi_Y(f) : Y \to \mathrm{Sym}^{n-1}_\bullet(X/A)$. This defines a function $\phi_Y : \mathrm{Sym}^n_\bullet(X/A,*)(Y) \to \mathrm{Sym}^{n-1}_\bullet(X/A)(Y)$. It is natural and so defines a map ϕ in $G\mathrm{Spc}'_\bullet(k)$ which is easily checked to be an isomorphism. □

Recall that the group of finite correspondences $\mathrm{Cor}_k(Y,X)$ is the free abelian group generated by integral closed subschemes $Z \subseteq Y \times X$ such that Z is finite over Y and dominates an irreducible component of Y. When G acts on X and Y there is an induced action on $\mathrm{Cor}_k(Y,X)$. The group of equivariant finite correspondences is defined to be $G\,\mathrm{Cor}_k(Y,X) := \mathrm{Cor}_k(Y,X)^G$. We write $\mathbb{Z}_{tr,G}(X)$ for the presheaf of equivariant correspondences, $\mathbb{Z}_{tr,G}(X)(Y) := G\,\mathrm{Cor}_k(Y,X)$. If $A \subseteq X$ is a closed invariant subscheme we define $\mathbb{Z}_{tr,G}(X/A) := \mathbb{Z}_{tr,G}(X)/\mathbb{Z}_{tr,G}(A)$.

The submonoid $\mathrm{Cor}_k^{eff}(Y,X)^G$ of effective equivariant correspondences consists of those equivariant correspondences $\sum n_Z Z$ such that all $n_Z \geq 0$. We write $\mathbb{Z}^{eff}_{tr,G}(X)$ for the corresponding presheaf of effective equivariant correspondences. Let $A \subseteq X$ is a closed invariant subscheme and define an equivalence relation $\sim$ on $\mathbb{Z}^{eff}_{tr,G}(X)(Y)$ by declaring $Z \sim Z'$ if $Z - Z' \in \mathbb{Z}_{tr,G}(A)(Y)$. Now define the presheaf $\mathbb{Z}^{eff}_{tr,G}(X/A) := \mathbb{Z}^{eff}_{tr,G}(X)/\sim$. We have that $\mathbb{Z}_{tr,G}(X) = (\mathbb{Z}^{eff}_{tr,G}(X))^+$ and $\mathbb{Z}_{tr,G}(X/A) = (\mathbb{Z}^{eff}_{tr,G}(X/A))^+$, where $(-)^+$ denotes group completion.

Consider the subset $L_n(X)(Y) \subseteq \mathrm{Cor}_k^{eff}(Y,X)$ of effective correspondences of degree n. Write $L_n^G(X)(Y) = (L_n(X)(Y))^G$ for the subset of equivariant correspondences of degree n. Now we consider a pointed G-scheme (X,x). The presheaf $L_n(X)$ is pointed by $n(Y \times x)$ in $L_n(X)(Y)$. Let $A \subseteq X$ be a closed invariant subscheme containing $x \in X$. Define $L_n^G(X/A)$ and maps $q_n : L_n^G(X) \to L_n^G(X/A)$ and $st_n : L_{n-1}^G(X/A) \to L_n^G(X/A)$ inductively as follows. Let $L_0(X/A) = *$ and define $L_1^G(X) \to L_1^G(X/A)$ to be the quotient map $X \to X/A$ in $\mathrm{sPre}_\bullet(G\mathrm{Sch}/k)$. Write π_j for the composition

$$L_j^G(A) \times L_{n-j}^G(X) \xrightarrow{p_2} L_{n-j}^G(X) \xrightarrow{q_{n-j}} L_{n-j}^G(X/A).$$

Now we define $L_n^G(X/A)$, q_n, and st_n by the pushout diagram in $G\mathrm{Spc}_\bullet(k)$

$$\begin{tikzcd}
\coprod_{j=1}^n L_j^G(A) \times L_{n-j}^G(X) \arrow[r] \arrow[d, "\pi"'] & L_n^G(X) \arrow[d, "q_n"] \\
L_{n-1}^G(X/A) \arrow[r, "st_n"] & L_n^G(X/A).
\end{tikzcd} \tag{A.24}$$

Note that the map st_n is a monomorphism and the addition map induces an addition map $L_i^G(X/A) \times L_j^G(X/A) \to L_{i+j}^G(X/A)$.

