Topics
The final will be comprehensive; it covers all topics discussed during the term. These correspond roughly to 1.7 - 1.8, 1.10, 2.1 - 2.7, 3.1 - 3.2, 5.1 - 5.2, 5.4, 6.1 - 6.4, in the text.

Note: The review/problems below covers only the material we’ve gone over after the last midterm exam. Be sure to go back over the review sheets for the midterm exams as well.

1. Be able to give precise and correct definitions of the important concepts from lectures. Know the basic examples and properties and relations between concepts. This includes:
   - ring, ring with 1, commutative ring, field, group of units
   - irreducible polynomial, root of a polynomial
   - ideals, principal ideals, maximal ideal
   - quotient rings
   - field extensions

2. Homomorphism theorem for rings (Theorem 6.3.4)

3. Determine when $R$ is a ring / ring with 1 / commutative ring / a field / etc...

4. Determine when $I \subseteq R$ is an ideal / principal ideal / maximal ideal.

5. If $K$ is a field and $p(x) \in K[x]$ is irreducible, construct a field $F$ such that $p$ has a root in $F$.

6. Be able to reproduce simple proofs from class such as Proposition 5.4.2, Propoistion 6.2.15, Proposition 6.2.29(c), Proposition 1.8.8, Proposition 1.8.22, Corollary 1.8.24.
Practice problems

See homeworks, worksheets, and review sheets for the midterms. Here are a few further practice problems.

1. Let $R$ be a commutative ring with 1 and $I \subseteq R$ an ideal. Show that the following are equivalent.
   
   (a) $I = R$
   
   (b) $I$ contains a unit (i.e. $I \cap R^\times \neq \emptyset$)
   
   (c) $1 \in I$.

2. Let $K$ be a field. Determine (with proof) all possible ideals of $K$.

3. If $R$ is a unital subring of a field $K$, must $R$ also be a field?

4. Let $K$ be a field. Show that either $\mathbb{Q} \subseteq K$ or $\mathbb{Z}_p \subseteq K$ for some $p$.

   (Hint: Recall that there is a canonical ring homomorphism $\phi: \mathbb{Z} \to K$. Show that if $\phi$ is injective, then $\mathbb{Q} \subseteq K$. If $\phi$ is not injective, then $\mathbb{Z}/\ker(\phi) \cong \phi(\mathbb{Z})$, so you need to show that $\mathbb{Z}/\ker(\phi) = \mathbb{Z}_p$ for some $p$.)

5. Which are fields? Why/why not?

   (a) $\mathbb{Z}_{13}$
   
   (b) $\mathbb{Z}_{16}$
   
   (c) $\mathbb{Z}_3[x]/(x^2 + 1)$
   
   (d) $\mathbb{Z}_5[x]/(x^2 + 1)$

6. Show that $F = \mathbb{Q}[x]/(x^2 - 2)$ is a field. Is $x^2 - 2$ irreducible in $F[x]$? Determine the multiplication in $F$.

7. Construct a field $K$ which contains $\mathbb{Z}_5$ and such that $x^2 - 3$ has a root. Determine the multiplication in $K$. How many elements does $K$ have?

8. Construct a field $K$ which contains $\mathbb{Z}_7$ and such that $x^2 - 2$ has a root. Determine the multiplication in $K$. How many elements does $K$ have?