More complicated counting than last time:

**Example:** How many distinct necklaces can be made out of
3 blue beads, 2 white beads, 1 red bead?

We consider 2 arrangements of the beads to be the same
if one is obtained from the other by:

- sliding beads around the wire
- turning the wire over

This is an action of dihedral group $D_6$
on the set of arrangements of the beads.

We know how to count the
number of arrangements of these beads

= number of ways to arrange the beads around the necklace

now we need to count the number of
orbits under the action of the dihedral group $D_6$
on the set of arrangements.

This will require another theorem:

First notation: $\text{Fix}(g) := \{ x \in X \mid gx = x \}$ for $g \in G$
Proposition (Burnside's Lemma):

Let $G$ be a finite group and $X$ a set with a $G$-action. Then the number of orbits of the action is

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

"number of orbits = average number of fixed points" of $G$

Proof: next time.

Example: In the previous necklace example:

There are $\frac{(3+2+1)!}{3!2!1!} = \frac{6!}{3!2!} = 60$ ways to arrange the beads around the 6-gon.

Let $X$ be the set of such arrangements.

Let $D_6$ act on this set of arrangements via rotations and flips.

The number of orbits of the action = number of distinct necklaces

(Why?)
BURNSIDE'S LEMMA TELLS US:

\[ \text{# orbits} = \frac{1}{|D_6|} \sum_{g \in D_6} |\text{Fix}(g)|. \]

Now have to count \(|\text{Fix}(g)|\) for various \(g \in D_6\).

- \(g = e\) \(\text{Fix}(e) = \{ \text{everything} \} = X\), \(|\text{Fix}(e)| = 60\)
- \(g = \text{rotation}\) \(\text{Fix}(g) = \emptyset\) (e.g. the red bead will always be moved by a rotation)
- \(g = \text{flip}\) \(\cdot\) a flip has 2 fixed vertices and interchanges the 2 remaining pairs of vertices.

For the flip to fix the array of beads one fixed vertex must be red and the other blue (so there are 4 pairs of remaining colors:

- A pair of W
- A pair of B

So the number of order 2 pairs is 1 pair blue, 1 pair white.

\[ \text{# ways to order the fixed vertices} = 1 \text{ red} \cdot 1 \text{ blue} = 2 \]

\[ \text{all in all: there are } 2 \cdot 2 \cdot 4 \text{ fixed arrangements for a flip.} \]
\[ |\text{Fix}(g)| = 4 \text{ if } g \text{ is a flip.} \]

\[ D_6 \text{ has: 5 nonidentity rotations, 6 flips.} \]

So using Burnside's lemma:

\[
\text{# orbits of } D_6 \text{ acting on } X \quad \frac{1}{|D_6|} \sum_{g \in D_6} |\text{Fix}(g)| = \frac{1}{12} \left( 60 + 5 \cdot 0 + 6 \cdot 4 \right)
\]

\[
= \frac{84}{12} = 7.
\]

**Proof of Burnside's lemma:**

want to show \( \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| = \text{# orbits} \)

Recall: \( x \in X \mapsto \text{Stab}(x) = \{ g \in G \mid g \cdot x = x \} \)

\( g \in G \mapsto \text{Fix}(g) = \{ x \in X \mid g \cdot x = x \} \)

combine to get the set:

\[ F = \{ (g, x) \in G \times X \mid g \cdot x = x \} \]

This can be expressed 2 ways:

\[ F = \bigcup_{g \in G} \text{Fix}(g) \quad \text{and} \quad F = \bigcup_{x \in X} \text{Stab}(x) \times x \]
$$\sum_{g \in G} |\text{Fix}(g)| = |F| = \sum_{x \in X} |\text{Fix}_{\text{stab}}(x)|$$

Dividing by $|G|$, we have:
$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| = \sum_{x \in \text{orb.}} \frac{1}{|\text{stab}(x)|}$$

This is the sum over the distinct orbits $\mathcal{O}$ of the term:
$$\sum_{x \in \mathcal{O}} \frac{1}{|\text{stab}(x)|} = |\mathcal{O}| \cdot \frac{1}{|\text{stab}(x)|} = 1$$

So the total of
$$\sum_{x \in \text{orb.}} \frac{1}{|\text{stab}(x)|}$$

is just the # of orbits! \qed