Assignment

1. Suppose that $N \leq G$ is a normal subgroup and $A \leq G$ any subgroup. Show that $N \cap A$ is normal in $A$.

2. Let $G$ be a finite group, $N \leq G$ a normal subgroup, and $A \leq G$ a subgroup. Show that

$$|AN| = \frac{|A||N|}{|A \cap N|}.$$

3. Let $n \geq 3$ be a natural number and suppose that $k$ divides $n$. Show that $\langle r^k \rangle$ is a normal subgroup of $D_n$ and that

$$D_n/\langle r^k \rangle \cong D_k.$$

4. Let $N \leq G$ be a normal subgroup. Prove that the order of $gN \in G/N$ is the least natural number $k$ such that $g^k \in N$.

5. The Klein 4-group is the order 4 group $V = \{e, a, b, c\}$ where

\[
\begin{array}{cccc}
  & e & a & b & c \\
 e & e & a & b & c \\
a & a & e & c & b \\
b & b & c & e & a \\
c & c & b & a & e
\end{array}
\]

(a) Show that $V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) Show that if $|G| = 4$ then either $G$ is cyclic or $G \cong V$.

6. Let $G$ be a group. Define $\text{Aut}(G) := \{f : G \to G \mid f \text{ is an isomorphism}\}$.

(a) Check that $\text{Aut}(G)$ is a group (the operation is composition of functions).

(b) Given $g \in G$, define $c_g : G \to G$ by $c_g(x) = gxg^{-1}$. Show that the assignment $g \mapsto c_g$ defines a homomorphism

$$c : G \to \text{Aut}(G).$$

(c) Show that $Z(G) = \ker(c)$. 