Recall: A subgroup \( N \leq G \) is **normal** if \( gN g^{-1} = N \).

Can check: \( N \) is normal \( \iff \) \( gN = Ng \)

**Example:**

\( G = S_3 = \{ e, (12), (13), (23), (123), (132) \} \).

Which are the normal subgroups?

Subgroups?

\[ S_3 \]

\[ \langle (12) \rangle \quad \langle (13) \rangle \quad \langle (23) \rangle \quad \langle (123) \rangle = \langle (132) \rangle \]

\[ \langle e \rangle \]

\( H = \langle (12) \rangle \), cosets are:

\( (13)H = \{ (13), (123) \} \)

\( H(13) = \{ (13), (132) \} \)

\( \Rightarrow \) These are different

\( \Rightarrow \) \( H \) not normal.

Similarly: \( \langle (13) \rangle \), \( \langle (23) \rangle \) are not normal.

\( K = \langle (123) \rangle = \{ e, (123), (132) \} \)

\( (12)K = \{ (12), (23), (13) \} \)

\( K(12) = \{ (12), (13), (23) \} \) equal

Check: \( \sigma K = K \sigma \quad \forall \sigma \in S_3 \) \( \Rightarrow \) so \( K \) is normal.
Note: \([S_3 : K] = 2\) above.

It turns out every index 2 subgroup is normal (HW)

Why care about normal subgroups?

Example: The group \(\mathbb{Z}_n\) was defined as follows:

\[ \mathbb{Z}_n = \{ [a] \mid a \in \mathbb{Z} \} \text{ set of congruence classes} \]

equivalently:

\[ [a]_n = a + n\mathbb{Z} \]

\[ [a]_n \times [b]_n = [a + b]_n \]

so

\[ \mathbb{Z}_n = \{ [a + n\mathbb{Z}] \} \text{ set of cosets} \]

The operation was:

\[ [a]_n + [b]_n = [a + b]_n \]

in terms of cosets

\[ (a + n\mathbb{Z}) + (b + n\mathbb{Z}) = (a + b) + n\mathbb{Z} \]

Can we do this in general?

Say we have subgroup \(H \leq G\).

Can we equip set of (left) cosets \(G/H\) with an operation?

Try:

\[ (aH)(bH) = abH. \]
Example: $S_3$, $H = \langle (12) \rangle = \{ e, (12) \}$

$\langle (13) \rangle H = \{ e \} \langle (13) \rangle$, $\langle (123) \rangle H = \langle (123) \rangle H$

$\langle (23) \rangle H = \{ e \} \langle (23) \rangle$, $\langle (132) \rangle H = \{ 132 \} \langle (132) \rangle H$

try:

$\langle (13) \rangle H \langle (23) \rangle H = \langle (13) \rangle \langle (23) \rangle H = \langle (132) \rangle H \not= \{ 132 \}, \langle (23) \rangle H$

should equal:

$\langle (123) \rangle H \langle (23) \rangle H \not= \langle (123) \rangle \langle (23) \rangle H = \langle (12) \rangle H = \langle 1 \rangle H = \{ e \} \langle (12) \rangle H$

whooh! not well defined!

Conclusion: Not always possible to define operation on $\{ e \} H \not= 3$

by above formula.

Soon we'll see that this operation is well-defined if the subgroup is normal.

(first a digression about equivalence relations)
**Equivalence relations**

Recall: An equivalence relation on a set $X$ is a binary relation with the properties:

1) **reflexive** : $x \sim x \quad \forall x \in X$

2) **symmetric** : $x \sim y \iff y \sim x \quad \forall x, y \in X$

3) **transitivity** : if $x \sim y$ and $y \sim z$ then $x \sim z \quad \forall x, y, z \in X$

**Example:**

1. equality $\equiv$ is an equivalence relation on any set $X$.
2. Let $n \in \mathbb{N}$, congruence mod $n$ is an equivalence relation on $\mathbb{Z}$. (check!)
3. Let $H \leq G$ subgroup. Define $a \sim b$ provided $aH = bH$.

   - reflexive: $a \sim a$ \quad $aHa^{-1} = aH$
   - symmetric: $a \sim b \quad b \sim a$ \quad $bH = aH$
   - transitive: $a \sim b, b \sim c \quad a \sim c$ \quad $aH = bH, bH = cH \implies aH = cH$

4. $\sim$ on $\mathbb{Z} \times \mathbb{Z}$ by $(a,b) \sim (c,d)$ provided $ad = bc$ \quad (check!) \quad $a \sim b \iff \frac{a}{b} = \frac{c}{d}$ in $\mathbb{Q}$.

5. $f : X \to Y$ a function. Define $\sim$ on $X$

   $x \sim y \iff f(x) = f(y)$.

   (we'll see every equivalence relation arises in this way.)