Example: $G$ a group, $X = \text{set of subgroups of } G$

$$= \{ H | H \leq G \}$$

$G$ acts on $X$ by conjugation:

$$G_g(H) = gHg^{-1}$$

Orbit $O_g(H) = \{ K | K = gHg^{-1} \} = \text{set of subgroups conjugate to } H$

Stab($H$) = $\{ g \in G | gHg^{-1} = H \}$ = $N_G(H)$

-called the normalizer of $H$ in $G$.

This is the largest subgroup of $G$ in which

$H$ is normal.

Example: $D_4 = \{ e, r, r^2, r^3, j, rj, r^2j, r^3j \}$

$$jrj = r^{-1}$$

$N_{D_4}(H) =$ ?

$$j^3 = j$$

$$j^2 r^i j r^i = j r^i r^i = j r^{2i} = j \quad \iff \quad i = 0, 2$$

$N(H) = \{ e, r^2, j, r^2 j \}$

$\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle r^2 \rangle \times \langle j \rangle$
Conjugacy class of \( H \):

\[
(H) = \mathcal{O}(H) = \{ H, j^1 r^2 \}
\]

\[
\begin{align*}
(j r)^{j_1} &= j^2 r^3 j = j^{-1} r^{-1} j^{-1} r^2 & \Rightarrow & & r^1 H r^{-1} &= j^1 H j^{-1} \quad & (r^2)^2 &= j^2 H r^{-2} \\
(r^3)^3 &= j^3 r^3 j = j r^{-3} r^{-3} j^{-1} r^{-2} & \Rightarrow & & r^3 H r^{-3} &= j^3 H j^{-1}
\end{align*}
\]

Note: \( |D_4| = 8 = |O(H)| = |N(H)| \)

\[
2 \cdot 4
\]

Example: Let \( G \) act on itself by conjugation

\[
G \rightarrow \text{Aut}(G)
\]

\[
g \mapsto c_g
\]

Orbits:

\[
G \cdot h = \{ g h g^{-1} \} \quad \text{conjugacy class of } h
\]

\[
\text{Stab}(h) = \{ g \in G \mid g h g^{-1} = h \} = \{ g \in G \mid g h = h g \} =: \text{Cent}_G(h)
\]

"Centralizer of \( h \)"

Kernel of the action:\n
\[
Z(G) = \{ g \in G \mid g h g^{-1} = h \forall h \in G \}
\]

Center of \( G \).
Example: $G = S_4$, $\sigma = (12)(34)$

From HW we know $\tau \bar{\tau} = (\tau(1) \tau(2))(\tau(3) \tau(4))$

The conjugacy class of $\sigma$ is all permutations which are a product of 2 disjoint 2-cycles:

$$[\sigma] = \{ (12)(34), (13)(24), (14)(23) \}$$

Centralizer of $\sigma$:

$$\{ \tau \mid (\tau(1) \tau(2))(\tau(3) \tau(4)) = (1 2)(3 4) \}$$

$$\{ e, (12), (34), (1 3)(2 4), (14)(23), (14 23), (1 3 2 4) \}$$

$$\{ (12)(34) \}$$

$|\text{Cent}(\sigma)| = 8$ elements.

Note: $|S_4| = 24 = |\text{Cent}(\sigma)| \cdot |[\sigma]|$

$$8 \cdot 4$$
Proposition: Let $G$ act on $X$, then there is a bijection

$$\Phi: g/\text{stab}(x) \rightarrow G \cdot x$$

$$a \text{ Stab}(x) \rightarrow ax$$

Moreover for any $g \in G$, we have $\Phi(g \cdot (a \text{ Stab}(x))) = g \cdot \Phi(a \text{ Stab}(x))$

Proof: This is well-defined since $a \text{ Stab}(x) \cdot b \text{ Stab}(x)$

$$\Rightarrow b^{-1}a \in \text{ Stab}(x)$$

$$\Rightarrow b^{-1}ax = x$$

$$\Rightarrow ax = bx$$

This is injective by the argument just given.

If $y \in G \cdot x$, then there is $g \in G$ such that $gx = y$.

Then $\Phi(g \cdot \text{Stab}(x)) = g \cdot x = y$, so $\Phi$ is onto as well.

Lastly, $\Phi(g \cdot a \cdot \text{Stab}(x)) = \Phi(g \cdot \text{Stab}(x)) = ga \cdot x = g \cdot (ax) = g \cdot \Phi(a \cdot \text{Stab}(x))$

Corollary: If $G$ is a finite group and $G$ acts on $X$.

Then $\Phi x \cdot x$,

$$|G \cdot x| = |G|/|\text{Stab}(x)| = |G; \text{Stab}(x)|$$

In particular,

$$|G \cdot x| \text{ divides } |G|.$$

Proof: Follows from the previous result and Lagrange's theorem.