Dihedral groups

Another important class of examples:

Dihedral group $D_n$ (for $n \geq 3$)
in the group of symmetries of the regular $n$-gon
in the plane.

$n=3$

$n=4$

$n=5$

$n=6$

rotate around the center of mass
flip about lines of symmetry
Example: $D_3 \leq D_6$

$D_6 = \{ e, r^i, r^j \}$

- rotate by $2\pi/6$
- flip around

The triangle preserved by $j$ is flipped around, preserving by rotation by $\frac{2\pi}{3} = \pi/2$.

So the subgroup

$D_3 = \{ e, r^2, r^4, j, r^2j, r^4j \} \leq D_6$.
Let $R_\theta$ be the rotation about the origin by angle $\theta$ and $J_\theta$ be flip about the line which makes angle $\theta$ with the positive x-axis.

These are represented by matrices:

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$J_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

$D_n$ consists of:

$$D_n = \{ e, r, r^2, \ldots, r^{n-1}, j, rj, r^2j, \ldots, r^{n-1}j \}$$

Check:

$$J\phi R_\theta = r^{-\phi} J_\theta$$

$$J_\theta J_\phi = r^{2(\phi-\theta)}$$

$$R_\theta J_\phi R_\theta^{-1} = J_\phi + \theta$$

The relations determine the group structure of $D_n$.

Write $j = j_0$ and $r = r^{2\pi/n}$, we have two flip about x-axis.

Note: $|D_n| = 2n$ have cyclic subgroup of order $n$.

$C_n = \{ e, r, \ldots, r^{n-1} \} \subseteq D_n$ generated by rotations. [Redacted: $D_n = \{ e, r, r^2, \ldots, r^{n-1}, j, rj, r^2j, \ldots, r^{n-1}j \}$]