Burnside's Lemma

Theorem (Burnside's Lemma): Let $G$ be a finite group, $X$ a set with $G$-action. Let $k = \# \text{ of orbits of this action. Then,}$

$$k = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|.$$

Recall: $\text{Fix}(g) = \{x \in X \mid gx = x \}$.

(Historical note: Originally due to Frobenius, also known to Cauchy.)

Example: How many ways to color vertices of square with 2 colors? How many ways with 3 colors? 4? $\ldots$?

$D_4 = \{e, r, r^2, r^3, j, rj, r^2j, r^3j\} = \text{symmetries of square.}$

Rotations

Reflections

2 colorings are the same if there is a symmetry taking one to another.

E.g.

$D_4 \text{ acts on } X = \{\text{colorings of square with ordered vertices}\}.$

Orbits of this action = colorings of square.

Use Burnside to count the orbits.
We need to compute \( |\text{Fix}(g)| \) for \( g \in D_4 \).

When is an element of \( X \) fixed by \( g \)?

Let \( V \) be set of vertices of square

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}
\]

A configuration in \( X \) is fixed by \( g \)

\( \iff \) for each \( v \in V \), the vertices

\( v, gv, g^2v, \ldots \) all have same color.

Orbits of \( \langle g \rangle \)-action on \( V \)

\( g = \) orbits of \( \langle g \rangle \) on \( V \)

\( e = \{13, 23, 33, 43\} \)

\( n \) \( \iff \) identity fixes every elt of \( X \)

\( r = \{1, 2, 3, 4, 3\} \)

\( n^2 \) \( \iff \) \( r \) only fixes configurations where every vertex has same color

\( r^2 = \{133, 243\} \)

\( n^2 \) \( \iff \) \( r^2 \) fixes configurations where pairs of opposing vertices have same color

\( r^3 = \{1, 2, 3, 4\} \)

\( n \) \( \iff \) \( r^3 \) fixes configurations where \( r \) action on \( 2, 4 \)

\( f = \{243, 3343\} \)

\( n^3 \) \( \iff \) \( f \) fixes configurations where \( \text{Section 2.4} \)

\( r f = \{123, 343\} \)

\( n^2 \) \( \iff \) \( r f \) fixes configurations where \( \text{Section 2.4} \)

\( r^j = \{133, 243\} \)

\( n^3 \) \( \iff \) \( r^j \) fixes configurations where \( \text{Section 2.4} \)

\( r^j = \{233, 143\} \)

\( n^2 \) \( \iff \) \( r^j \) fixes configurations where \( \text{Section 2.4} \)

So # of colorings = \( \frac{1}{|D_4|} \sum_{g \in D_4} |\text{Fix}(g)| \)

\[
= \frac{1}{8} \left( n^4 + n^3 + n + n^3 + n^2 + n^2 + n^3 + n^2 \right)
\]

\[
= \frac{1}{8} \left( n^4 + 2n^3 + 3n^2 + 2n \right)
\]
For \( n = 2 \) there are 6 colorings:

\[ \text{[Images of colored squares]} \]

For \( n = 3 \) there are \( \frac{1}{6} (81 + 2 \cdot 27 + 3 \cdot 9 + 6) = 21 \) colorings ...

Proof of Burnside's Lemma:

want to show: \( \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| = \# \text{orbit} \).

strategy: combine \( \text{stab}(x) = \{ g \in G \mid g \cdot x = x \} \)

\[ \text{Fix}(g) = \{ x \in X \mid g \cdot x = x \} \]

let \( F = \{ (g, x) \in G \times X \mid g \cdot x = x \} \).

Then \( F = \bigcup_{g \in G} g \times \text{Fix}(g) \) on the one hand and

\[ F = \bigcup_{x \in X} \text{Stab}(x) \times \{ x \} \] on the other.

Thus

\[ \sum_{g \in G} |\text{Fix}(g)| = \sum_{x \in X} |\text{Stab}(x)| \]

\[ \Rightarrow \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| = \frac{1}{|G|} \sum_{x \in X} |\text{Stab}(x)| = \sum_{x \in X} \frac{|\text{Stab}(x)|}{|G|} = \sum_{x \in X} \frac{1}{|G(x)|} \]

\[ = \sum_{O \text{orbits}} \sum_{x \in O} \frac{1}{|G|} = \sum \frac{1}{|G|} = \# \text{orbit} \]
Example: How many necklaces can be made out of 3 blue, 2 red beads, 1 blue,
arrange beads as vertices of regular 6-gon, 2 necklaces are same if there is symmetry taking 1 arrangement
(right side) slide beads around necklace = rotation 
turn once over = reflection, are all equivalent.

This problem is similar to previous one, but we impose restrictions on 
the colorings:

\[ X = \text{set of arrangements of the beads.} \]

\[ |\text{Fix}(g)| \text{ for } g \in D_6? \]
\[ g = e \text{ or } |\text{Fix}(g)| = \frac{(3+2+1)!}{3! \cdot 2! \cdot 1!} = 60 \]
\[ g = \text{any non-identity rotation} \text{ or } |\text{Fix}(g)| = 0 \text{ (since white bead will not be fixed.)} \]
\[ g = \text{flip: 2 kinds } \begin{array}{c} \circ \end{array} \text{ or } \begin{array}{c} \bullet \end{array} \]

A flip has line of reflection which
(i) passes through 2 vertices (3 of time) or
(ii) through middle of 2 parallel sides (3 of time).
in case (i) the flip has 2 fixed vertices and interchange 2 pairs of vertices.

For an arrangement to be fixed: must fix blue vertex and white vertex and interchange pair of blue and pair of red

2 ways to order fixed vertices
2 ways to order pair of interchanged vertices

\[ \text{no. of fixed arrangements} = 4 \]

in case (ii) no fixed arrangement (white never fixed)

So from Burnside we get:

\[ \# \text{ orbits} = \frac{1}{12} \left( \begin{array}{c} 60 + 5 \cdot 0 + 3 \cdot 4 + 3 \cdot 0 \end{array} \right) = \frac{72}{12} = 6 \]

from identity rotation flip type (i) flip type (ii)