Proposition A.25. *Let X be a pointed, semi-normal, quasi-projective G-scheme over k and $A \subseteq X$ an invariant closed reduced subscheme containing the basepoint. There are isomorphisms $L_n^G(X) \cong \mathrm{Sym}^n(X)$ in $G\mathrm{Spc}_\bullet(k)$ which induce isomorphisms $L_n^G(X/A) \cong \mathrm{Sym}^n_\bullet(X/A)$.*

Proof. There is a natural isomorphism $\psi : \mathrm{Sym}^n_\bullet(X) \to L_n(X)$ by [SV96, Theorem 6.8], which is given as follows. Let $W_n \subseteq \mathrm{Sym}^n(X) \times X$ be the image of the closed subscheme $X^{n-1} \times \Delta_X \subseteq X^{n+1}$ under $\pi_n \times \mathrm{id}_X$, where $\pi_n : X^n \to \mathrm{Sym}^n(X)$ is the quotient. If $f : Y \to \mathrm{Sym}^n(X)$ is a map, with Y smooth, then $\psi(f) = (f \times \mathrm{id}_X)^*(W_n)$ is defined to be the pullback cycle. This is an equivariant isomorphism. The image of $\coprod_{j=1}^n L_j^G(A) \times L_{n-j}^G(X) \to L_n^G(X)$ is represented by $\mathrm{Sym}^n_\bullet(X,A)$ under the above isomorphism. Comparing the pushout squares A.23 and A.24 and induction thus yield the isomorphisms $L_n(X/A) \cong \mathrm{Sym}^n_\bullet(X/A)$. □

The maps $L^G_n(X) \to \mathbb{Z}^{eff}_{tr,G}(X)$ induce maps $L^G_n(X/A) \to \mathbb{Z}^{eff}_{tr,G}(X/A)$ which are compatible with the stabilization $L^G_n(X/A) \to L^G_{n+1}(X/A)$. We thus have an induced map $\operatorname{colim}_n L^G_n(X/A) \to \mathbb{Z}^{eff}_{tr,G}(X/A)$.

Proposition A.26. *Let (X, x) be a pointed smooth quasi-projective G-scheme over k and $A \subseteq X$ a closed reduced subscheme which contains the basepoint $x \in X$. Then we have isomorphisms in $G\mathrm{Spc}_\bullet(k)$*

$$\mathrm{Sym}^\infty_\bullet(X/A) \xleftarrow{\cong} \operatorname*{colim}_n L^G_n(X/A) \xrightarrow{\cong} \mathbb{Z}^{eff}_{tr,G}(X/A).$$

Proof. The left hand isomorphism follows from the previous proposition. The map $\operatorname{colim}_n L_n(X) \to \mathbb{Z}^{eff}_{tr,G}(X)$ is immediately seen to be an isomorphism. The map $f : \operatorname{colim}_n L^G_n(X/A) \to \mathbb{Z}^{eff}_{tr,G}(X/A)$ is surjective and we check injectivity. Let $[W], [V] \in L^G_n(X/A)(Y)$ and represent them by elements in $L^G_n(X)(Y)$ which we again write as W, V. We can write $W = W_A + W'$ and $V = V_A + V'$ uniquely as a sum of effective cycles with W_A, V_A supported on $Y \times A$ and no component of the supports of W', V' contained in $Y \times A$. If $f([W]) = f([V])$, then we have that $W' = V'$. It follows from the definition of $L^G_n(X/A)$ that $[W] = [V]$. □

Proposition A.27. *Let X be a pointed smooth quasi-projective G-scheme over k and V a representation which contains a copy of the trivial representation. Then the map $\mathrm{Sym}^\infty_\bullet(T^V \wedge X) \to \mathbb{Z}_{tr,G}(T^V \wedge X)$ is an equivariant motivic weak equivalence.*

Proof. If E is a presheaf of sets, write C_*E as usual for the presheaf of simplicial sets defined by $C_nE(X) = E(X \times \Delta^n_k)$. The map $E \to C_*E$ is an $\mathbb{A}^1$-weak equivalence.

