Let $K$ be a field.

$p \in K[x]$ a poly. s.t. $p(x) = 0$ has no solution in $K$.

Q? Is there a larger field $K \leq F$ such that $p(x) = 0$ has a solution in $F$?

**Construction:** Suppose that $p(x)$ is irreducible. (If it isn't, restrict attention to each irreducible factor.)

$F := K[x]/(p)$

We know this is a field from last time since:

$F$ field $\iff$ $(p) \subseteq K[x]$ is maximal $\iff$ $p$ is irreducible.

The composite $K \subseteq K[x] \xrightarrow{\pi} K[x]/(p) = F$ is injective.

$\implies K$ is a subfield of $F$.

In particular we can view $K[x]$ as a subring of $F[x]$.

$K[x] \subseteq F[x]$

$\sum a_i x^i \mapsto \sum \pi(a_i) x^i$

Write $d = \pi(x) \in F$.

Then we get $K[x] \xrightarrow{\pi} K[x]/\langle p \rangle = F$

$x \mapsto d$.
Example. \( K = \mathbb{Q} \quad p(x) = x^2 - 3 \) is reducible

[Fact: If \( \deg(f) = 2 \) then \( f \) is reducible \( \iff \) \( f \) has a root.]

\[ F = \mathbb{Q}[x] / (x^2 - 3) = \{ a + bx + I \mid a, b \in \mathbb{Q} \} \quad I = (x^2 - 3) \]

recall: \( \mathbb{Q}(\sqrt{3}) \subseteq \mathbb{R} \)

\[ \exists a + b\sqrt{3} \mid a, b \in \mathbb{Q} \]

Can identify \( F \) with \( \mathbb{Q}[x] / (x^2 - 3) \) (why?)

\( \mathbb{Q}[x] \rightarrow \mathbb{R} \)

\[ x \mapsto \sqrt{3} \]

Image: \( \mathbb{Q}(\sqrt{3}) \)

Kernel is \( x^2 - 3 \)

in particular: \( p = x^2 - 3 \) has root in \( F \).

multiplication:

\[(a + b\sqrt{3})(c + d\sqrt{3}) = (ac + 3bd) + (ad + bc)\sqrt{3}\]
Proposition: With notation as above, \( \alpha \in F \) is a root of \( p(x) \in F[x] \)

**Proof:**

Write \( p(x) = \Sigma a_n x^n \in K[x] \). Then \( p(x) \in F[x] \) is

\[
p(x) = \Sigma \pi(a_n) x^n
\]

\[
\Rightarrow p(\alpha) = \Sigma \pi(a_n) \alpha^n = \Sigma \pi(a_n) \pi(x)^n = \Sigma \pi(a_n x^n) = \pi(\Sigma a_n x^n) = \pi(p(x))
\]

Recall: we have identified \( K[\overline{x}] / (p(x)) \) with the set of cosets

\[
\{ a_0 + a_1 \alpha + a_2 \alpha^2 + \ldots + a_{n-1} \alpha^{n-1} \mid a_i \in K \}
\]

This can be written as

\[
\{ a_0 + a_\alpha \alpha + a_\alpha^2 + \ldots + a_{n-1} \alpha^{n-1} \} \quad \text{where } \alpha = \pi(x)
\]

Indeed:

\[
\alpha = \pi(x) = x + (p(x)) \in F = K[\overline{x}] / (p(x))
\]

\[
\Rightarrow F \text{ is a dim } n \text{ vector space over } K \text{ with basis } 1, \alpha, \alpha^2, \ldots, \alpha^{n-1}
\]
Example \( K = \mathbb{F}_2, \quad x^3 - 1 \) not irreducible \( \text{has not} \ 1 \)

\[ x^3 - 1 = (x-1)(x^2 + x + 1) \]

\( p(x) = x^2 + x + 1 \) is irreducible \((x \neq 1)\)

\[ F = \mathbb{F}_2[x]/(x^2 + x + 1) =: \mathbb{F}_2(\theta) \]

Multiplication \( \times F? \)

\[ \Theta \text{ satisfies:} \quad \Theta^2 + \Theta + 1 = 0 \quad \text{in} \ \mathbb{F}_2 \]

\[ \Rightarrow \Theta^2 = -1 = \Theta + 1 \]

\[ F = \{ a + b\Theta \mid a, b \in \mathbb{F}_2 \} \subseteq \mathbb{F}_2 \text{ v.s} \rho \quad \text{so} \quad |F| = 4 \]

\[(a + b\Theta)(c + d\Theta) = ac + (ad + bc)\Theta + bd\Theta^2\]

\[ = ac + (ad + bc)\Theta + bd(\Theta + 1)\]

\[ = (ac + bd) + (ad + bc + bd)\Theta\]

Observe in \( F \) the poly \( x^3 - 1 \) splits as \( \Theta \)

\[ x^3 - 1 = (x - 1)(x - \Theta)(x - \Theta^2) \]

Over \( \mathbb{C} \) \( x^3 - 1 \) splits as \( (x - 1)(x - \xi_3)(x - \xi_3^2) \)

\[ \xi_3 = e^{2\pi i/3} \]

Note: \( (F_1)^+ \) has 4 elements \( (F_1)^+ \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \)

\((\text{if} F \text{ v.s} \rho) \)