Lecture 13: Asymptotics for Ordinary Differential Equations: Regular Perturbations

A regular perturbation can be informally defined to be a perturbation that does not change the nature of the problem. Consider, for example, a system of ordinary differential equations

\[ \frac{dx}{dt} = f(t, x, \epsilon) \quad x(0) = x_0 \]

where \( f(x, \epsilon, t) \) is Lipschitz in \( x \) and \( C^1 \) in \( \epsilon \). A standard result in the theory of ordinary differential equations says that (for small times!) one has that the solution \( x(t, \epsilon) \) is a \( C^1 \) function of \( \epsilon \).

**Theorem 1.** Suppose that \( f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is a continuous function, and is continuously differentiable with respect to \( x \) and \( \epsilon \). Then the solution to

\[ \frac{dx}{dt} = f(t, x, \epsilon) \quad x(0) = x_0 \]

is a \( C^1 \) function of \( \epsilon \). Further if one denotes \( y = \frac{dx}{d\epsilon} \) then one has that \( y \) solves the following linear equation

\[ \frac{dy}{dt} = \frac{\partial f}{\partial x} (t, x, \epsilon) y + \frac{df}{d\epsilon} \]

The equation satisfied by \( y \) is the linearized flow, and follows formally from simply differentiating the equation with respect to the parameter \( \epsilon \). Note that once you have this for a single derivative you automatically have it for higher derivatives.

This is the basic result for perturbation theory for ordinary differential equations. Let me start with one example to illustrate both the strengths and the weaknesses of the method.

**Example 1.** Find the approximate solution to the problem

\[ x'' - \epsilon x' + x = 0 \quad x(0) = 0 \quad x'(0) = 1 \]

The leading order solution \( x^{(0)}(t) \equiv x(t, \epsilon = 0) \) solves

\[ x^{(0)''} + x^{(0)} = 0 \quad x^{(0)}(0) = 0 \quad x^{(0)'}(0) = 1 \]

which has the solution \( x^{(0)}(t) = \sin(t) \). The next order \( y(t) \equiv \frac{dx}{d\epsilon} (t, \epsilon = 0) \) solves

\[ y'' - x^{(0)'} + y = 0 \quad y(0) = 0 \quad y(0) = 0 \]

using the fact that \( x^{(0)}(t) = \sin(t) \) we find that \( y(t) = \frac{t}{2} \sin(t) \). This gives the solution to leading order as

\[ x(t, \epsilon) = (1 + \epsilon \frac{t}{2}) \sin(t) \]

Of course we can just solve this directly. The solution is given by

\[ x(t, \epsilon) = \frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2} \]

where \( r_1, r_2 \) are the roots of \( r^2 - \epsilon r + 1 = 0 \). For small \( \epsilon \) we have \( r_{12} = \pm i + \epsilon/2 + O(\epsilon^2) \). This gives

\[ x(t, \epsilon) \sim \frac{e^{(i+\epsilon/2) t} - e^{(-i+\epsilon/2) t}}{2i + O(\epsilon^2)} = e^{\frac{\epsilon}{2} t} \sin(t) + O(\epsilon^2) \]

Taylor expanding the exponential \( e^{\frac{\epsilon}{2} t} \sim 1 + \frac{\epsilon}{2} t \) shows that the two results agree to leading order.
Some comments: First note that the approximation is only good for times such that $\epsilon t \ll 1$: the approximation is not uniform in time but only holds on compact sets. This is to be expected. Secondly notice that the (approximate) exponential solution is in some sense better than the approximation that we derived from perturbation theory: it "knows" that the solution grows exponentially, while our approximation grows only linearly. We will later learn a technique (the method of multiple time-scales) to fix this.

Let’s do another example

**Example 2.** Find an approximate solution to the boundary value problem

$$y'' = -\lambda (1 + \epsilon x) y \quad y(0) = 0 \quad y(1) = 0$$

(Note: this can be exactly solved in terms of Airy functions, which we have discussed.)

Note that we know that the problem above only has solutions for certain values of $\lambda$. The unperturbed problem is

$$y'' = -\lambda y \quad y(0) = 0 \quad y(1) = 0$$

This has a non-zero solution if and only if $\lambda = n^2 \pi^2$. Otherwise the only solution is $y = 0$. These are the eigenvalues of the problem. Generally when epsilon changes the eigenvalues will change as well. So we assume $\lambda = \lambda(0) + \epsilon \lambda(1) + \ldots$ and $y = y(0) + \epsilon y(1) + \ldots$ and expand we find

$$y''_0 = \lambda_0 y_0 \quad y(0) = y(1) = 0$$

so $\lambda_0 = \pi^2$ (the lowest eigenvalue) and $y = \sin(\pi x)$. The next order equation is

$$y''_1 = \lambda_0 y_1 + \lambda_1 \sin(\pi x) + x \sin(\pi x)$$

Solving this with $y_1(0) = 0$ gives

$$y_1 = \frac{-\pi x (2 \lambda_1 + x) \cos(\pi x) + (2 \lambda_1 + 4 \epsilon \pi + B x) \sin(\pi x)}{4 \pi^2}$$

where $c$ is an arbitrary constant of integration. Substituting $x = 1$ in the above gives

$$y_1(1) = \frac{2 \lambda_1 + 1}{4 \pi}.$$

This should be zero, so this tells us that $\lambda_1 = -\frac{1}{2}$.

We will also see later that there are much better and easier ways to solve eigenvalue problems perturbatively, methods that utilize the algebraic structure.