Write $F_1 = \mathbb{Z}^{eff}_{tr,G}(T^V \wedge X)$ and $F_2 = \mathbb{Z}_{tr,G}(T^V \wedge X)$ and let $\phi : F_1 \to F_2$ be the group completion. Under the isomorphism of Proposition A.26, the map in the statement of the proposition is identified with ϕ. It suffices to show that $\phi : C_*F_1 \to C_*F_2$ is a motivic weak equivalence. In fact, we show that if S is a point for the equivariant Nisnevich topology, then $\phi : C_*F_1(S) \to C_*F_2(S)$ is a weak equivalence of simplicial sets. We claim that $\pi_0 C_*F_1(S) = 0$. Granted this, since ϕ is the group completion of a free commutative monoid, by [FM94, Theorem Q4] the map ϕ is a homology equivalence. Since ϕ is a map between simple spaces, this implies that ϕ is in fact a homotopy equivalence.

It remains to see that $\pi_0 C_*F_1(S) = 0$. We have that $S = G \times_H \mathrm{Spec}(R)$ where $H \subseteq G$ is a subgroup and R is a smooth, local Henselian ring with H-action, see [HVØ15, Theorem 3.14]. Write $\rho_H : G\mathrm{Sch}/k \to H\mathrm{Sch}/k$ for the functor which restricts the action. Note that $\rho_H(\mathrm{Sym}^\infty_\bullet(Y)) = \mathrm{Sym}^\infty_\bullet(\rho_H Y)$ for any based quasi-projective G-scheme Y and so for any H-scheme Z

$$\mathrm{Hom}_{G\mathrm{Sch}/k}(G \times_H Z, \mathrm{Sym}^\infty_\bullet(Y)) = \mathrm{Hom}_{H\mathrm{Sch}/k}(Z, \mathrm{Sym}^\infty_\bullet(\rho_H Y)).$$

Thus, replacing G with H and S with $\mathrm{Spec}(R)$, we may assume that S is a local Henselian G-scheme.

Write

$$F'_1 = \mathbb{Z}^{eff}_{tr,G}\left(\mathbb{P}(V \oplus 1)/(\mathbb{P}(V \oplus 1) \setminus \mathbb{P}(1))\right).$$

First note that since $\mathbb{P}(V) \subseteq \mathbb{P}(V \oplus 1) \setminus \mathbb{P}(1)$ has an equivariant section, that

$$\pi_0 C_*F_1(S) \subseteq \pi_0 C_*F'_1(S).$$

(In fact, $C_*F_1(S)$ and $C_*F'_1(S)$ are weakly equivalent, but the weaker statement is enough for our purposes here.) Therefore, it suffices to show that $\pi_0 C_*F'_1(S) = 0$.

An element of $C_dF'_1(S)$ is represented by an effective invariant cycle $\sum n_iZ_i$ on $\Delta^d_S \times \mathbb{P}(V\oplus 1)$ such that Z_i is finite over S and has nontrivial intersection with $\Delta^d_S\times\mathbb{P}(1)$.

Let $\mathcal{Z} = \sum n_iZ_i$ be an element of $C_0F'_1(S)$. Since the Z_i are finite over S, they are local. Since Z_i has nontrivial intersection with $\mathbb{P}(1)_S$ the closed points are supported in $\mathbb{P}(1)_S$ and Z_i are contained in the open $\mathbb{A}(V)_S\subseteq \mathbb{P}(V\oplus 1)$ (the point $\mathbb{P}(1) = 0$ in $\mathbb{A}(V)$). By assumption, V contains a copy of the trivial representation so we can write $V = 1\oplus V'$. Consider the equivariant map $\phi : \mathbb{A}^1\times\mathbb{A}(1\oplus V')$ given by $(t, x, y)\mapsto (x-t, y)$. Let Φ be the cycle on $\mathbb{A}^1\times\mathbb{A}(V)_S$ obtained by pull-back of $\mathcal{Z}$ along ϕ. Then $\Phi\in C_1F'_1(S)$ and has the property that $\Phi|_0 = \mathcal{Z}$ and $\Phi|_1$ is supported in $(\mathbb{P}(V\oplus 1)\setminus\mathbb{P}(1))_S$ (the closed points lie in $\{1\}_S\subseteq\mathbb{A}(V)_S$). Thus $\mathcal{Z} = 0$ in $\pi_0C_*F'_1(S)$. □

