Lecture 3: Math 553

Burger’s Equation:

The (inviscid) Burger’s equation is the following first order quasi-linear differential equation:

\[ u_t + uu_x = 0 \]

The characteristic equations are:

\[ \frac{dt}{ds} = 0 \quad t(0) = 0 \quad (1) \]
\[ \frac{dx}{ds} = u \quad x(0) = \alpha \quad (2) \]
\[ \frac{du}{ds} = 0 \quad u(0) = f(\alpha) \quad (3) \]

Integrating these up we find that

\[ t = s \quad (4) \]
\[ x = f(\alpha)s + \alpha \quad (5) \]
\[ u = f(\alpha) \quad (6) \]

Eliminating \( \alpha \) gives

\[ u = f(x - ut) \]

This defines \( u \) implicitly as a function of \( x, t \), although not explicitly. If we think of this as a change of coordinates from \( x, t \) to \( \alpha, s \) it makes sense to look where the change of coordinates breaks down. This occurs where the Jacobian of the map vanishes. The Jacobian of the map is given by

\[
\begin{vmatrix}
\partial(x, t) \\
\partial(\alpha, s)
\end{vmatrix} = \left| \begin{array}{cc}
f'(\alpha)s + 1 & f(\alpha) \\
0 & 1
\end{array} \right| = -(1 + f'(\alpha)s)
\]

The Jacobian vanishes when \( s = -\frac{1}{f'(\alpha)} \). Note that \( s = t \), so the Jacobian never vanishes for positive times if \( f(\alpha) \) is increasing. If \( f(\alpha) \) is decreasing then the Jacobian will vanish at some positive time. If \( f' \) is somewhere negative then the first time that the Jacobian is going to vanish is at time

\[ t = -\frac{1}{\min_\alpha f'(\alpha)} \]

This meshes nicely with our intuition about the solution. The Burger’s equation represents a transport equation where regions of higher density travel faster than regions of lower density. Thus if the function is decreasing (there is a region of higher density behind a region of lower density) then one expects that the higher density stuff will catch up to the lower density stuff and over take it. This corresponds to the solution going from single valued to triple valued.

One way to understand what is going on is to look at the characteristics projected onto the \( (x, t) \) plane.

This illustrates very clearly the development of what are called caustics. The characteristics behind have smaller slope, and those ahead have larger slope, so
they must cross. They do so in the “fold” or caustic region which should be clear from the graph.

If we look at the solution as a surface in $\mathbb{R}^3$ it looks like this

**Example:** To see clearly what is going on, let’s take very special initial data. Let $g(z)$ be the function $g(z) = -(z^3 + z)$. This function is obviously invertible. Take $f(x) = g^{-1}(x)$ (meaning the inverse function, not the reciprocal!). The function $f$ looks like

\[
\text{BY the formula we derived we have that } u \text{ satisfies }
\]
\[
u = f(x - ut)
\]

or equivalently
\[
g(u) = x - ut
\]

which is the same as
\[
-u^3 - u + ut = x
\]
This is a cubic equation in \( u \). It is not hard to see that this cubic has one real root if \( t - 1 < 0 \) and three real roots if \( t - 1 > 0 \). Thus the solution becomes triple valued at \( t = 1 \). By our Jacobian formula it is easy to see that, by implicit differentiation, \( \min_\alpha f' = -1 \), so the Jacobian calculation gives the same answer: the solution ceases to be well-behaved at \( t = 1 \).

**Exercise:** Suppose that \( t > 1 \). Compute the interval in \( x \) for which the solution \( u \) is multiply-valued.

**Envelope:**

I want to make a slight divergence to talk about the concept of an envelope. This idea arises naturally in characteristics and will be useful in treating the case of a fully nonlinear equation. Suppose that one has a one-parameter family of curves in the plane:

\[ F(y, x; \alpha) = 0 \]

For fixed \( \alpha \) this gives a curve. The envelope to the family of curves is defined to be the simultaneous solution to

\[
\begin{align*}
F(y, x; \alpha) &= 0 \\
\frac{\partial F}{\partial \alpha} &= 0
\end{align*}
\]

Since there are two equations and three unknowns one can eliminate \( \alpha \) and get \( y \) as a function of \( x \). This curve is called the envelope. There are a cou-
ple of ways to think about this curve. The first is as the intersection of two
infinitesimally nearby curves in the family

(Algebra)

Another way to think about it is as a curve which is tangent to each member
of the family at the point of intersection.

