Exercises of the form "Exercise a.b.c" refer to exercises in *Algebra: Abstract and Concrete* by Goodman, available for free from the link on the course website.

**Assignment**

1. Use Burnside's lemma to count the number of necklaces which can be made using 1 red bead, 2 blue beads, and 2 white beads.

2. Let $G$ be a finite group.
   
   (a) Suppose that $G/Z(G)$ is cyclic. Show that $G$ is abelian.
   
   (b) Suppose that $G$ is nonabelian and $|G| = 21$. Show that $Z(G) = \{e\}$.

3. Suppose that $|G| = p^3$, where $p$ is a prime. Show that either $G$ is abelian or that $|Z(G)| = p$.

4. Let $G$ be a finite group and suppose that there is a unique Sylow $p$-subgroup for each prime $p$ dividing $|G|$. Show that $G$ is the direct product of its Sylow $p$-subgroups.

5. Show that the groups $\mathbb{Z}_{30}$, $D_{15}$, $\mathbb{Z}_3 \times D_5$ and $\mathbb{Z}_5 \times D_3$ are all distinct (that is, these are mutually nonisomorphic groups).

   
   (a) Let $P$ be a Sylow 2-subgroup and $Q$ a Sylow 3-subgroup. Up to isomorphism what are all the possibilities for $P$? For $Q$?
   
   (b) According to the Sylow Theorems, what are all possible values for the pair of integers $(n_2, n_3)$?
   
   (c) From the previous part you have a list of potential values of $(n_2, n_3)$ for a group of order 12. Which of these values actually occur? For each of the values from part (b), either give an example of a group $G$ with that many Sylow 2-subgroups and Sylow 3-subgroups or explain why there is no such group.