Ring of polynomials over a field:

\[ f = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n \]

Say that: \( \deg(f) = n \), 
leading coefficient is \( a_n \)

\( \deg(0) = -\infty \)
\( \deg(a) = 0 \) if \( a \in K = K \setminus \{0\} \) (constant polynomial)

Example:

\[ f(x) = \pi^{15} x^3 + 2x^2 - x + \sqrt{2} \]

\( \deg(f) = 3 \)
leading coeff. is \( \pi^{15} \)

Proposition: If \( f, g \in K[x] \)

\[ \deg(fg) = \deg(f) + \deg(g) \]
\( \deg(fg) \leq \max \{ \deg(f), \deg(g) \} \)

Proof:

\[ f = \sum_{i=0}^{n} a_i x^i \quad g = \sum_{j=0}^{m} b_j x^j \]

\[ fg = \sum_{i+j=n} a_i b_j x^{i+j} \]
\( \Rightarrow \deg(fg) = n + m \)

\[ f + g = \sum \max \{ a_i + b_j \} x^i \]
\( \Rightarrow \deg(f + g) = \max \{ \deg(f), \deg(g) \} \)
Corollary: \((K\bar{K}x)^x = K^x\)

Proof: \(K^x = (K\bar{K}x)^x\)

If \(f(x) \in (K\bar{K}x)^x \Rightarrow \exists \; f^{-1}\) such that

\[1 = f(x) \cdot f^{-1}(x)\]

\[\Rightarrow 0 = \deg(1) = \deg(f^{-1}) - \deg(f)\]

Since \(\deg > 0 \Rightarrow \deg(f) = 0\)

\[\Rightarrow f \in K \Rightarrow f \in K^x.\]

Def: Say that \(f \in K\bar{K}x\) is irreducible if \(\deg(f) > 0\) and whenever \(f = gh\) either \(g \in K^x\) or \(h \in K^x\).

Example: \(K = \mathbb{R}\), \(f(x) = x^2 + 1\) is irreducible:

- If \(f = gh\) with \(\deg(g) > 0\) and \(\deg(h) > 0\) \(\Rightarrow g, h \notin K^x\)

- \(x^2 + 1\) has no real roots \(\Rightarrow\) not possible.

\(K = \mathbb{C}\), \(f(x) = x^2 + 1\) is reducible: \(x^2 + 1 = (x - i)(x + i)\)
Theorem: Any \( f \in \mathbb{K}[x] \), \( \deg(f) > 0 \) is a product of irreducible polynomials, unique up to multiplication by a unit, if

\[ f = p_1 \cdots p_k = q_1 \cdots q_l \]

then \( k = l \) and after reindexing, \( p_i = a_i q_i \) for some \( a_i \in \mathbb{K}^\times \).

Remark: Similar to prime factorization in \( \mathbb{Z} \), similar proof.

Proof:

**Existence:** induction on degree.

If \( \deg(f) = 1 \), we're done since \( f \) must be irreducible

\[ \deg(f) = \deg(g) + \deg(h) \]

Suppose true \( \forall g \in \mathbb{K}[x], \deg(g) < n \).

Let \( g \in \mathbb{K}[x] \) have degree \( n + 1 \).

If \( g \) is irreducible, we're done otherwise

\[ g = f \cdot h \]

Applying IH, we can write \( f, h \) as a product of irreducibles.

**Uniqueness:** on \( + \), similar to case of primes in \( \mathbb{Z} \).

\( 1.8.21 \)
Def: If \( f, g \in K[x] \) say that \( f \mid g \) if \( K[x] \) such that \( f = gh \).

Proposition: Let \( f, d \in K[x] \), \( \deg(d) > 0 \).
Then \( \exists \ q, r \in K[x] \) st.
\[ f = qd + r \quad \text{where} \quad \deg(r) < \deg(d). \]

Proof: This is polynomial long division.

Example: \[ f(x) = 2x^4 + x^3 - 2x^2 + 2x + 1 \quad K = \mathbb{Z}_3 \]
\[ d(x) = x^2 + 2 \]

\[
\begin{array}{c|ccccc}
& 2x^2 & + x \\
\hline
x^2 + 2 & 2x^4 & + x^3 - 2x^2 & + 2x & + 1 \\
- & (2x^4 & + 4x^2) \\
\hline
& x^3 & - 6x^2 & + 2x & + 1 \\
- & (x^3 & + 2x) \\
\hline
& & & & 1
\end{array}
\]

So \[ 2x^4 + x^3 - 2x^2 + 2x + 1 = (2x^2 + x)(x^2 + 2) + 1 \quad \text{in} \quad \mathbb{Z}_3[x] \]
Def: \( f, g \in K[x] \), \( \text{gcd}(f, g) \) is in \( K[x] \) st.

\( \text{if } f \text{ and } g \text{ and if } h \in K[x] \text{ is any other polynomial such that } \text{gcd}(f, g) \text{ divides \( h \) as well.} \)

Theorem: \( \text{for } f, g \in K[x] \), then \( \text{gcd}(f, g) \) exists and

\[ \text{gcd}(f, g) = sf + tg \text{ for some } s, t \in K[x] \]

Proof: omitted. use long division (see 1.8.16 in text)

Def: For \( f, g \in K[x] \), let

\( \langle f, g \rangle \subseteq K[x] \) be defined by

\[ \langle f, g \rangle = \{ sf +tg \mid s, t \in K[x] \} \]

called ideal generated by \( f \) and \( g \)

Proposition: \( \langle f, g \rangle \subseteq K[x] \) is a subring (without 1) of \( K[x] \)

and if \( h \in \langle f, g \rangle \) then

\[ h \in \langle f, g \rangle \Rightarrow h \in K[x]. \]
Proposition: If \( p \) is irreducible, and \( pf + pg \) in \( \mathbb{K}[X] \) with \( \deg(f) < \deg(g) \), then \( p \mid f \) or \( p \mid g \).

Proof: If \( p \nmid f \) then since \( p \) is irreducible we write \( \gcd(f, p) = 1 \).

\[ \Rightarrow \quad \text{we can write } 1 = sf + tp \]

\[ \Rightarrow \quad g = sfg + tpg \]

\( \Rightarrow \quad p \mid g \). \qed