A.9. **Realization of Eilenberg-MacLane spectra.** Our next goal is to show that the topological realization, in both the complex and real case, of the Bredon motivic cohomology spectrum $\mathbf{M}\underline{A}$ is the topological Bredon cohomology spectrum (for the constant Mackey functor $\underline{A}$). The construction of the motivic G-spectrum $\mathbf{M}\underline{\mathbb{Z}}$ is spelled out in detail in Section 3.2 for the case $G = C_2$ and the construction for a general finite group G is similar. In brief, we set $\mathbf{M}\underline{\mathbb{Z}}_n := \mathbb{Z}_{tr,G}(T^{n\rho_G})$ and structure maps are defined by $\mathbf{M}\underline{\mathbb{Z}}_n\wedge T^{\rho_G}\to\mathbf{M}\underline{\mathbb{Z}}_n\wedge\mathbb{Z}_{tr,G}(T^{\rho_G})\to\mathbf{M}\underline{\mathbb{Z}}_{n+1}$, the first map being induced by the inclusion $T^{\rho_G}\to\mathbb{Z}_{tr,G}(T^{\rho_G})$ together with the addition-of-cycles map.

Lemma A.28. (1) *Let V be a complex representation which contains a trivial summand. For any X in GSch/$\mathbb{C}$ the natural map*

$$\mathbb{L}\mathrm{Re}_{\mathbb{C}}(T^V\wedge X_+)\to S^{V(\mathbb{C})}\wedge X(\mathbb{C})_+$$

is an equivariant equivalence in $G\mathrm{Top}_\bullet$.

(2) *Let V be a real representation which contains a trivial summand. For any X in GSch/$\mathbb{R}$ the natural map*

$$\mathbb{L}\mathrm{Re}_{\mathbb{C},\,\Sigma_2}(T^V\wedge X_+)\to S^{V(\mathbb{C})}\wedge X(\mathbb{C})_+$$

is an equivariant equivalence in $(G\times\Sigma_2)\mathrm{Top}_\bullet$.

Proof. We treat the complex case, the real case is similar.

Over a field of characteristic zero there are equivariant resolutions of singularities, see e.g., [Kol07, Proposition 3.9.1]. We may thus find a simplicial scheme $X_\bullet\to X$ over X such that $X^H_\bullet\to X^H$ is a proper *cdh*-hypercover for every subgroup $H\subseteq G$. We have $|X_\bullet|^H = |X^H_\bullet|$ and since $V^H\neq 0$, it follows from [Voe10b, Theorem 4.2] that the map $(T^V\wedge|X_\bullet|_+)^H\to(T^V\wedge X_+)^H$ is a motivic weak equivalence in $\mathrm{Spc}_\bullet(\mathbb{C})$. Therefore the map $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(T^V\wedge|X_\bullet|_+)\to S^{V(\mathbb{C})}\wedge X(\mathbb{C})_+$ is an equivariant weak equivalence in $G\mathrm{Top}_\bullet$, see Remark A.14.

Since each X_n is smooth, $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(T^V\wedge|X_\bullet|_+)\simeq S^{V(\mathbb{C})}\wedge|X(\mathbb{C})_\bullet|_+$. To complete the proof it remains to show that $S^{V(\mathbb{C})}\wedge|X(\mathbb{C})_\bullet|_+\to S^{V(\mathbb{C})}\wedge X(\mathbb{C})_+$ is an equivariant weak equivalence. It suffices to show that this map is a weak equivalence on all fixed points. Note that $X^H(\mathbb{C}) = X(\mathbb{C})^H$. Applying $(-)^H$ to the map above we obtain the map $S^{2|V^H|}\wedge|X^H(\mathbb{C})_\bullet|_+\to S^{2|V^H|}\wedge X^H(\mathbb{C})_+$. Since $X^H(\mathbb{C})_\bullet\to X^H(\mathbb{C})$ is a proper hypercover, it is a universal cohomological descent hypercover [Del74, 5.3.5]. It follows that $H^*_{sing}(|X^H(\mathbb{C})_\bullet|, A)\to H^*_{sing}(X^H(\mathbb{C}), A)$ is an isomorphism for all abelian groups A. It follows that $S^{2|V^H|}\wedge|X^H(\mathbb{C})_{\bullet+}|\to S^{2|V^H|}\wedge X^H(\mathbb{C})_+$

is a homology isomorphism. Since $|V^H| \geq 1$, these are simply connected spaces and thus this homology isomorphism is a weak equivalence. $\square$

Lemma A.29. (1) *Let W,V be complex representations with V containing a trivial summand and X a smooth quasi-projective complex variety with G-action. Then for $n \geq 0$,*

$$\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\Sigma_{T^V}\mathrm{Sym}^n_\bullet(\Sigma_{T^W}X_+)) \to \Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}^n_\bullet(\Sigma_{S^{W(\mathbb{C})}}X(\mathbb{C})_+)$$

is an equivariant weak equivalence in $G\mathrm{Top}_\bullet$.

