Lecture 1: First examples.

- What is this class about anyway? Abstract Algebra.

- Algebra: what do we think about when we hear the word?
  (Student supplied examples:)
  - integers, natural numbers
  - polynomials
  - \( \mathbb{R}, \mathbb{C} \)
  - vector spaces
  - matrices
  - ...?

- Abstractness: identify commonalities.
  Study those things which share the common feature.

- What are the commonalities of the objects above?
  - "addition" of elements, "subtraction" + operations
  (Student supplied)
  - neutral element
  - inverses (in \( \mathbb{N} \), doesn't have inverses)
  - order of addition matters? Some examples: yes in some, no in others
  - both addition and multiplication? Few operations: cubes, not always present.

- Nailing down precisely which commonalities to focus on is not easy.
  Takes time, thought, and distillation through the lessons of history.
We abstract in order to clarify and analyze, understand better, examples, at interest. 

In turn we find new examples in new places we didn’t see before.

Useful abstraction should perform this function: clarify examples we knew apply to new examples.

Important interplay:

Concrete example $\leftrightarrow$ abstraction

Lessons of history: Basic object here is a group.

Definition: A group $G$ is a set $G$ together with an operation

$*: G \times G \rightarrow G$

$(a,b) \mapsto a*b$

which satisfies the following properties

1) **Associativity**: $a*(b*c) = (a*b)*c$ for all $a,b,c \in G$

2) **Identity**: $e \in G$ such that

$e*a = a*e = a \quad \forall a \in G$

3) **Inverse**: For any $a \in G$, there exists an element $a^{-1}$ such that

$a*a^{-1} = a^{-1}*a = e$. 
Examples / nonexamples:

- \((\mathbb{Z}, +)\) check in detail: assoc.
- \((\mathbb{R}, +), (\mathbb{C}, +)\)
- \((\mathbb{Q}, +)\) no.
- \((\mathbb{N}, +)\) no.
- \((\mathbb{R}, \cdot)\)
- \((\mathbb{Q}, \cdot)\) no.
- \((\text{Mat}(\mathbb{R}), +)\)
- \((\text{GL}_n(\mathbb{R}), \cdot)\)

- \(\emptyset\)? No: A group must have an elt.
- \(X\) a nonempty set.

\[\text{Sym}(X) = \{ f : X \to X \mid f \text{ is bijective} \}\]

operation: \(f \circ g \) composition.

check: \(f, g \in \text{Sym}(X) \Rightarrow f \circ g \in \text{Sym}(X)\)

check: group axioms satisfied.