Lecture 2: More examples - symmetry

\((\text{GL}_n(\mathbb{R}), \circ)\)

\(\{ A \in \text{M}_n(\mathbb{R}) \mid \det(A) = 0 \} = \{ A \in \text{M}_n(\mathbb{R}) \mid A \text{ is invertible} \}\)

\(\det(AB) = \det(A) \det(B) \neq 0 \) if \(\det(A) \neq 0 \neq \det(B)\)

\(\Rightarrow AB \in \text{GL}_n(\mathbb{R}) \) if \(A, B \in \text{GL}_n(\mathbb{R})\).

- Associative: \(\forall A, B, C \in \text{GL}_n(\mathbb{R})\), \((AB)C = A(BC)\)
- Identity: \(I = \text{I}_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\)
- \(\text{O}_n(\mathbb{R}) = \{ A \in \text{GL}_n(\mathbb{R}) \mid A^T A = I \}\)
- \(\text{SO}_n(\mathbb{R}) = \{ A \in \text{O}_n(\mathbb{R}) \mid \det(A) = 1 \}\)

Symmetry of a geometric object:

- Example: consider a rectangle (unequal sides)
- A symmetry = motion returning the object back to its original position.
- \(C = \text{identity}\)
- \(r_1, r_2, r_3\) rotation by \(\pi\) around \(x, y, z\)-axis's respectively.

(Note: regard rotation by \(2\pi\) to be same as identity)

Multiplication table:
Matrix representations: the actions are the restriction of linear transformations of $\mathbb{R}^n$ (in fact, distance-preserving, orientation-preserving).

So have matrix representations (in fact: of $\text{SO}_3$).

What are they?

\[
e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad r_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad r_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad r_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\]

Example: consider a square.

What are the symmetries?

- $r =$ rotation by $\pi/2$ around $z$-axis
- $a,b,c,d =$ rotation by $\pi$ around $d$-axis

Also have multiplication table. (exercise)
Permutations:

Groups of the form $\text{Sym}(X)$ are very important, especially when $X = \{1, 2, \ldots, n\}$.

Example: 3 identical objects on a table:

![Diagram of 3 identical objects on a table]

This configuration has symmetries:
- Can interchange objects; these symmetries are exactly the bijections of $\{1, 2, 3\}$ with itself.

Write:

$S_n = \text{Sym}(\{1, 2, \ldots, n\})$ = symmetric group on $n$ elements.

Notation: If $\sigma: \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ is a bijection (= permutation), convert to write:

\[
\begin{array}{cccc}
1 & 2 & \cdots & n \\
\sigma(1) & \sigma(2) & \cdots & \sigma(n)
\end{array}
\]

So for example:

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$
Example: List the permutations of 3 objects:

\[
\begin{align*}
&\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\
&\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}
\end{align*}
\]