(2) *Let W,V be real representations with V containing a trivial summand and X a smooth quasi-projective real variety with G-action. Then for $n \geq 0$,*

$$\mathbb{L}\mathrm{Re}_{\mathbb{C},\,\Sigma_2}(\Sigma_{T^V}\mathrm{Sym}^n_\bullet(\Sigma_{T^W}X_+)) \to \Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}^n_\bullet(\Sigma_{S^{W(\mathbb{C})}}X(\mathbb{C})_+)$$

is an equivariant weak equivalence in $(G\times\Sigma_2)\mathrm{Top}_\bullet$.

Proof. We treat the complex case, the real case is similar.

We proceed by induction on n. The case $n=0$ is immediate and $n=1$ follows from Lemma A.28. Write $Y=\mathbb{P}(W\oplus 1)\times X$, $A=\mathbb{P}(W)\times X$. Smashing the homotopy pushout square obtained from Proposition A.22 by T^V, we obtain the homotopy pushout square

$$\begin{tikzcd}
\Sigma_{T^V}\mathrm{Sym}^n_\bullet(Y,A) \arrow[r] \arrow[d, "\pi_n"'] & \Sigma_{T^V}\mathrm{Sym}^n_\bullet(Y) \arrow[d] \\
\Sigma_{T^V}\mathrm{Sym}^{n-1}_\bullet(\Sigma_{T^W}X_+) \arrow[r, "st_n"] & \Sigma_{T^V}\mathrm{Sym}^n_\bullet(\Sigma_{T^W}X_+).
\end{tikzcd}$$

Applying $\mathbb{L}\mathrm{Re}_{\mathbb{C}}$ to this square we obtain a homotopy cartesian square in $G\mathrm{Top}_\bullet$ which we claim is equivariantly weak equivalent to

$$\begin{tikzcd}
\Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}^n_\bullet(Y,A)(\mathbb{C}) \arrow[r] \arrow[d, "\pi_n"'] & \Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}^n_\bullet(Y)(\mathbb{C}) \arrow[d] \\
\Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}^{n-1}_\bullet(\Sigma_{S^{W(\mathbb{C})}}X(\mathbb{C})_+) \arrow[r, "st_n"] & \Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}^n_\bullet(\Sigma_{S^{W(\mathbb{C})}}X(\mathbb{C})_+).
\end{tikzcd}$$

The displayed square is a homotopy cartesian square in $G\mathrm{Top}_\bullet$. By Lemma A.28, the map $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\Sigma_{T^V}\mathrm{Sym}^n_\bullet(Y,A)) \to \Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}^n_\bullet(Y,A)(\mathbb{C})$ is an equivariant weak equivalence and similarly for $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\Sigma_{T^V}\mathrm{Sym}^n_\bullet(Y)) \to \Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}^n_\bullet(Y)(\mathbb{C})$. By induction the map $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\Sigma_{T^V}\mathrm{Sym}^{n-1}_\bullet(\Sigma_{T^W}X_+)) \to \Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}^{n-1}_\bullet(\Sigma_{S^{W(\mathbb{C})}}X(\mathbb{C})_+)$ is an equivariant weak equivalence. It follows that

$$\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\Sigma_{T^V}\mathrm{Sym}^n_\bullet(\Sigma_{T^W}X_+)) \to \Sigma_{S^{V(\mathbb{C})}}\mathrm{Sym}^n_\bullet(\Sigma_{S^{W(\mathbb{C})}}X(\mathbb{C})_+)$$

is also an equivariant weak equivalence as desired. $\square$

Corollary A.30. (1) *Let X be a smooth quasi-projective complex variety with G-action. For any $n,m \geq 0$ the natural map*

$$\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\Sigma^\infty_{T^{\rho_G}}\mathrm{Sym}^n_\bullet(\Sigma^m_{T^{\rho_G}}X_+)) \to \Sigma^\infty_{S^{2\rho_G}}\mathrm{Sym}^n_\bullet(\Sigma^m_{S^{2\rho_G}}X(\mathbb{C})_+)$$

is a stable equivariant equivalence.