Example
Consider the family of lines

\[ \cos \theta x + \sin \theta y = 1 \]

The envelope of these lines is easily seen to be a circle.

(Algebra)

here is a picture.

Burger’s equation
Recall that last time we saw that the Burger’s equation

\[ u_t + uu_x = 0 \quad u(x, 0) = f(x) \]

fails to be single valued at some time whenever the initial data \( f(x) \) is somewhere
decreasing. It is interesting to calculate the envelope of the characteristics in a
particular case. For example \( f(x) = -\tanh(x) \). In this case the initial data
has a slope of \(-1\) at the origin, so we know from the Jacobian calculation of
last time that the solution will break down at time 1. We can also see that by
calculating the envelope of the family of characteristics. The characteristics are
given by

\[ t = s \]  
(9)

\[ x + \tanh(\alpha)s = \alpha \]  
(10)

\[ u = -\tanh(\alpha) \]  
(11)

The envelope is given by the simultaneous solution to the equations

\[ x + \tanh(\alpha)t = \alpha \]  
(12)

\[ \text{sech}^2(\alpha)t = 1 \]  
(13)

After some algebra

we get the following formula for the envelope curve:

\[ x = \frac{1}{2} \ln\left(\frac{2 - 4t \pm \sqrt{(2 - 4t)^2 - 4}}{2}\right) - \sqrt{t(t - 1)} \]

Note that this curve only begins at \( t = 1 \), where it comes to a cusp. This marks the beginning of the region which is triply covered by caustics.

**After Breaking**

Ok, this raises the question of what we need to do to make sense of the
solution after the time of first breaking.

Picture

The idea of a weak solution is this: we allow a discontinuity into the solution to restore single-valuedness. We have to do this in a very particular way if the solution is to remain physical. The solution can no longer solve the differential equation but it can remain a solution of the integral form of the equation. This is referred to as a weak solution. Recall that the weak form of the equation

\[ u_t + (g(u))_x = 0 \]

is given by

\[ \frac{d}{dt} \int_a^b ud\xi + g(u(b,t)) - g(u(a,t)) = 0 \]

The basic physical principle we use is the conservation of mass. We are going to make this the basis of our discontinuous solution. We demand that the discon-
tinuity respect the mass conservation. Suppose that the shock or discontinuity is located at $\eta(t)$. Then we have that in the neighborhood of a shock $a < \eta < b$

The first term must vanish if mass is to be conserved. This gives a condition called the Rankine-Hugoniot jump condition which relates the speed of propagation of the shock to the values on either side.

As an example we consider the problem of the Burgers’ equation with step initial data:

$$u_t + (u^2/2)_x = 0 \quad u(x,0) = H(x)$$

The characteristics look like this

so the wedge between the line $x = 0$ and $x = t$ is doubly covered by characteristics. The shock location $\eta(t)$ satisfies the equation

$$\eta' = 1/2$$

so the solution is given by $u(x,t) = H(x - t/2)$
It is worth emphasizing that this procedure is not unique. There are many possible weak formulations. For example we can multiply through by \( u \) to get

\[
\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x = 0 \quad u(x,0) = H(x)
\]

The corresponding Rankine-Hugoniot equation is

\[
\eta' = \frac{2(u_+^3 - u_0^3)}{3(u_+^2 - u_0^2)} = 2/3
\]

so the shock speed has changed. What gives?

The thing to understand is that by multiplying through by \( u \) we have changed the thing that is conserved. We are now conserving \( u^2 \) instead of \( u \). In the case where the solution is smooth both of these (and more) are conserved. But when we go to a weak solution we are no longer able to conserve all of them, so we must somehow pick which conserved quantity to keep. Different choices involve different physics and lead to different shock speeds.

If we are allowing discontinuous initial conditions there is a second kind of structure we have to allow, called a rarefaction fan. Consider the Burger’s equation with initial data \( u(x,0) = H(-x) \). If we draw the characteristics we have the following

So there is a wedge \( 0 < x < t \) where there are no characteristics. In a physical problem it doesn’t make sense to just say that the solution is not defined in this region. Hence we look for a special type of solution. First note that the following function

\[
u(x,t) = \frac{x - x_0}{t - t_0}
\]

is a solution of the Burger’s equation for \( t > t_0 \). This solution is singular at \( t = t_0 \) and it is constant along lines \( (x - x_0) = c(t - t_0) \). We can think of these as being the characteristics. Thus the characteristics form a fan.

The basic idea is that we use this solution to “patch” the hole where we have no characteristics.