Product groups: direct products, semidirect product

Let $A, B$ be groups:

Define $A \times B = \{ (a,b) \mid a \in A, b \in B \}$

- Operation: $(a,b)(a',b') = (aa', bb')$
- Identity: $(e_A, e_B)$
- Inverse: $(a,b)^{-1} = (a^{-1}, b^{-1})$

Example: $Z_a \times Z_b$, $Z_{ab} \cong Z_a \times Z_b$ (a, b relatively prime)

Observe: $A \times B$ contains normal subgroups isomorphic to $A$ and $B$:

$A \times e \cong A$, $e \times B \cong B$

$A \times e \leq A \times B$

$e \times B \leq A \times B$

Also: $(A \times e)(e \times B) = A \times B$

And: $A \times e \cap e \times B = \{e\times e\}$
Proposition: Suppose $G$ has 2 normal subgroups $A \triangleleft G$ and $B \triangleleft G$, and $A \cap B = \{e\}$ and $G = AB$.

Then $G \cong A \rtimes B$.

Proof: Define $f : A \rtimes B \rightarrow G$ by $(a, b) \mapsto ab$; check that this is a homomorphism.

Here $f((a, b) (a', b')) = f((a a', b b')) = a a' b b'$

$f((a, b) (a', b')) = aba'b'$,

so need to see that $ba' = a'b$.

This holds since we have:

$ab a'^{-1} b^{-1} \in A \cap B = \{e\}$ (Why?).

Here that $f : A \rtimes B \rightarrow G$ is onto. If $g(a, b) \in \text{Ker}(f)$ then

$ab = e$

$\Rightarrow a = b^{-1}$

$\Rightarrow a \in A \cap B \Rightarrow a = e = b$.

$\Rightarrow f$ is one-to-one.

Thus $A \rtimes B \cong G$. \[\Box\]
Example: We can prove \( \mathbb{Z}_{ab} \cong \mathbb{Z}_a \times \mathbb{Z}_b \) (\(a, b\) relatively prime) in a different way:

\[ A = \langle a \rangle \leq \mathbb{Z}_{ab} \]
\[ B = \langle b \rangle \]

\[ |A| = a, \quad |B| = b \]

Then \( A \cap B = \{0\} \) since \( |A \cap B| \) divides both \( |A|, |B| \)

\[ \Rightarrow |A \cap B| = 1. \]

So \( AB \cong \mathbb{Z}_a \times \mathbb{Z}_b \) by the previous result.

But this means \( \mathbb{Z}_a \leq \mathbb{Z}_{ab} \Rightarrow AB = \mathbb{Z}_{ab} \Rightarrow \mathbb{Z}_a \times \mathbb{Z}_b \cong \mathbb{Z}_{ab} \).

The previous result can be extended to recognize product of normal groups:

**Proof:**

If \( N_1, N_2, \ldots, N_r \) are normal subgroups of \( G \) and

\[ G = N_1 \times N_2 \times \cdots \times N_r \]

and \( G = N_1 \cap (N_2 \times \cdots \times N_r) = e \)

then

\[ G \cong N_1 \times N_2 \times \cdots \times N_r. \]

\( N_1, N_2, \ldots, N_r \) are normal subgroups of \( G \).
Warning: It is not enough for \( N_i \cap N_j = \emptyset \).

Example:
\[ G: \mathbb{R} \times \mathbb{R}, \quad N_1 = \mathbb{R} \times \mathbb{R}^3, \quad N_2 = \mathbb{R}^3 \times \mathbb{R}, \quad N_3 = \{(x, x) \mid x \in \mathbb{R}^3\} \]
\[ \text{But } N_1 + N_2 + N_3 \rightarrow \mathbb{R} \times \mathbb{R} \]
\[ (x, y, z) \mapsto x + y + z \]
\[ \text{not an isomorphism.} \]

Semidirect products

Define a group \( \text{Aff}_n \) as follows:
\[ \text{Aff}_n = \left\{ \text{transformations of } \mathbb{R}^n \text{ of the form} \right\}
\[ T(x) = Ax + b \]
\[ \text{where } A \in \text{GL}_n(\mathbb{R}), \quad b \in \mathbb{R}^n \]

Check: this is a group:
Write \( T_{ab} \) for \( T(x) = Ax + b \)
\[ T_{ab} T_{a'b'} = T_{ab' + b} \]
\[ (T_{ab})^{-1} = T_{a^t, -b^t} \]

Now consider a subgroup
\[ N = \left\{ T_{I_1, b} \mid b \right\} \leq \text{Aff}_n \]
In identity matrix. So \( N \) is the subgroup of transformations of \( \mathbb{R}^n \) of \( \text{Aff}_n \):
\[ N = \{ x + b \} \]
$N \leq \text{Aff}_n$ is normal: since
\[
T_{Ax} T_{I,b} T_{A,a}^{-1} = T_{Ax} T_{I,b} T_{A, a^{-1}x}
\]
\[
= T_{Ax} T_{A^{-1}, b} A^{-1}x + b
\]
\[
= T_{I, -b + Ab + x} = T_{I, Ab}
\]

Also have the subgroup
$GL_n(\mathbb{R}) \leq \text{Aff}_n$

\[
\{ T_{A, 0} \}
\]

We have: $N \cap GL_n(\mathbb{R}) = \text{Aff}_n$, $N \cap GL_n(\mathbb{R}) = \{ I \}$

Even have $\text{Aff}_n / N \cong GL_n(\mathbb{R})$ (use homomorphism theorem)

But $\text{Aff}_n \neq N \cap GL_n(\mathbb{R})$,

$N, GL_n(\mathbb{R})$ don't commute:

\[
A T_{I,b} = T_{I,b} A
\]

for $A \in GL_n(\mathbb{R})$

Here's a way groups can be put together to form something similar to, but different from direct product.