(2) *Let X be a smooth quasi-projective real variety with G-action. For any $n,m \geq 0$ the natural map*

$$\mathbb{L}\mathrm{Re}_{\mathbb{C},\Sigma_2}(\Sigma^\infty_{T^{\rho_G}}\mathrm{Sym}^n_\bullet(\Sigma^m_{T^{\rho_G}}X_+)) \to \Sigma^\infty_{S^{2\rho_G}}\mathrm{Sym}^n_\bullet(\Sigma^m_{S^{2\rho_G}}X(\mathbb{C})_+)$$

is a stable equivariant equivalence.

Define

$$(\Sigma^{\infty}_{T^{\rho_G}} X)^{eff}_{tr} = (\mathrm{Sym}^{\infty}_{\bullet}(X_+), \mathrm{Sym}^{\infty}_{\bullet}(\Sigma_{T^{\rho_G}} X_+), \mathrm{Sym}^{\infty}_{\bullet}(\Sigma^{2}_{T^{\rho_G}} X_+), \ldots)$$

with obvious structure maps.

Theorem A.31. *There is an isomorphism* $\mathrm{Re}_{\mathbb{C}}(\mathbf{M}\underline{A}) \cong \mathrm{H}\underline{A}$ *in* SH_G*, for any abelian group* A*. Similarly, there is an isomorphism* $\mathrm{Re}_{\mathbb{C},\Sigma_2}(\mathbf{M}\underline{A}) \cong \mathrm{H}\underline{A}$ *in* $\mathrm{SH}_{G\times\Sigma_2}$*.*

Proof. We treat the complex case. The real case is similar.

Since $\mathbf{M}\underline{A} = \mathbf{M}\underline{\mathbb{Z}} \wedge \mathcal{M}A$ and $\mathrm{H}\underline{A} = \mathrm{H}\underline{\mathbb{Z}} \wedge \mathcal{M}A$, where $\mathcal{M}A$ is a Moore spectrum for A, and $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\mathcal{M}A) = \mathcal{M}A$, it suffices to establish the result for $A = \mathbb{Z}$. The map $(\Sigma^{\infty}_{T^{\rho_G}}(S^0))^{eff}_{tr} \to \mathbf{M}\underline{\mathbb{Z}}$ is a stable equivalence by Proposition A.27.

It follows from [dS03, Proposition 3.7] that the spectrum $\{\mathbb{Z}S^{2n\rho_G}\}_{n\geq 0}$ is a $S^{2\rho_G}$-spectrum model for $\mathrm{H}\underline{\mathbb{Z}}$. It follows from [Dug05, Proposition A.6] that the natural map $(\Sigma^{\infty}_{S^{2\rho_G}}(S^0))^{eff}_{tr} := \{\mathrm{Sym}^{\infty}_{\bullet}(\Sigma_{S^{2n\rho_G}} S^0)\} \to \{\mathbb{Z}S^{2n\rho_G}\}_{n\geq 0}$ is an equivariant stable equivalence. It thus suffices to see that $\mathbb{L}\mathrm{Re}_{\mathbb{C}}(\Sigma^{\infty}_{T^{\rho_G}}(S^0))^{eff}_{tr} \to (\Sigma^{\infty}_{S^{2\rho_G}}(S^0))^{eff}_{tr}$ is a stable equivalence.

We have the natural isomorphism $\mathrm{colim}_n(\Sigma^{\infty}_{T^{\rho_G}} E_n)[n] \cong E$ in $\mathrm{SH}_k(G)$, where $D[n]$ is the shifted T^{ρ_G}-spectrum given by $(D[n])_i = D_{i-n}$. Similarly, we have the natural isomorphism $\mathrm{colim}_n(\Sigma^{\infty}_{S^{2\rho_G}} F_n)[n] \cong F$ in SH_G. Since $\mathbb{L}\mathrm{Re}_{\mathbb{C}}$ preserves homotopy colimits and shifts, the result follows from Corollary A.30. □

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E-mail address: `jeremiahheller.math@gmail.com`

Department of Mathematics, University of Illinois, Urbana-Champaign

E-mail address: `m.voineagu@unsw.edu.au`

UNSW Sydney, NSW 2052 Australia

E-mail address: `paularne@math.uio.no`

Department of Mathematics, University of Oslo, Norway