Topics
The final will be comprehensive; it covers all topics discussed during the term. These correspond roughly to 1.7 - 1.8, 1.10, 2.1 - 2.7, 3.1 - 3.2, 5.1 - 5.2, 5.4, 6.1 - 6.4, in the text. Review the topics from the midterm review sheets. Additionally:

1. Be able to give precise and correct definitions of the important concepts from lectures. Know the basic examples and properties and relations between concepts. This includes:
   - Sylow subgroup
   - ring, ring with 1, commutative ring, field, group of units
   - irreducible polynomial, root of a polynomial
   - ideals, principal ideals, maximal ideal
   - quotient rings
   - field extensions

2. Know what our important theorems say, know how and when to use them:
   - Burnside’s Lemma
   - Class equation
   - Sylow’s theorems
   - Homomorphism theorem for rings (Theorem 6.3.4)

3. Know how to use the orbit stabilizer theorem and Burnside’s lemma for counting arguments.

4. Determine when $R$ is a ring / ring with 1 / commutative ring / a field / etc...

5. Determine when $I \subseteq R$ is an ideal / principal ideal / maximal ideal.

6. Classify groups of order $p$, $p^2$, $pq$ ($p, q$ primes).

7. If $K$ is a field and $p(x) \in K[x]$ is irreducible, construct a field $F$ such that $p$ has a root in $F$.

8. Be able to reproduce simple proofs from class such as Proposition 5.4.2, Propoistion 6.2.15, Proposition 6.2.29(c), Proposition 1.8.8, Proposition 1.8.22.
Practice problems

See homeworks, quizzes/workheets, and review sheets for the midterms. Here are a few further practice problems.

1. Let $G$ be a finite group and consider the conjugation action on itself. Use Burnside’s lemma to compute the number of orbits:
   - $G = S_3$. (Confirm your answer by direct computation)
   - $G$ an abelian group. (Is there another way to arrive at the answer?)

2. Let $R$ be a commutative ring with 1 and $I \subseteq R$ an ideal. Show that the following are equivalent.
   (a) $I = R$
   (b) $I$ contains a unit (i.e. $I \cap R^\times \neq \emptyset$)
   (c) $1 \in I$.

3. Which are fields? Why/why not?
   (a) $\mathbb{Z}_{13}$
   (b) $\mathbb{Z}_{16}$
   (c) $\mathbb{Z}_3[x]/(x^2 + 1)$
   (d) $\mathbb{Z}_5[x]/(x^2 + 1)$

4. Show that $x^2 - 2$ is irreducible in $\mathbb{Q}[x]$. Can you find a field $F$ such that $\mathbb{Q} \subseteq F$ and $x^2 - 2$ has a root in $F$? Is $x^2 - 2$ irreducible in $F[x]$?

5. Show that $x^2 - 2$ is not irreducible in $\mathbb{Z}_7[x]$.

6. If $G$ is abelian and $p$ is a prime, what are the possible values for $n_p$, the number of $p$-Sylow subgroups?

7. Suppose that $G$ is a group of order 88. Let $H \leq G$ be an 11-Sylow subgroup and $K$ a 2-sylow subgroup. Must $H$ be normal? What about $K$? (In each case, give an argument for normality or provide a counterexample).

8. For which $a$ is there a semi-direct product $\mathbb{Z}_5 \rtimes_a \mathbb{Z}_a$ (not isomorphic to the direct product)?


10. What is the normalizer of $\langle j \rangle \in D_4$?

11. Let $H \leq G$ be a subgroup. Is $N_G(H)$ normal in $G$? Is $H$ normal in $N_G(H)$?