Quotient rings:

Let \( R \) be a ring, \( I \) a \( R \) ideal. Since \( I \) is a subgroup of \((R, +)\), we can form quotient group \( R/I \). The set of cosets \( \{ r+I \} \)

**Q:** When \( I \) is a ring so that \( \pi: R \rightarrow R/I \) is a **ring homomorphism**.

\[
\begin{align*}
\pi(rs) &= rs + I \\
\pi(r + I) \cdot \pi(s + I) &= (r + I)(s + I)
\end{align*}
\]

Only thing that helps is the formula

\[
(b) \quad (r + I)(s + I) = rs + I
\]

**Theorem:** If \( I \) is an ideal in a ring \( R \), then

(a) *define* a ring structure on \( R/I \) and

\[
\pi: R \rightarrow R/I \quad \text{is a ring homomorphism.}
\]

(b) If \( R \) is commutative, so is \( R/I \).

If \( I \) has \( 1 \) then \( 1 + R \) is the identity for \( R \).

**Proof:** Exercise:

\[
\begin{align*}
(a+I)(b+I)(c+I) &= (a+I)(bc+I) = a(bc)+I = (ab)c+I = (ab+I)(c+I) = ((a+I)(b+I))(c+I) = \quad \text{so associative} \quad 2
\end{align*}
\]

\[
\begin{align*}
(a+I) + (b+I)(c+I) &= (a+b+I)(c+I) = (a+b)c+I = (ac+bc)+I = (ac+I)(c+I) + (bc+I)
\end{align*}
\]

\[+ \ldots \]

\[ \Box \]
Example: \( R \leq 2 \rightarrow \frac{R}{n^2} \) has ring structure.

Example: \( R[x] \)

\[
I = (x^2 + 1)
\]

\[
\frac{R[x]}{I} = \frac{R[x]}{(x^2 + 1)}
\]

given a coset: \( g + (x^2 + 1) = h + (x^2 + 1) \) \( \Rightarrow \) \( g - h \in (x^2 + 1) \)

so if choose for any \( g + (x^2 + 1) \) \( r \in I \) \( \deg(r) < 2 \)

\[
g + (x^2 + 1) = r + (x^2 + 1)
\]

(namely: write \( g = \frac{g(x^3 + 1)}{q(x^3 + 1)} \) via division algorithm.)

\[
\frac{R[x]}{I} = \left\{ ax + b + (x^2 + 1) \right\} \ a, b \in R
\]

\[
\left( a + bx + (x) \right) \left( c + dx + (x) \right) =
\]

\[
a c + adx + bcx + bdx^2 + (x)
\]

\[
= a c + (ad + bc)x + bd(x^2 + 1) - bd + (x)
\]

\[
= (ac - bd) + (ad + bc)x + (x).
\]
More generally,

K a field \((f) \in K[t]\), ideal gen by nonzero \(f\).

\(K[t]/(f) = \{ r + (f) \mid \deg(r) < \deg(f) \}\)

can be \(g + (f)\), write \(g = f + r\), where \(\deg(r) < \deg(f)\).

\((r + (f))(s + (f)) = rs + (f) = a + (f)\).

**Homomorphism theorem for rings**

Let \(\phi: R \to S\) be a surjective homomorphism of rings with \(\ker(\phi) = I\).

Then \(\exists \overline{\phi}: R/I \to S\), isomorphism such that

\[\overline{\phi}[r] = \phi(r)\]

\[R \xrightarrow{\phi} S\]

\[\pi \downarrow \overline{\phi}\]

\[R/I \]

**Proof:** See Theorem 6.3.4 in text.

**Example:** Define \(\phi: \mathbb{R}[x] \to \mathbb{C}\) by \(\phi(g(x)) = g(i)\).

This is surjective. (every elt of \(\mathbb{C}\) is of form \(z = a + bi\), \(a, b \in \mathbb{R}\).

\(z = \phi(i)\) where \(g = a + bx\).

What is \(\ker(\phi)\)?

\(\{ g(x) \in \mathbb{R}[x] \mid g(i) = 0 \}\)

Note: \((x^2 + 1) \in \ker(\phi)\).
We know that \( \ker(\phi) \subseteq \K[x] \) is a principal ideal (last time) and it is generated by a nonzero polynomial of least degree in \( \ker(\phi) \).

Suppose \( a+bx \in \ker(\phi) \), if \( b=0 \Rightarrow a \in \ker(\phi) \Rightarrow a = 0 \).

If \( b \neq 0 \Rightarrow a+bi=0 \Rightarrow i = -\frac{a}{b} \in \R \).

So there are no degree 0 polynomials in \( \ker(\phi) \) except possibly \( i \) or \( 2 \Rightarrow \ker(\phi) = (x^2+1) \).

By Homomorphism Theorem we conclude that

\[
\frac{\K[x]}{(x^2+1)} \cong \C.
\]