Topics

Material covered in class corresponding to Sections 2.4-2.7, 3.1-3.2, 5.1.

1. Be able to give precise and correct definitions of the important concepts from these lectures. Know the basic examples and properties.
   - kernel of a homomorphism
   - normal subgroup
   - (left/right) coset
   - equivalence relation, equivalence class, partition
   - quotient group
   - direct product, semi-direct product
   - group action, orbit, stabilizer, centralizer, normalizer

2. Know what our important theorems say, know how and when to use them:
   - Lagrange’s theorem
   - the homomorphism theorem
   - the orbit-stabilizer theorem

3. Know and understand the other significant isomorphism theorems (the correspondence theorem (Proposition 2.7.13) and the diamond isomorphism theorem (Proposition 2.7.19)).

4. Recognize direct products (Proposition 3.15).

5. Recognize semi-direct products (Proposition 3.23).

6. Be able to prove results proved in class such as:
   - \( \text{Ker}(f) = \{e\} \) if and only if \( f \) is injective.
   - The operation on \( G/N \) is well-defined when \( N \) is a normal subgroup of \( G \).
   - If \( \sim \) is an equivalence relation on \( X \), then for any \( x, y \in X \),
     (a) \( x \sim y \) if and only if \( [x] = [y] \).
     (b) Either \( [x] = [y] \) or \( [x] \cap [y] = \emptyset \).
   - If \( A, B \) are normal subgroups of \( G \) such that \( A \cap B = \{e\} \), then \( \varphi : A \times B \to G \) defined by \( \varphi(a, b) = ab \) is a homomorphism.
   - The orbit-stabilizer theorem.
Practice problems

Here are a few further practice problems.

**Question 1** Consider the subgroups of $D_5$, $H = \langle r \rangle$ and $K = \langle j \rangle$. Are these normal subgroups? Is $D_5$ a direct product $H \times K$? Is it a semi-direct product of $H$ and $K$?

**Question 2** Suppose that $d|n$. Show that $f : \mathbb{Z}_n \to \mathbb{Z}_d$ defined by $f([a]_n) = [a]_d$ is a well-defined homomorphisms. What is its kernel? What is $\mathbb{Z}_n/\text{Ker}(f)$?

**Question 3** Is $\mathbb{Z}_9 \to \mathbb{Z}_5$, $[a]_9 \mapsto [a]_5$ well-defined?

**Question 4** Is $SO_n \leq O_n$ normal? If so what is $O_n/\text{SO}_n$? Is $O_n \leq GL_n$ normal? If so, what is $GL_n/O_n$?

**Question 5** Show that the $G \to \text{Aut}(G)$, $g \mapsto c_g$ is a homomorphism (where $c_g$ is the conjugation homomorphism, $c_g(x) = gxg^{-1}$).

**Question 6** What are the orbits of the conjugation action of $\mathbb{Z}_6$ on itself?

**Question 7** Does the formula $aH \cdot bH$ define a group structure on the set of cosets $G/H$ for:

(a) $G = S_3$, $H = \langle (12) \rangle$

(b) $G = S_3$, $H = \langle (132) \rangle$

**Question 8** How many elements does the subgroup of even permutations in $S_n$ have?

**Question 9** Show that $\mathbb{Z}_{10}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_5$.

**Question 10** Show that $D_n$ is isomorphic to a semi-direct product $\mathbb{Z}_n \rtimes \mathbb{Z}_2$.

**Question 11** Use the correspondence theorem to show that every subgroup of $\mathbb{Z}_n$ is cyclic.

**Question 12** Is there a subgroup of $\mathbb{Z}_{10} \times \mathbb{Z}_{10}$ of order 8? of order 4?

**Question 13** Suppose that $G \cong N \rtimes \phi K$ and $G' \cong N' \rtimes \psi K'$. If $N \cong N'$ and $K \cong K'$ then does it follow that $G \cong G'$?